Quantum Field Theory

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Module - 3
Free Field Theory Quantization Spinor and Vector Fields
Lecture - 15
Quantuzation of Electromagnetic Field I

We have been discussing quantization of a scalar fields, although some of the results that we have derived is a much more general than scalar field. So, it goes beyond scalar fields, but mostly we were discussing a free scalar fields as well as self interacting scalar fields and a bunch of scalar fields interacting with each other. That is not what happens in the real world, in real world there are particles which have non zero spin, and then they interact among each other, electromagnetic field interacts with radiation field with electron and so on. So, we need to consider beyond scalar fields one of the classical example of a quantum field which is not a scalar field is the radiation field or electromagnetic field, what we will discuss now is we will discuss the quantization of free radiation field.

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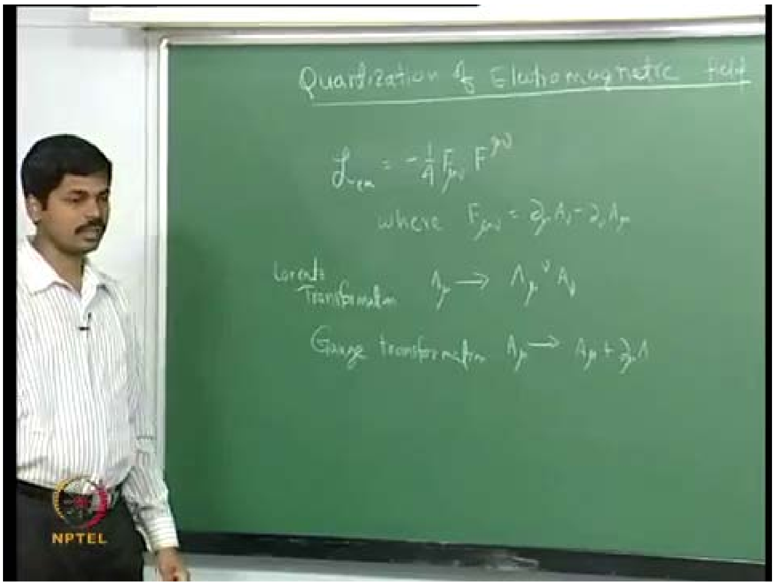

So, let us consider the Quantization of Electromagnetic Field, normally when you have a bunch of scalar fields, let us say this scalar fields are label by pi i then what do you do you. You consider the Lagrangian density and then you derive the corresponding conjugate momentum, let us say you have a free field then pi i phi i of x t pi j of y t. So, you postulate this equal time commutation relation to be i delta i j for this label psi n j and delta x minus y.

Now, we have the electromagnetic field which I will denoted as a mu the Lagrangian density, which governs the dynamics of this field is given by L electromagnetic is a minus 1 4th F mu nu F mu nu, where F mu nu equal to del mu A mu minus del mu A mu. The gauge field A mu is a vector field, which amongst to saying that under Lorentz transformation, this field A mu goes to lambda nu mu A mu that is why it is a vector field. And because it transforms nontrivially under Lorentz transformation, it has non zero spin and you can work out the details and it turns out that, this represents a spin one field that is why we consider photons to represent by this field.

In addition you have also a gauge transformation, under which this field A mu goes to A mu plus del mu lambda, where lambda is any arbitrary function of space time. You can notice that under this transformation this field's strength F mu nu remains invariant correspondingly the electric as well as magnetic field, remain invariant under this gauge transformation. And hence your equations of motion remain invariant under this gauge transformation.

So, we will consider the quantization of this electromagnetic field or the gauge field. How do you do that, we will see what happens when we try to do it in the usual way that is to find the corresponding conjugate momentum for this Lagrangian and then look at the equal time commutation relations.

