Quantum Field Theory
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Module - 2
Interacting Quantum Field Theory
Lecture - 14
Interacting Field Theory – VII

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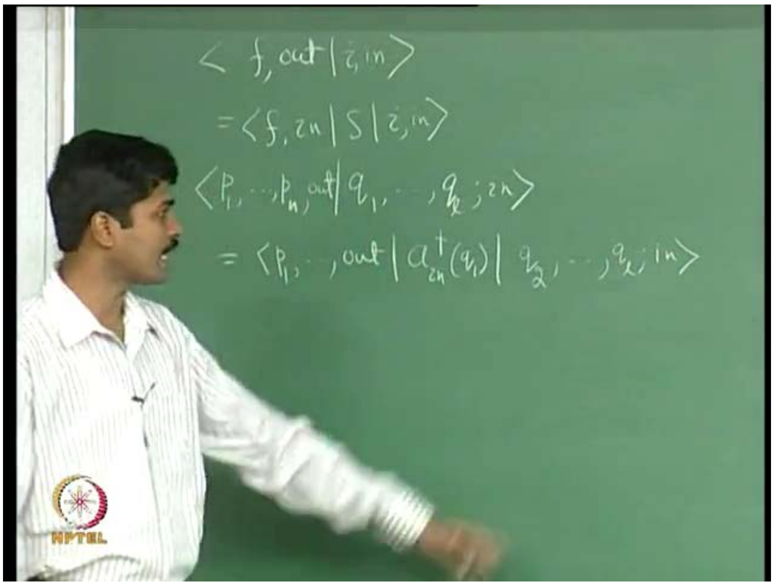

So, we are discussing about the transition amplitude from some initial state i to some final state f, which is also given by f in, S i in. What we would like to do is we would like to consider this transition amplitude and express it, in terms of the n point correlation function. So, this is what we have been trying to do, what we did is we considered a initial state, which is coming momentum q 1 to some q l, let say. What is the transition amplitude where this state going to some state, which is p 1 to p n out. This is what we are working and we would like to obtained a reduction formula for this.

So, to do this what we did is we started with this in state. Then we have seen this we can write this as p 1 some a i n dagger of q 1 acting on q 2 up to q l in. So, this is l this is a l minus 1 particle state this creation operator will give it give us n l particle state, which is give by this, we started with this.

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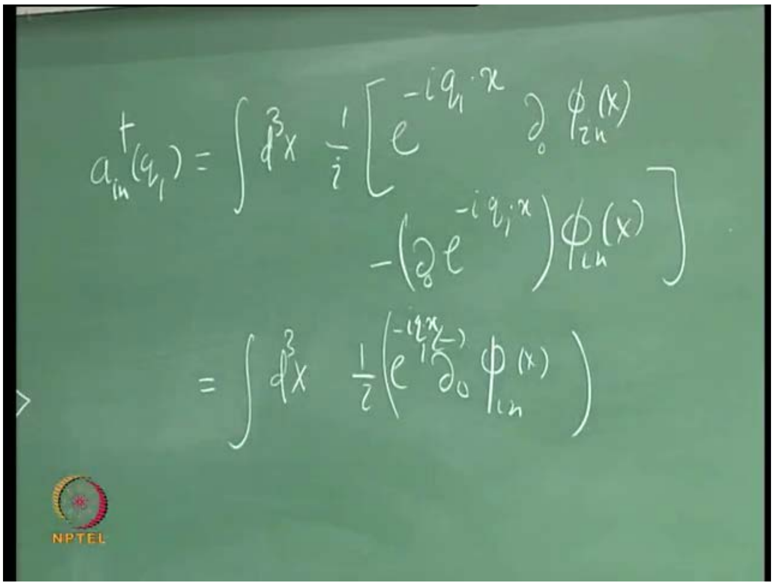

Since, a in dagger is the creation operator for a free field we can use this formula, which is given by a in for free fields a in dagger q 1 is given by 1 over i e to the power minus i phi 1 dot x times del 0 phi in of x minus del 0 e to the power minus i q 1 dot x as phi in of x. We will use this notation such that this is given by 1 over i del 0 acting, both the side e to the power minus i q 1 dot x phi in of x. This is what is the expression for a dagger in of q 1, we will substitute this expression here, when we substitute this what we get is p 1 with an integration over d q x 1 over i e to the power minus i q 1 dot x del 0 phi in of x.

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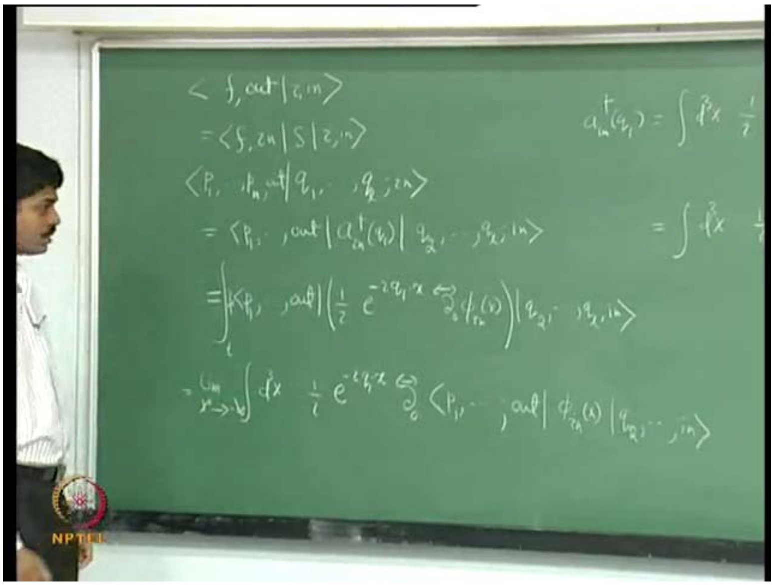

Now, notice that this is time independent. So, this expression here also when you carry out the integration it will turn out to be time independent. So, although for in other words here in this expression both e to the power i q 1 dot x in the phi of x. In both of these are time independent, but ultimately when you carry out this integration you, will get a dagger in of q 1, which is independent of time.

So, because of this you can evaluate this integration at any given time. So, this is evaluated at any arbitrary time. Especially, you can evaluate this integration in the limit t goes to minus infinity or in the limit x 0 goes to minus infinity. So, let say what we get this is equal to d q x 1 over i e to the power minus i q 1 dot x del 0. This quantity itself is equal to this same quantity in the limit x 0 tends to minus infinity, but in the limit x 0 tends to minus infinity.

