Quantum Field Theory
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Interacting Quantum Field Theory

Lecture - 12
Interacting Field Theory - V

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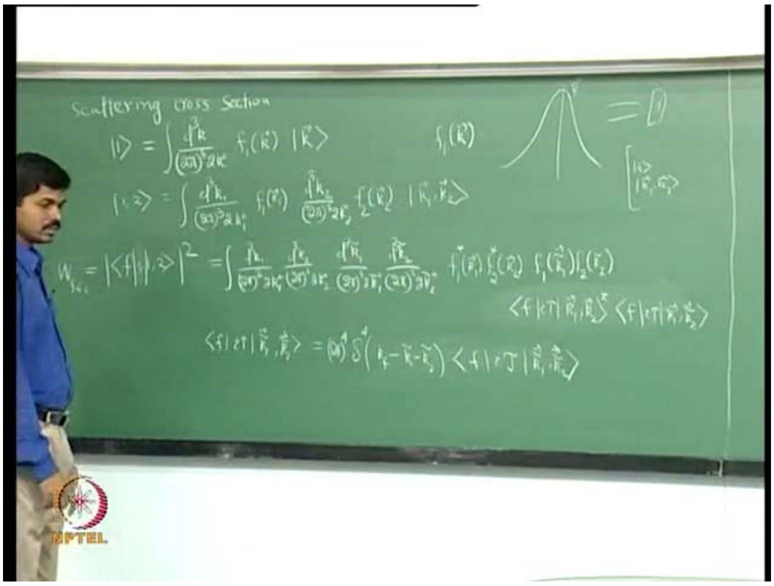

We are interested in finding a formula for the scattering cross section to get this. We constructed we will construct wave packets let us say you consider the particle one. It will be localized in the position space at some point. Also, it will be localized that in the momentum space, that there is going to be distribution angle all this 2 pi cube 2 k zero f 1 of k. Then so this function f 1 of k determines the distribution in the momentum space for particles, which are Eigen state of the momentum, this f 1 is delta k minus k 1 bar. So, f 1 of k will have some distribution in the momentum space. Then I will assume that this as a spread in the momentum I will along some value, which I will call as k bar. If I consider two particles then the wave packet for the two particles system, which I will call as particle one. Particle two will be given by d cube k 1 over 2 pi q to k 1 0 f 1 k 1, then d cube k 2 over 2 pi cube 2 k 2 0 f 2 k 2 k 1 k 2.

We will like to find the transition probability of this state evolving to some final state f as T goes to infinity. So, what I need is the transition probability for this process, which is given by the mode square of this quantity. I will call this as W for some initial state i going to a final state f.

So, this transition probability we can work it out let's do it in step by step. This is basically d cube k 1 over 2 pi cube 2 k 1 0 d cube k 2 over 2 pi cube 2 k 2 0. Then there will be complex conjugate of this quantity, which will give matrix element d cube k 1 tilde over 2 pi cube 2 k 1 tilde 0 d cube k 2 tilde over 2 pi cube 2 k 2 tilde 0. Then I will have Fourier quantities they are f 1 k 1 f 2 k 2. Then I will put a star here, which has a mode square f 1 k 1 tilde f 2 k 2 tilde. Then the inner product of f with this quantity f k 1 k 2 star and f k 1 k 2 k 1 tilde k 2 tilde star.

However, that is one point the second point is that this the inn space is not same as the out space. So, the transition probability will also involve the s matrix. Therefore, I will have what I am actually interested to compute is either everything I can work in the inhale bird space or I can work everything in the out hale bird space. Therefore, the probability amplitude actually involves a factor of S here. So, trick less, so actually what I will have is f s k 1 k 2 star times f s k 1 k 2 tilde, where in this expression. I am assuming that everything the state f this state's k 1 k 2 all are state vectors of the inhale bird space the inn space.

Actually we do not even need this, because this S matrix contains an identity piece. Then the remaining terms i, I am denoting that by i times some operator T the first term gives the forward scattering part, so even in presence of interaction if I consider a beam incident on a target. There is a finite probability that the particles will just pass through. So, there without interacting and that is what is the forward scattering part that is given by this identity term here. So, I am not interested in the forward scattering part of the cross section. So, I will forget about the one term here. Therefore, I have to i times T here similarly i t. So, this is what we want to compute to do this lets introduce this identity.

So, we will do one more thing this matrix elements here both these matrix elements, they describe the transition probability. If I forget about the forward scattering part they describe the transition probability from some state k 1 tilde k 2 tilde to some final state f. This transition will conserve energy momentum, therefore the matrix elements here f i T k 1 tilde k 2 tilde this will contain a delta. We do know the exact form of the matrix element here, but we know for sure, that whatever it is it will contain a delta function, which will conserve the energy momentum in this process.