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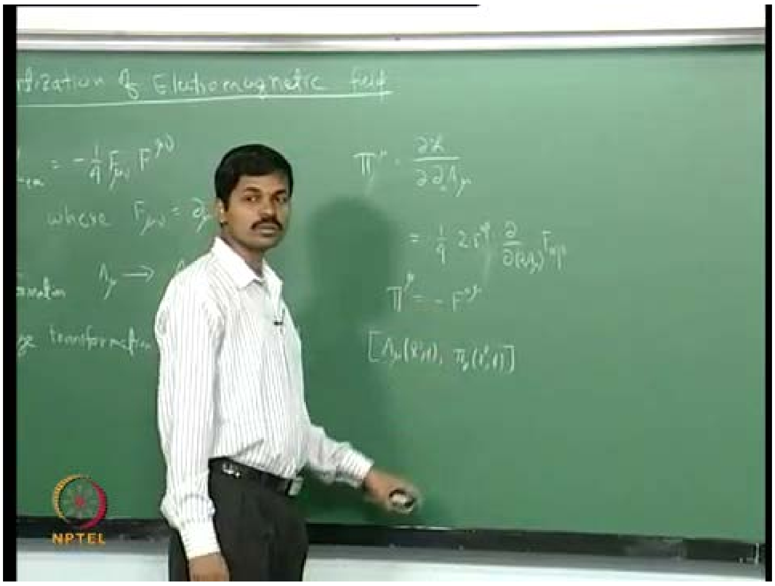

So, the conjugate momentum which I will denote as pi mu or pi mu equal to del l over del del 0 A mu right, there are four fields A 1, A 2, A 3, A 4, four independent quantities. So, they correspondingly you have four conjugate momentum that I am labeling this index mu here and I mu is given by this. From the expression here, you can see that it is just a minus 1 4th twice f alpha beta del over del del 0 A mu f right; so this will give me minus f 0 mu.

So, therefore, we have pi mu the conjugate momentum is given by this you might think of considering this commutation relations A mu pi mu A mu of x t pi mu x t. However, it is not, so trivial the reason is following, if you consider pi 0, the conjugate variable to the field A 0, then because of anti-symmetry of f mu nu pi 0 vanishes. Therefore, you have a difficulty in considering the commutation relation here, because one of the conjugate momentum actually vanishes so how to deal with that.

There are many ways you can handle this problem one of them is by exploiting the gauge transformation here, because the theory is invariant under this gauge transformation you can what you can do is, you can fix a gauge and then you can quantize. Or you can start with this Lagrangian you can add an additional terms here and then you can quantize it in different way.

So, we will what we will see is that, we will quantize it into two different ways first one is we will first fix the gauge and then we will quantize with theory and then what is we will consider the covariant quantization, which we will perform without fixing any gauge. So, let us first do not worry about the Lorentz covariance or gauge invariance or anything like that we will just do the gauge fixing, so how do you do that. Let us write the equations of motion corresponding to this Lagrangian in a non covariant way.

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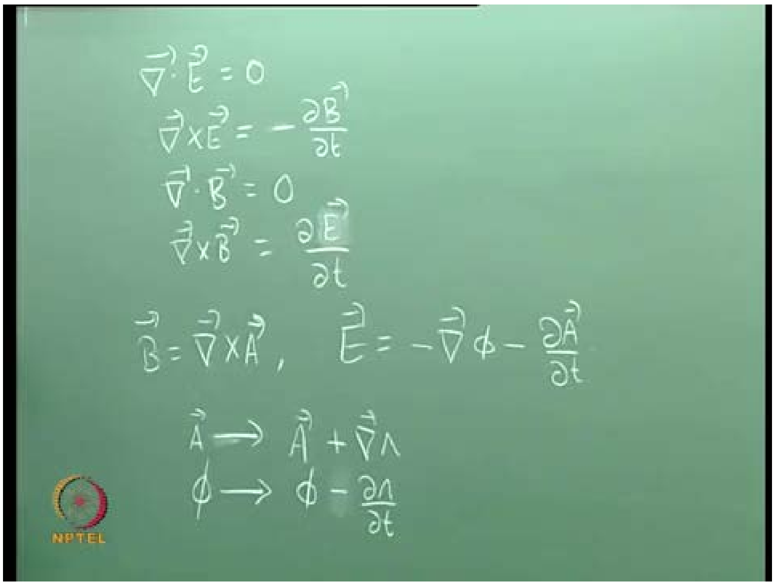

So, what are the equations of the motion you have del dot e equal to 0, because we are considering the Maxwell’s equation in free space and you have del cross E which is a minus del B over del t del dot B equal to 0 right is this minus sign here correct. And in this notation the statement about gauge transformation turns out to be, first of all this the electric and magnetic fields are given by B equal to del cross A and E equal to minus grade phi minus del A over del t you can check that from these two equations. This implies that B has to be curl of a vector field and now you can plug this forming in this one. And then you can see that it has to be at this point or you can plug it here and then you can see that the electric field has to here this form.