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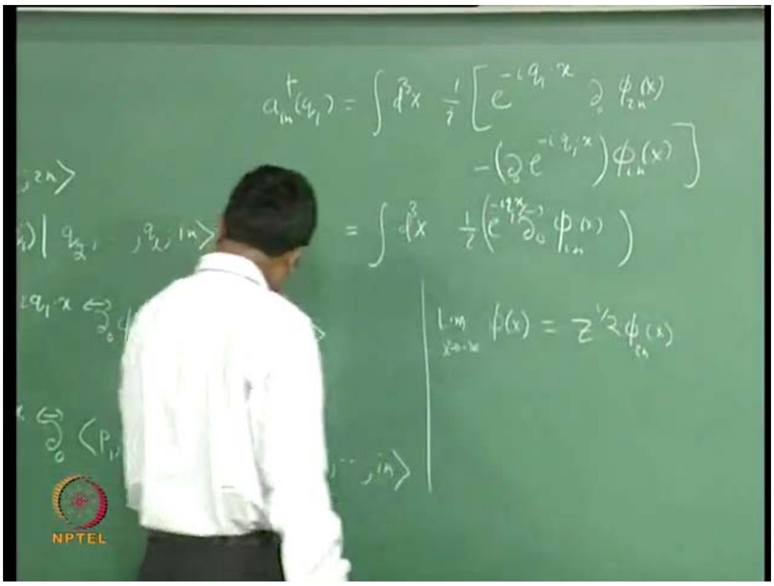

The interacting field phi of x is equal to z to the power half phi in of x. So, therefore, in this limit I can substitute this relation here. In place of phi in of x I can write the interacting field phi of x. So, when I do that what I get is the transition amplitude is given by limit x 0 tends to minus infinity z to the power minus half integration d q x e to the power i q 1 dot x del 0 p 1 out and phi of x instead of phi in of x q 2 in.

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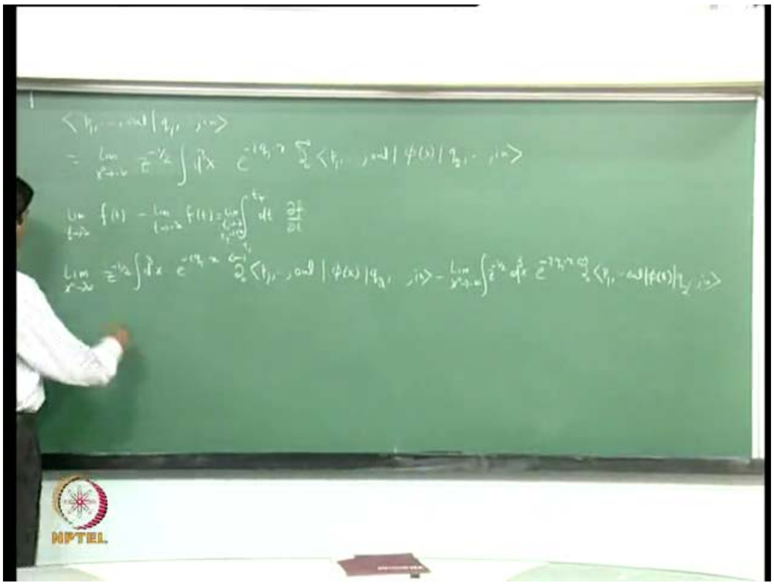

So, this is what we have got. Now, what we will do is we will use this relation that if you consider any arbitrary function f of t. You take this of that limit t tends to infinity f of t minus limit t tends to minus infinity f of t. Then this quantity is nothing but integration d t del f over del t for any arbitrary function of time, where the integration is carried out from some t i to t f in the limit t i tends to minus infinity t f tends to plus infinity.

So, we will use this relation here remember although this quantity here is independent of time, when you replace phi in of x by phi of x this quantity need not to be independent of time. So, only in the limit x 0 tends to minus infinity this is time independent. This becomes equal to the other quantity. So, now what we will do is that we will consider this same quantity. Then we will consider it at x 0 tends to plus infinity and x 0 tends to minus infinity we will take the difference.

The difference now will be equal to limit x 0 tends to infinity z to the power minus half d cube x minus i q 1 dot x del 0 phi one out p 1 out phi of x q 2 in minus. The same quantity in the limit x 0 tends to minus infinity z to the power minus half d cube x e to the power minus i q 1 dot x del 0 p 1. This quantity, now will be equal to integration. Here, now integration d t d cube x, where t ranges from minus infinity to plus infinity and show as x is just integration of the d 4 x del over del t is simply del.

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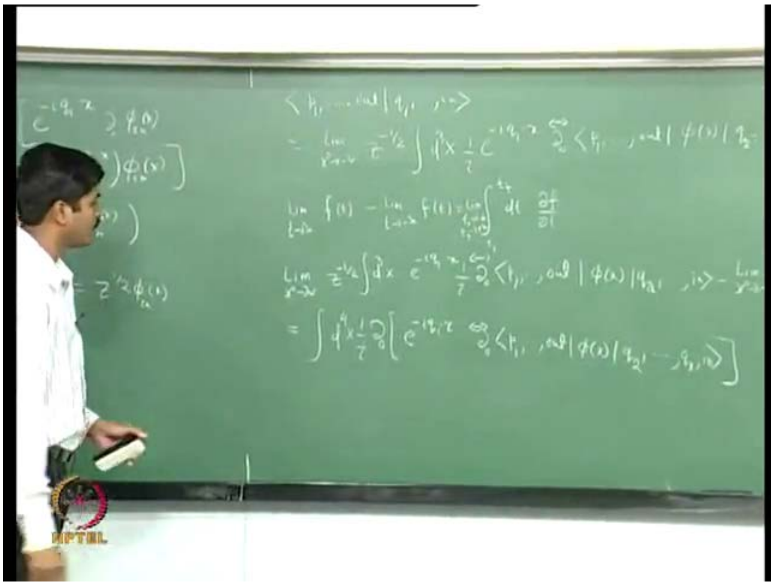

Now, f of t is just equal to this quantity. So, del 0 acting on e to the power minus i q 1 dot x del 0 p 1 out phi of x q 2 q l in. Therefore, this quantity the transition amplitude here will, now be equal to this quantity minus this quantity, this minus this will give you this, which is nothing but minus transition amplitude. So, I guess I missed one over i factor somewhere in the first step itself, there is 1 over I. So, is it? The transition amplitude, which is p 1 p n out q 1 up to q n in.

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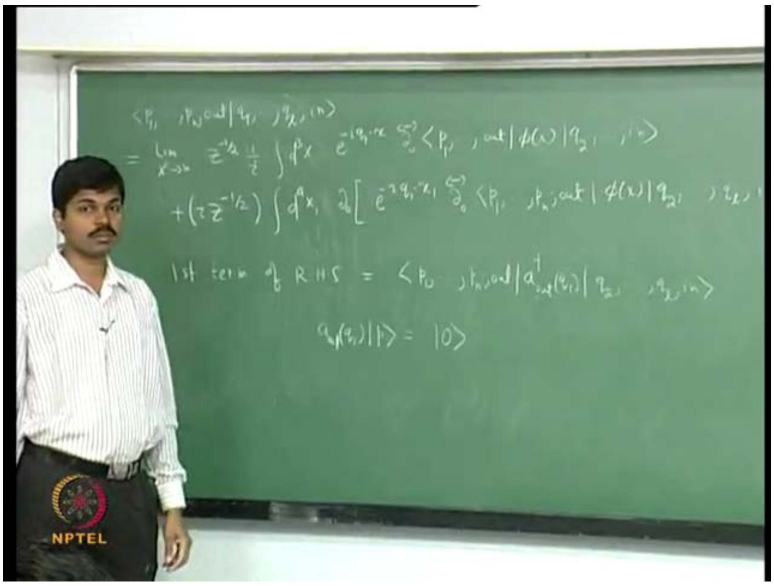

This is equal to this quantity limit x 0 tends to infinity z to the power minus half one over i d q x e to the power minus i q 1 x del 0 p 1, up to out phi of x in minus this quantity here. So, minus n 1 over I will make me plus i n also, there is a z to the power minus half. So, what I will get here is this plus i z to the power minus half integration d 4 x del 0. Let we call this variable as x 1 in the integration itself the dummy variable. So, I can change x to x 1 so this is integration over x 1 e to the power minus i q 1 dot x 1 del 0 p 1 up to p l out phi of x q 2 up to q l in.

Now, let us concentrate on the first term first in the first term you take limit x 0 tends to plus infinity. Therefore, again you can consider this integration here, this will give you a dagger out q 1. So, therefore, the first term of the right hand side is simply equal to p 1 up to p n out a dagger out of q 1 q 2 up to q l in. So, what we started is we started with the transition amplitude, which is equal to the same quantity with this creation operator acting on in state. Then we have used this technique to express in terms of a dagger out of q 1.

Now, a dagger out of q 1 will act on this state here what will you get when a dagger act acts on this? It will annihilate one of these one particle state one of these momentum to do that. Let's consider the conjugate of this, which is what you get when you consider a state a out of q 1. You act on a one particle state let say this is someone particle state of momentum p. Will you get simply the vacuum? You will get you will get something else? You will get some normalization factor and the delta function. So, let us just, so even one p 1 equal to q you have to be a bit careful. So, let us do that in in slightly more detail this is a out q 1. This quantity I can write it as a dagger of p a dagger out of p, p out in this is p out acting on the out vacuum.

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Now, this quantity I can write it as commutator of a out q 1 a dagger out p plus a dagger out p, a out q 1 acting on vacuum, but this quantity will be annihilate the vacuum. So, the second term will give you zero the first term is a sheet number. We are using the canonical commutation relation, where this is equal to 2 pi cube 2 p 0 times delta p minus q 1 times the vacuum state. Now, instead of the one particle state if you have n particle state what would you get is this. Now, what we show is this is equal to 2 pi cube 2 p 0 delta of p minus q 1.

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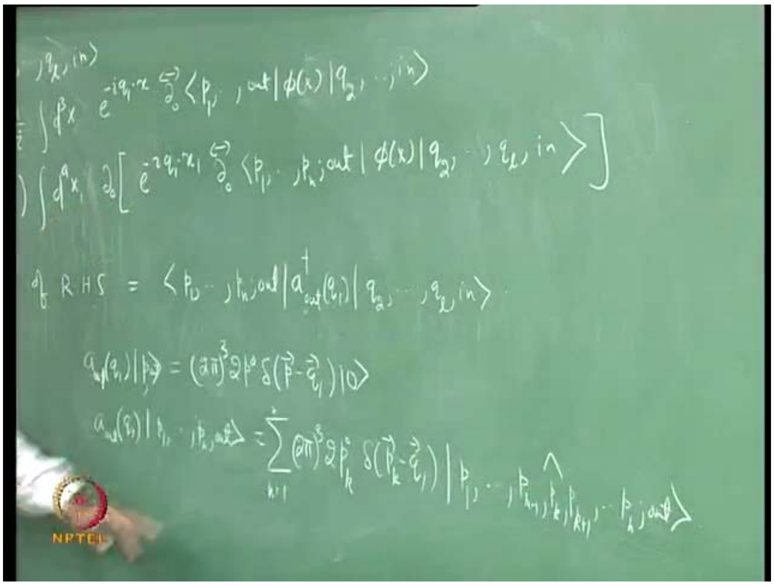

What we have here is n particle state. So, if you consider n particle state an act a out of q 1 on such an n particle state p 1 to p n out. What we will get is this normalization this overall factor times and n minus 1 particle state. This is there are n variable n such momentum, here you this can act on any of these things, so you have to sum. So, sum over k runs from 1 to n 2 pi cube 2 p k 0 delta of p k minus q 1 and this is p 1 up to p k minus 1 p k i, I will put a hat here to indicate that this momentum is absent, p k plus 1 up to p n out.

So, what you have here is just the formation conjugate of this quantity. Now, when I substitute it the first term will become a sum of n terms, which contents a delta function times sum normalization factor times the amplitude, which involves l incoming states. It involves l minus 1 incoming state and n minus outgoing particles. So, initially you considered the process for particles, which are for a process, which involves l incoming particles and n outgoing particles this is the amplitude for this process and this amplitude.

Now, what you are showing is that can be expressible in terms of sum of this quantity here, which we will be simplifying in a moment. The another quantity, which involves the amplitude for a process, which has l minus 1 incoming particles and n minus 1 outgoing particles, so at least one of the particles here.