The gauge transformation says that the vector potential A goes to A plus some gradient of lambda whereas, the scalar potential phi goes to phi plus del lambda over del t you can see that under this transformation I think one of this has to be minus. If you make such a transformation, then the electric field as well as the magnetic field remain invariant, this phi you can identify it with the 0th component of the gauge field a mu, and the i th components are identical to the i th components of this vector potential here.

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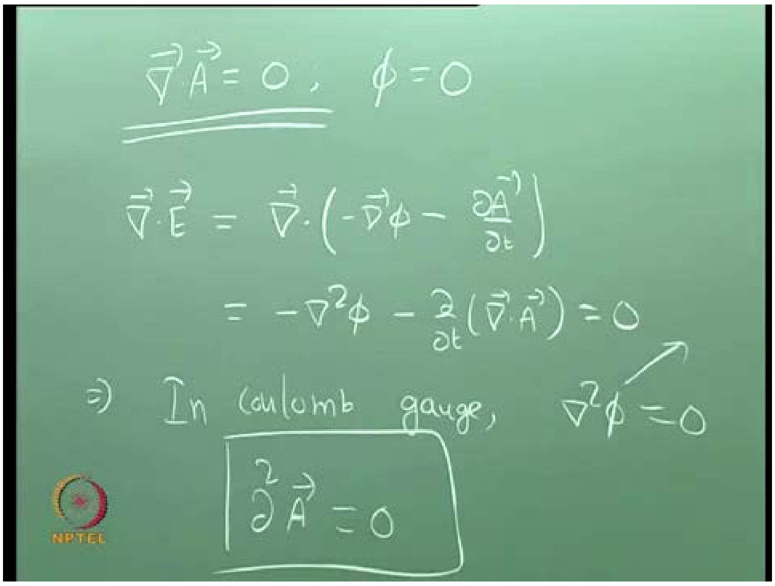

So, you can exploit this gauge transformation to consider this gauge del dot A equal to 0, this is known as the coulomb gauge. So, what we will do is that, we will work with this coulomb gauge, look at the equations of motion you can consider, let us say this equation here del dot E, E is minus grade phi minus del A over del t. So, this is nothing but minus del square phi minus del over del t del dot A, so in the gauge del dot A equal to 0, this quantity here vanishes. Therefore, this equal to 0 implies in the coulomb gauge, del square phi equal to 0, if in addition you impose the boundary condition that this scalar potential vanishes at infinity, then it amongst to saying that this vanishes throughout.

So, with suitable boundary condition, what you have seen is that, we can just set these and phi equal to 0, now you are left with only the A field. Now, you can consider the weather equations of motion appropriately and then you can show that in the coulomb gauge, this is what is the equation of motion for the vector potential A. So, I leave it as a homework for you, you show that these equations with the gauge del dot A equal to 0 and phi equal to 0 implies that del number and acting on the vector potential A is equal to 0.

So, this equation looks quite familiar to you, this almost looks like the Klein Gordon equation, except that the field A is now a vector field, and also the mass term is no longer there.

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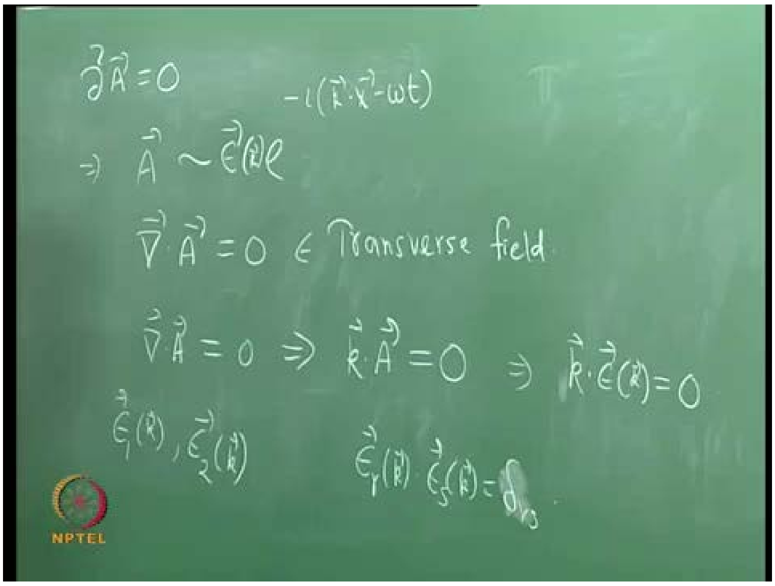

See what is the implication of that del square A equal to 0 means that, for this field A you will have a plane wave like solution, which goes like e to the power minus i k dot x minus omega t, the amplitude here will be sum vector epsilon of k. The field which satisfies this condition del dot A equal to 0 is known as a transverse field, which it so because del dot A is simply given by, this equal to 0 simply means k dot A equal to 0. Therefore, the field is orthogonal to the wave vector that is the reason you say that, this is a transverse field.