So, basically this delta function here says that this will be this will give you a non zero contribution. If and only one of these l incoming particles go without doing any interaction it’s completely on this term. On the other hand if none of these outgoing momentum, here coincide with any of these incoming momentum. Then this quantity will give you vanishing contribution. So, this will not give you any contribution if any outgoing these things do not mess any of these quasars, because of the presence of this delta function. I will call such a term as a disconnected term here in the sense that it essentially it has described a process, which is one less incoming particle than one less outgoing particle. So, now let us focus on the second term.

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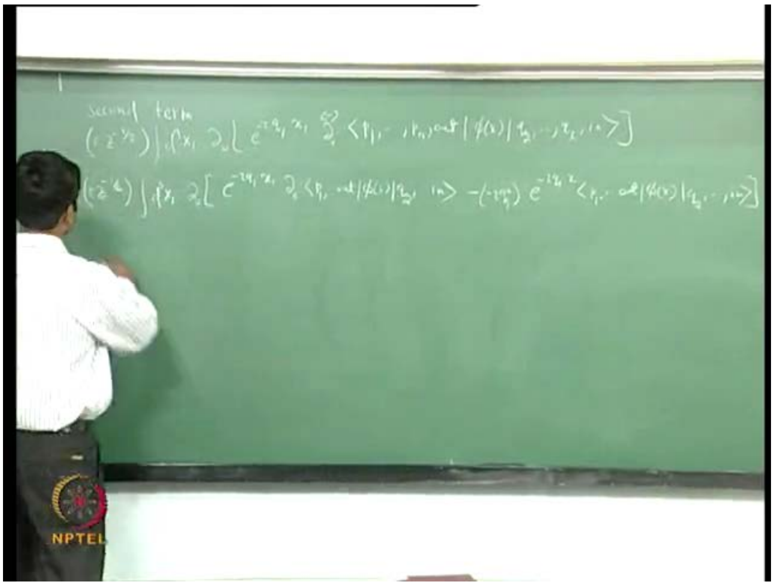

Basically, i z to the power minus half d 4 x 1 del 0 q 1 dot x 1 del 0. We will evaluate the time derivative here explicitly and when we add del 0 on this state. What we get is first I will write this in terms of sum of two terms. Then I will add del 0 on it i z to the power minus half d cube x 1 del 0 times e to the power minus i q 1 dot x 1 then del 0 acting on this term p 1 out phi of x q 2 in. Then minus del 0 acting on e to the power minus i q 1 dot x 1 will give you minus i q 1 0. Then this quantity e to the power minus i q 1 dot x times the same thing p 1 out phi of x q 2 in. Now, I will add del 0 on this as well as on this when I add del 0 on this it will give me first one term, which is just do it explicitly i z to the power minus half d cube x 1.

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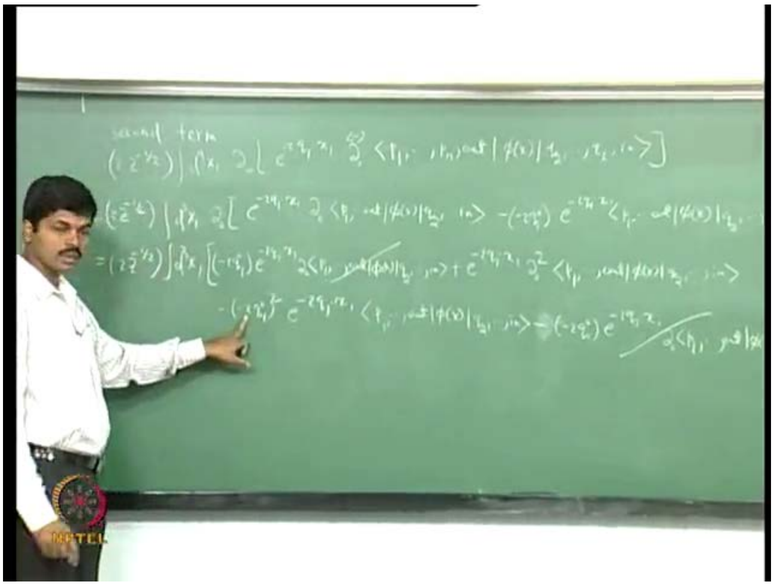

The first term will give me minus i q 1 0 e to the power minus i q 1 dot x 1 del 0 of p 1 phi q 2. Then what I what get is plus e to the power minus i q 1 dot x del 0 square acting on this p 1 phi x q 2 in. Then I have the second term. So, what I will get is minus i q 0 q 1 0 del 0 acting on this will give another factor same to that. So, I will get a square times e to the power minus i q 1 dot x 1 x 1 p 1 out phi of x q 2 that in that first term. Then I will get another term, which is minus i minus of minus i q 0 q 1 0 e to the power minus i q 1 dot x 1. Then del 0 will add on this state. So, del 0 add on p 1 out phi of x q 2 in.

Now, you can see the first term here is same as this term apart from this minus sign. So, this will cancel this what you are left with is this. This there are these two terms here, because there is a minus. Then this will make it plus q 1 0 square, because of this i square. This minus sign you will get plus q 1 0 square. Now can used this relation q 0 q 1 0 square is basically q 1 dot q 1 plus m square.

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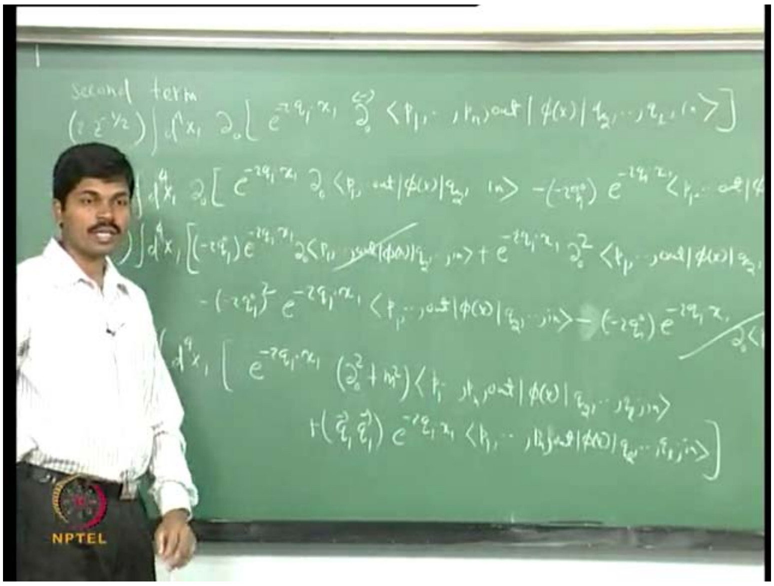

So, we will substitute this relation here what you get here is and i will keep this term, which is q 1 vector dot itself. Then m square term I will just add here. So, when I do that I get here is the second term simply becomes i z to the power minus half times its d 4 x 1 integration t 4 x 1 then e to the power minus i q 1 dot x 1 del 0 square plus m square. Because, this will give me one m square this acting on p 1 up to p n out phi of x q 2 q l in that is first term. Then the second term is simply plus q 1 dot q 1 times e to the power minus i q 1 dot x 1 this term here p 1 up to p n out phi of x q 2 up to q l in.

Now, the reason I keep it separate is the following, because of q 1 dot q 1 what I can do is I can consider this whole term here. I can write it as del 1 square acting on e to the power minus i q 1 dot x 1. What I am saying is del one square acting on e to the power minus i q dot x 1 is simply equal to minus q 1 dot q 1 times e to the power minus i q 1 dot x 1.

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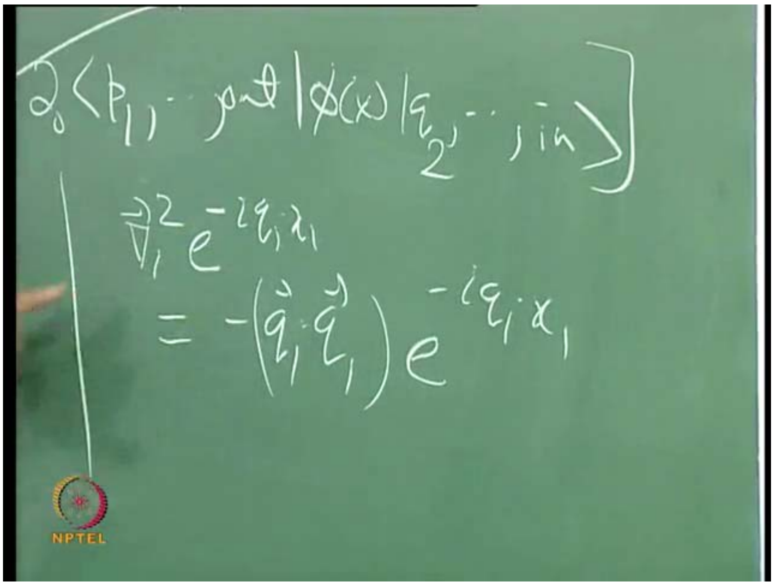

So, this quantity here, I can substitute as minus del 1 square.

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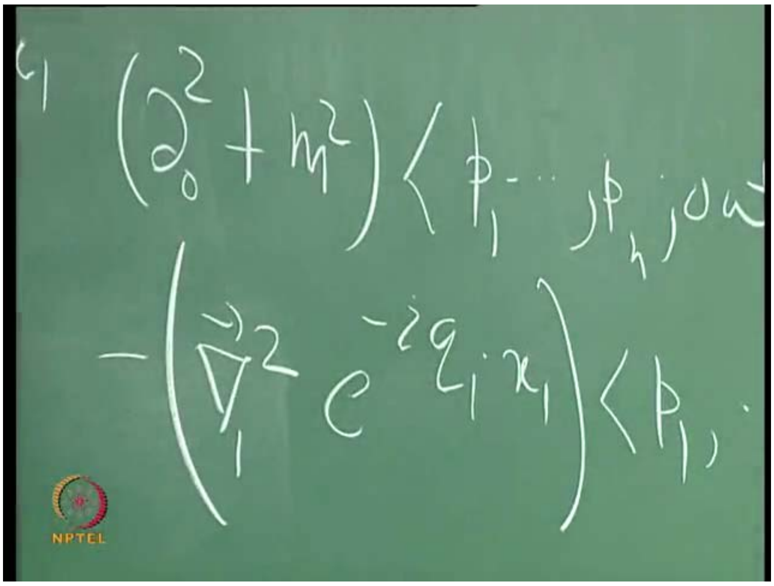

So, instead of q 1 dot q 1 I can write it here as minus del one square acting on this. Now, what you can do is you can make a partial integration twice, this surface term will vanish. Because, its especial infinity you are considering interaction of localized particles. So, therefore the field here is basically consist of several wave packets, which vanishes at especial infinity. So, when you do a partial integration you will get a surface term here. The surface term will vanish, because of the fact that you are considering interaction of particles, which are localized both in position space as well as in momentum space.

So, if you used that then this simply becomes the same quantity, but this del 4 square acting on this term here. So, therefore this quantity here I can, now write it as the same quantity, but this e to the power minus i q 1 dot x 1 and del 1 square acting on this.

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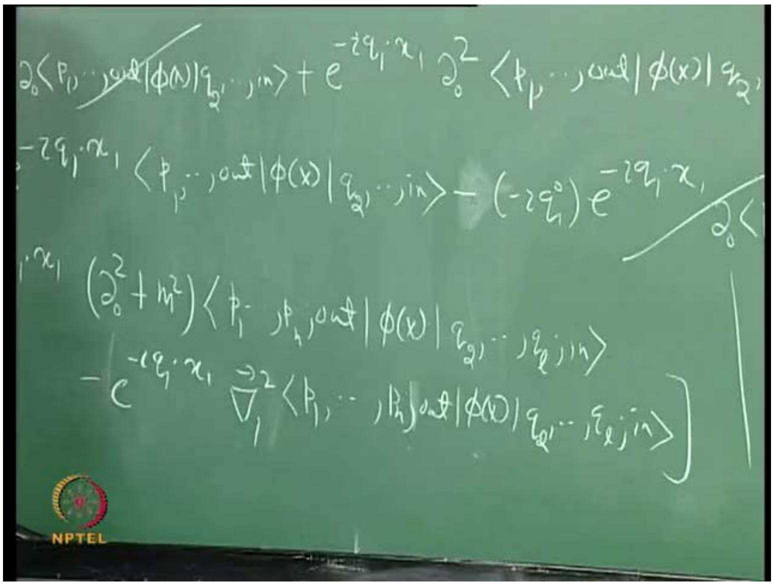

Now, I can combine this term and this term what I will get here is the del numberings operators appearing in this term. So, what you have seen is the second term here is nothing but when the surface term vanishes.