So, because the field is transverse, therefore there are two independent polarizations here, this epsilon here, this simply implies that k dot epsilon of k equal to 0, this simply implies that this epsilon k only have two independent components, so it is a 3 vector. But, it is not an arbitrary 3 vector, it is a orthogonal to the direction of propagation of the wave, therefore it will have only two linearly independent components. So, therefore, I will denote, therefore this admits two linearly independent solutions, I will denote them to be epsilon 1 of k and epsilon 2 of k. I can choose a linear combinations such that, they are orthogonal to each other, so epsilon r of k dot epsilon s of k equal to delta r s.

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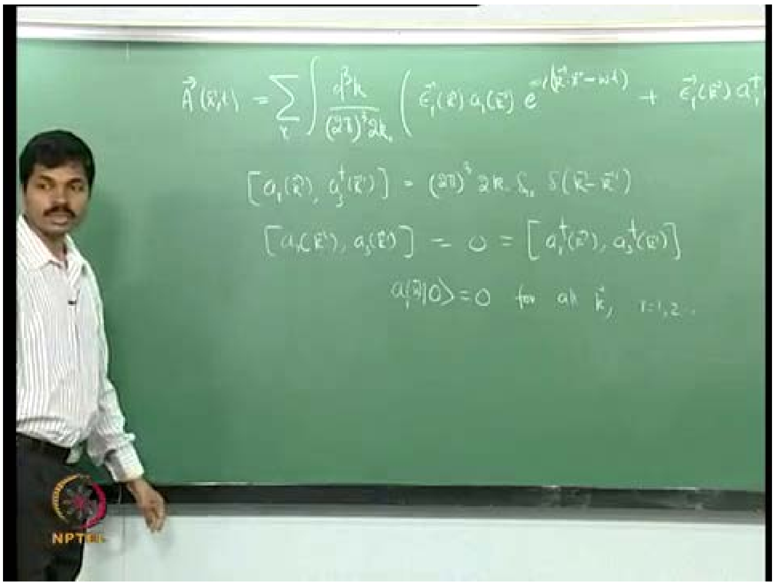

So, the most general solution for the vector field, this is summed over r integration d cube k over 2 pi cube 2 k 0 and then epsilon r of k a r k e to the power minus i k dot x minus omega t plus epsilon r of k a dagger r of k e to the power minus i k dot x minus omega t, this is the most general solution. Now, do not have to worry about, the conjugate momentum corresponding to the variable A 0, because we have set this gauge and the boundary condition. So, that A 0 equal to 0 identically, so this field does not appear at all.

So, you can now quantize it in the usual way, you will find that I mean the analysis is exactly identical to what we have discussed for a free scalar field. You can show that, this field a r of these creation at annihilations operators a r, and a r daggers satisfy this commutation relation a s dagger of k prime commutator is equal to 2 pi cube 2 k 0 delta r s delta of k a minus k prime whereas, a r k a s k prime equal to 0 equal to a r dagger of k a s dagger of k prime.

So, you can argue that, there exist a state which we will call as the ground state, and this is annihilated by a r of k for all k and r equal to 1, 2 then the n particles states are created by acting creation operators on this vacuum here. So, the entire analysis will be exactly identical to that of a real scalar field, but the difficulty here is that in quantizing this way, the covariance of the theory is lost now, this is no longer manifest Lorentz invariant.

So, because you have already chosen a gauge in which A 0 equal to 0, so Lorentz invariance is lost in this process, what you would like to see is, that you would like to see the quantization, where the Lorentz invariance is manifest to do that what we will do is, we will and then we will quantize.