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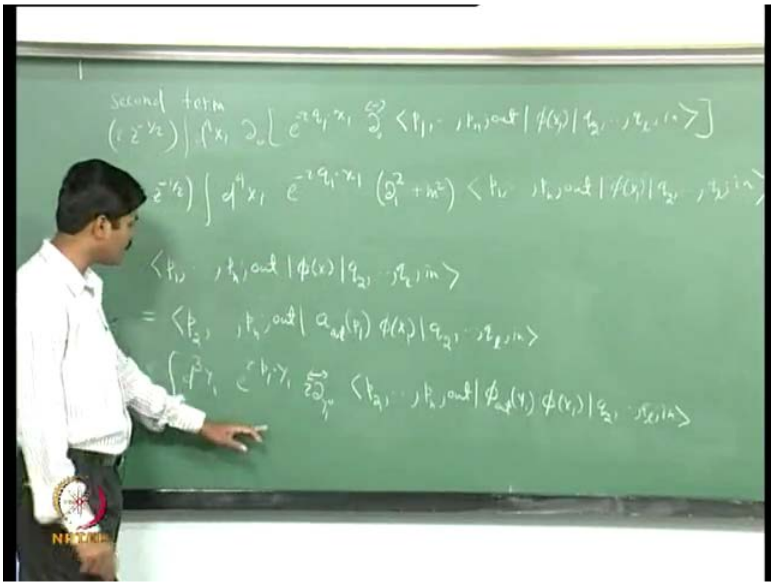

At the end of the day you will get this term as equal to i z to the power minus half integration d 4 x 1 e to the power minus i q dot x 1 del 1 square. I will denote plus m square acting on this state p 1 up to p n out in the phi of x q 2 up to q l in. Now, what I will do is I will consider this term here. Then I will see if I can do further reduction to this one. So, let us consider this term here inside the sanctification p 1 up to p n out phi of x q 2 up to q l in. Now, what I will do is I will pull a p 1 factor here from the out state. Then I will write it here a dagger out p 1. So, this is same as p 2 up to p n out a dagger out of p 1 then phi of x q 2 up to q l in again, because this it is not a dagger. It is simply a, because a acting on this state will create a 1 particle state.

So, this, now what I will do is this is a is an annihilation operator again for a free field. So, I can express it in terms of the phi outfield, therefore this is integration I will use y as the variable d cube y or y 1. Because, there is a p 1 here e to the power i p 1 dot y 1 times i del y 1 0 acting on this p 2 up to p n out phi out of y 1 this is x 1 here, phi of x 1 q 2 i 2 q l in. Again, this here is time independent. Therefore, I can evaluated at any arbitrary time especially I can evaluated in the limit y tends to plus infinity.

So, this quantity itself is equal to the same in the limit y 1 0 tends to plus infinity and in this limit what I can do is I can replace this by z to the power minus half times phi of y 1. In the limit y 1 0 tends to plus infinity phi y goes to minus z to the power phi y 1 goes to z to the power minus half where plus half y out y 1.

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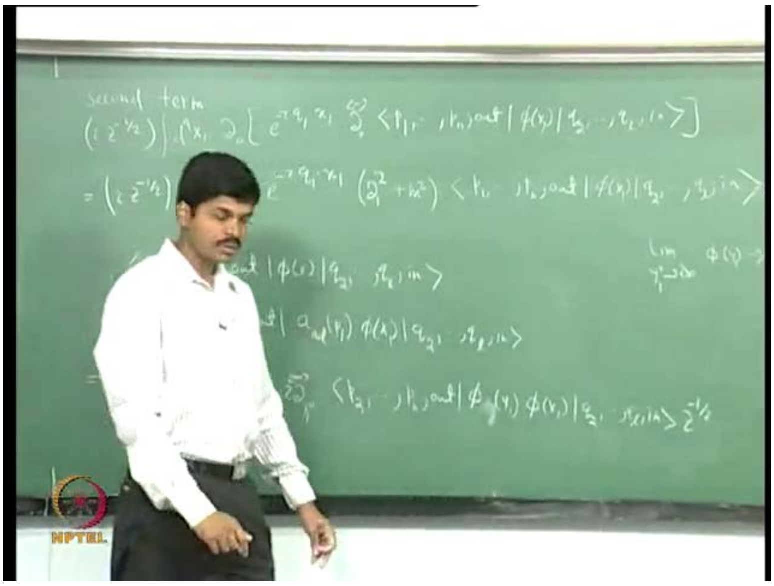

Similarly, I can use this relation here and what I will get here is essentially there is a factor of z to the power minus half, there is not phi out. So, this times z to the power minus half I will consider this i, I will take it out and this integration, then what I have is the amplitude p 1 to p n out phi of x 1 q 2 q l in.

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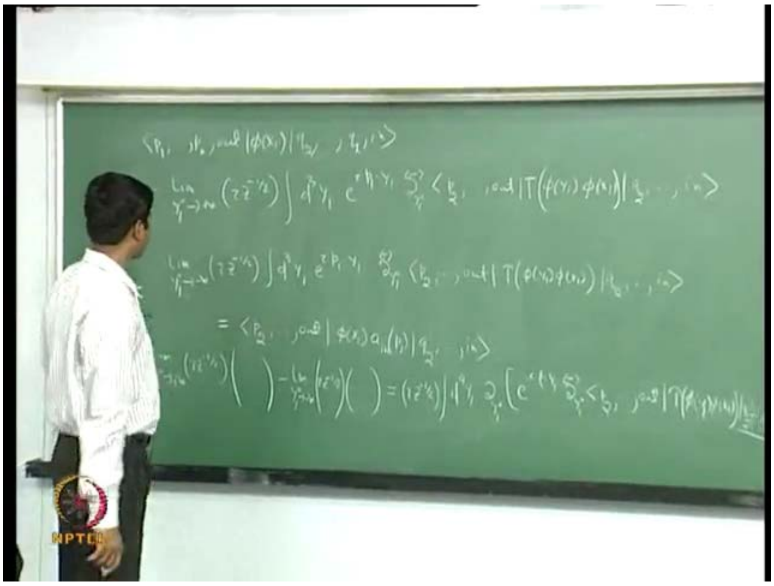

This is equal to limit y 1 0 tends to plus infinity i z to the power minus half integration d cube y 1 e to the power i p 1 dot y 1 del y 1 0 acting on p 2 out phi of y 1 phi of x 1 q 2 in this is what I get. Now, you see I am taking the limit y 1 0 tends to infinity. This operator here is left to this operator this is at any finite time. Here you are considering the limit, where y 1 0 tends to infinity. Therefore, this expression here is equal to time order product itself.

So, I can introduce a time ordering here, because of this fact that I am considering limit y 1 tends to plus infinity. Now, what you do is you consider limit y 1 0 tends to minus infinity of the same quantity. You take the difference, so limit y 1 tends to minus infinity of i z to the power minus half d cube y 1 e to the power i p 1 dot y 1 del y 1 0 p 2 out time order product of phi of y 1 phi of x 1 q 2 in. Now, you see why I wanted to emphasize on the time ordering, because when you consider the limit y tends to minus infinity the time ordering will makes sure that this phi of y 1 operator goes to the right of phi of x 1. So, it is on the right.

Now, you can carry out this integration over y 1 d cube y 1, when you carry out the integration over y 1, what you get is a of q a of p 1 acting on this. So, this will annihilate one of these in states. Therefore, this here is nothing but integration will go away, this is p two up to out phi of x 1 a in of p 1 acting on q 2 in. So, again this will give a disconnected fields. Now, you take the difference of this and this. So, this quantity itself is difference of this and this plus this term here.

Therefore, this is when I take the difference of this and this, again I will get a an integration over d 4 x. So, limit y 1 0 tends to plus infinity i z to the power minus half times whole quantity minus limit y 1 0 tends to minus infinity, i z to the power minus half times the whole quantity here, is equal to i z to the power minus half integration d 4 y 1 times del y 1 0 acting on the whole thing times e to the power minus i. It the plus i p 1 dot y 1 del y 1 0 p 2 out time ordering of phi of y 1 phi of x 1 q 2 in. So therefore, this quantity here is this plus this, so let us write it.

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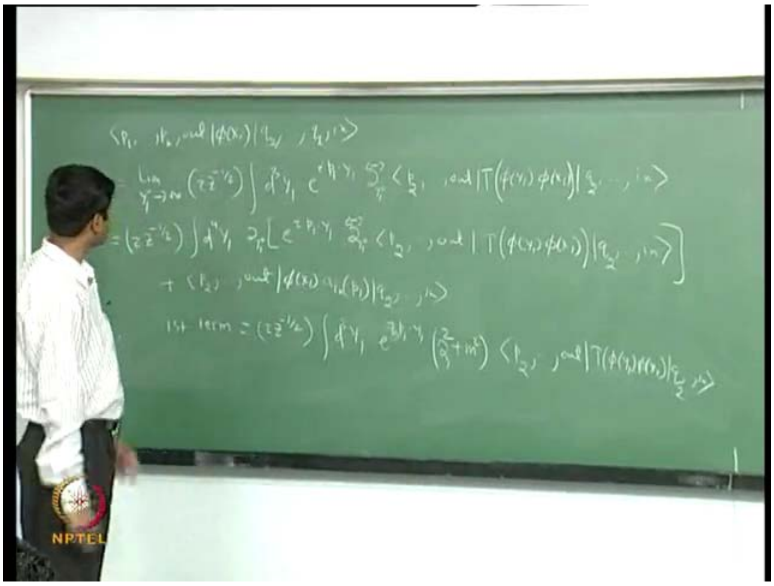

This is equal to i z to the power minus half d 4 y 1 del y 1 0 e to the power i p 1 dot y 1 del y 1 0 p 2 out time order product of phi of y 1 phi of x 1 q 2 in. Plus, this quantity plus p 2 out phi of x 1 a in of p 1, again I can consider this term here. This the first term here can be written as i z to the power minus half d cube y 1 e to the power i p 1 dot y 1 del y 1 square plus m square acting on this term p two out time order product of phi of y 1 phi of x 1 q 2 up to l. So, what I can do is I can substitute this result here in this expression. Then I will get a disconnected term plus.

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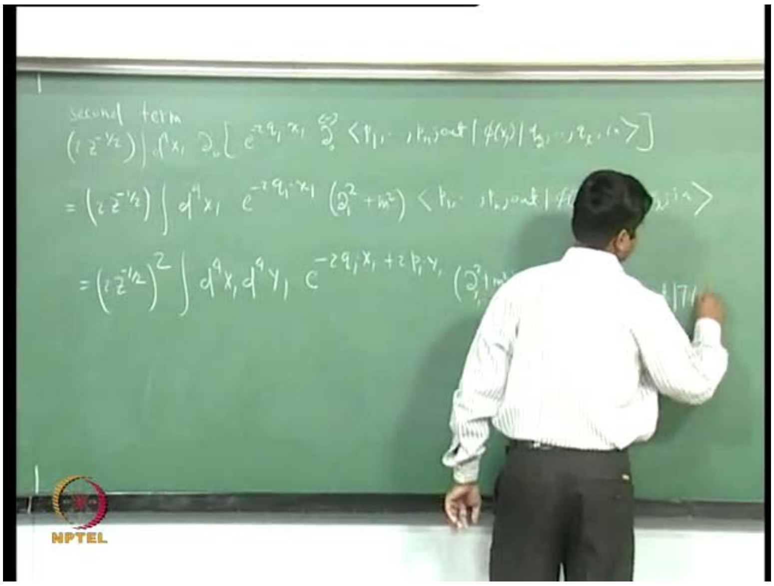

So, what I will get is i z to the power minus half square d 4 x 1 d 4 y 1 e to the power minus i q 1 dot x 1 plus i p 1 dot y 1.

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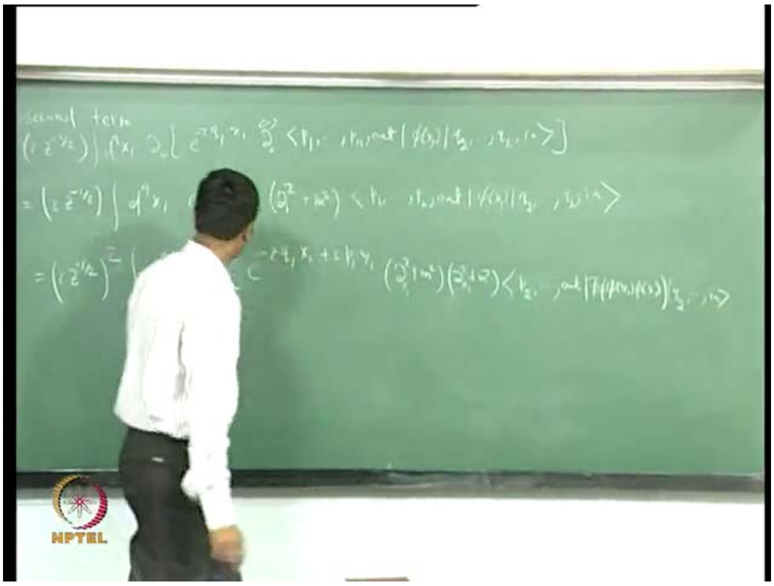

Then del y 1 square plus m square del x 1 square plus m square acting on p 2 out time order product of phi of y 1 phi of x 1 q 2 up to q in plus disconnected terms. I can keep on doing this the same process. When I do that at the end of the day what I get is.