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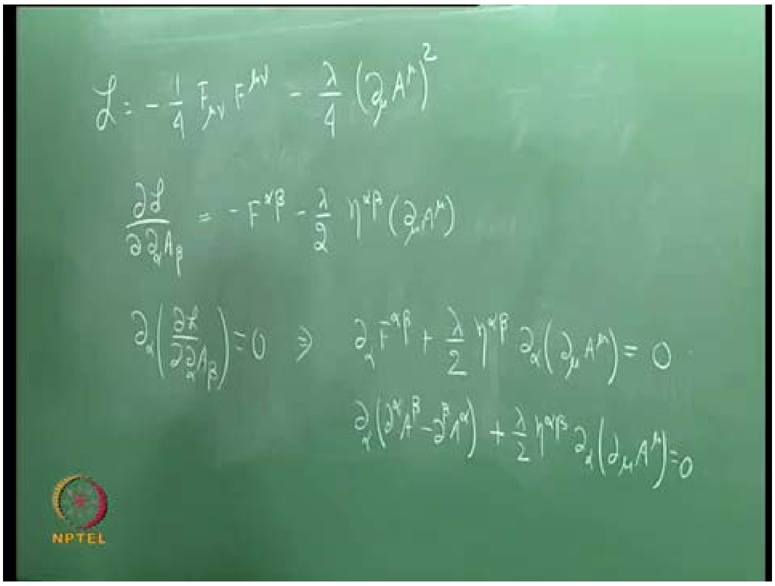

So, this is your Lagrangian density minus 1 4th F mu nu F mu nu to this I have added this term minus lambda over 4 del mu A mu square, so of course, this is not a Maxwell's Lagrangian, this is modified this term. Let us see what is the equation of motion corresponding to this Lagrangian density here, where del L over del del alpha A beta, from here you get minus F alpha beta. And this one will give us minus lambda over 2 times what you will get eta alpha beta del mu A mu.

Del alpha of del L over del del alpha A beta equal to 0, so this simply means that you have here del alpha F alpha beta minus lambda over 2 del alpha it is A plus. So, let us write it in the more simplified manner, this is del alpha of F alpha beta is del alpha A beta minus del beta A alpha plus lambda over 2 eta alpha beta del alpha del mu a mu equal to 0.

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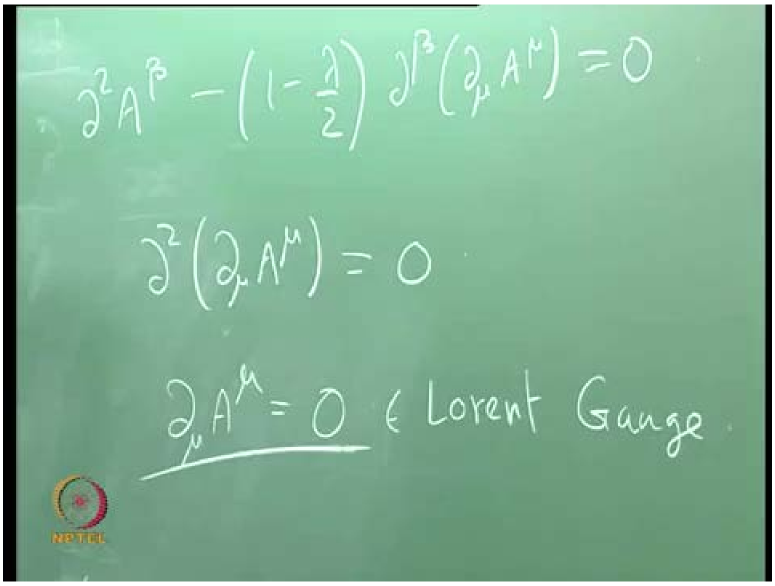

So, this simply means that you have, del square acting on A beta minus 1 minus lambda over 2 del beta of del mu A mu equal to 0, I mean it is trivial from this that, if you act further on del beta, then you see that del square of del mu Amu becomes 0. This also gives del mu, there is a del mu A mu here, del square del mu A mu, here again you get a del square del mu A mu and these two terms add there is some common coefficient apart from that this is 0.

So, this quantity del mu A mu, it acts like A free field, free scalar field you can you see that you recover Maxwell's equation, if you impose this gauge del mu A mu equal to 0, if del mu A mu equal to 0. Then this quantity the equation of motion coincides with the Maxwell's equations, so what we will do is that, we will first quantize the electromagnetic field here. And then later we will see what do you mean by imposing this condition del mu A mu equal to 0, what does this mean, this is known as the Lorentz gauge.

We can choose one specific value for this parameter lambda here, which is lambda equal to 1, lambda equal to 1 although it is not a gauge it is known as the Feynman gauge, so we will work with Feynman gauge and then quantize this theory. So, tomorrow we will start with this lambda equal to 1 gauge and then we will carry out the quantization and we will impose this Lorentz gauge condition and then we will see what it physically means.