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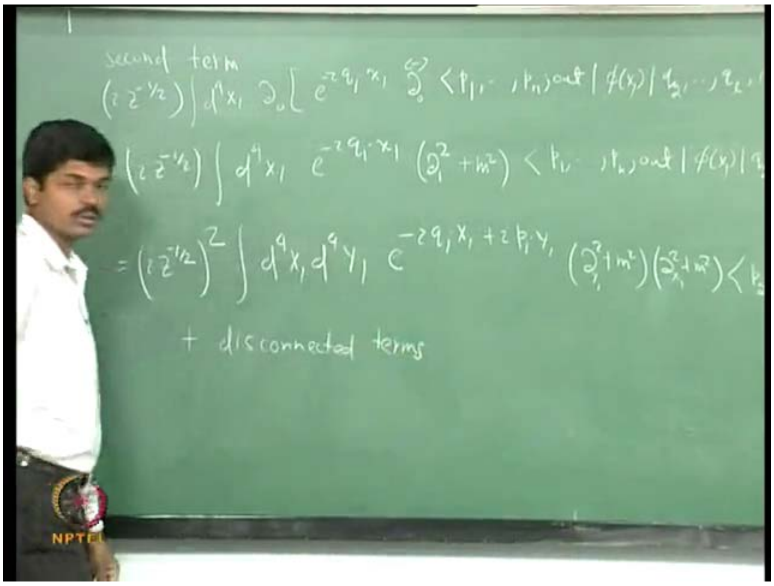

The amplitude for this process, where you have n outgoing state n outgoing particles and l incoming particles with momentum q 1 up to q l.

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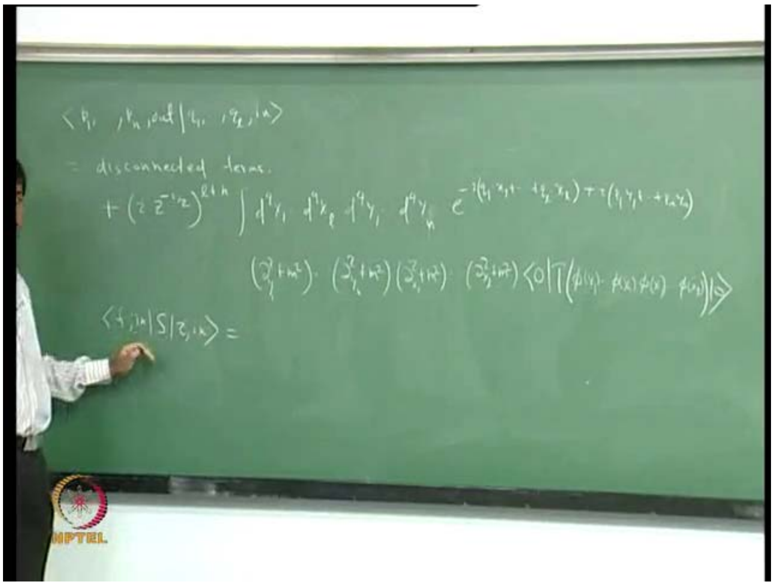

The amplitude is given by disconnected terms, plus i z to the power m minus half to the power l plus n d 4 x 1 up to d 4 x l d 4 y 1 up to d 4 y l y n e, to the power minus i q 1 dot x 1 plus q l dot x l plus i p 1 dot y 1 plus up to p n dot y n. This time del y 1 square plus m square up to del y n square 0 m square. Then del x 1 square plus m square up to del l square plus m square, acting on the vacuum expectation value of time order product of phi of y 1 up to phi of y n phi of x 1 up to phi of x l. This is what you will get.

So, to summarize what you can do is you can consider the s matrix element of some phi some final state with some initial state. If you evaluate the connected part of the s matrix element then the connected part is given by this. So, this is what is known as the connected part of the s matrix element. What you can do is you can consider the outgoing state to be union of.

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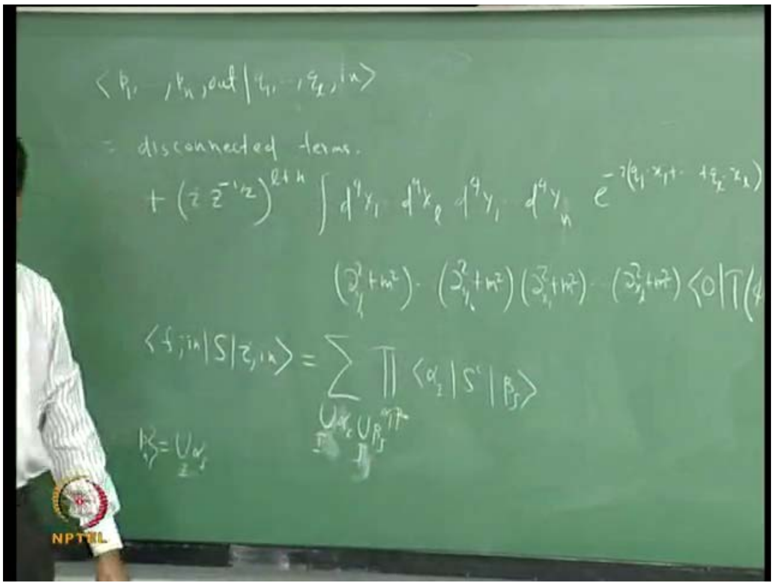

So, you can consider various subsets, this is your final state. Here, this is your incoming state this incoming state contents n incoming particles. You can consider all possible subsets here. Similarly, you can consider all possible subsets, here you can compute the connected part of the s matrix elements. Take their product and take the sum that will give you the final s matrix element here.

So, symbolically what I can write it here is I can write here is, some alpha union over alpha i and i alpha i union over beta j union over beta j, where alpha i and beta j are basically a subset of the outgoing, the subset of incoming state. Especially, this p 1 up to p n, That I can consider it as union of alpha i or various alphas or for various i's, various j's. Same about their incoming state and then the product over these states with a alpha i the connected part beta j here.