So, this just basically gives k f minus k 1 tilde minus k 2 tilde a four dimensional delta function for convenience. I will put a vector 2 pi to the power four here and then the remaining parts of the matrix element I will denote it as f i some script T k 1 tilde k 2 tilde. So, this is basically the definition of the matrix element here depending on the exact form of the interaction. I will have exact expression at least an infinite series expression for this operator i t. Correspondingly I will have an expression for the matrix elements here in the right hand side. So, I can substitute it in this expression and have a nice substitute. That is what I will get for the transition probability is w, for some initial state i going to final state f, which is mode square.

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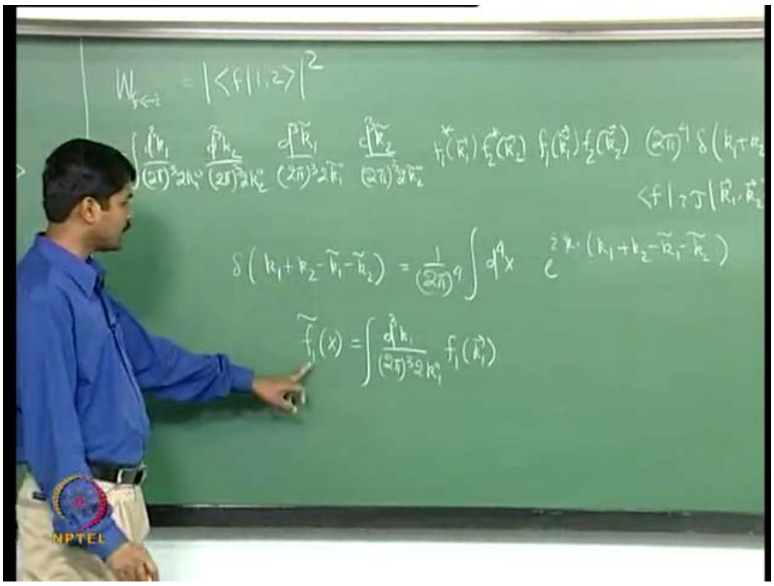

This quantity is nothing but d cube k 1 tilde over 2 pi cube 2 k 1 tilde 0 d cube k 2 tilde over 2 pi cube 2 k 2 tilde 0. Then f 1 k 1 star f 2 k 2 star and then f 1 k 1 tilde f 2 k 2 tilde. Then I will have 2 pi to the power fourth delta of k 1 plus k 2 minus k 1 tilde minus k 2 tilde. Then 2 pi to the power four delta k f minus k 1 tilde minus k 2 tilde. Where, in the first delta function instead of k f i have instead of k f. Here, I have ah k 1 tilde minus k 2 tilde that is because it is already multiplied by another delta function. This times the matrix elements here f i T k 1 k 2 star and f.

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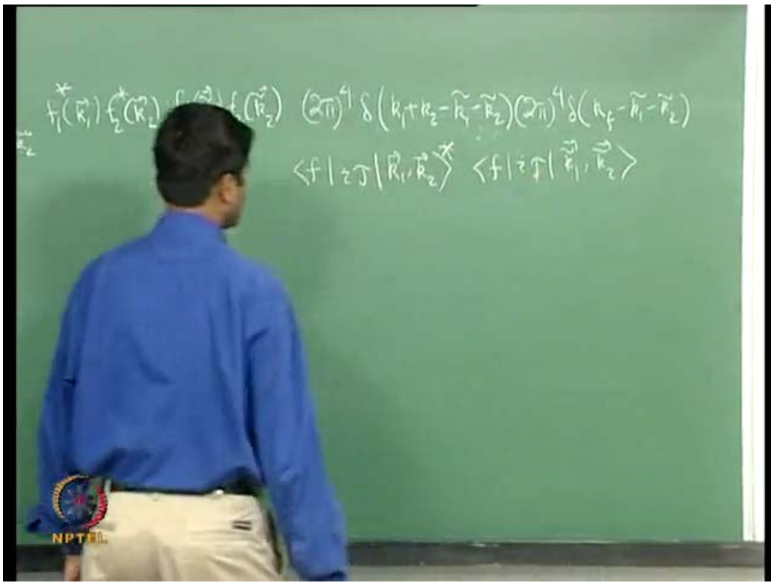

Now, I will make the assumption that these the distribution is peaked around certain loop k bar. So, f 1 k 1 has a peak around k 1 bar and same about f 2 k 2 it has spread and the momentum, but the spread the distribution has a peak around certain mean value. I will assume that the amplitude here are actually close to some quantity, which is f i top k 1 k 2 they are non zero. Only when they are significantly different from zero only when the momenta k 1 and k 2 are equal to k 1 bar and 2 k 2 bar and same about this quantity is here.

So, with this assumption what I will do is that in this in both these expressions I can write instead of k 1 k 2 here. Instead of k 1 tilde k 2 tilde I can write here k 1 bar k 2 bar and so on and. The second thing is I can write this, I can consider this representation for this delta question, which is delta q n plus k 2 minus k 1 tilde minus k 2 tilde is equal to one over 2 pi to the power 4, integration d 4 x e to the power i x dot k 1 plus k 2 minus k 1 tilde minus k 2 tilde. I will substitute it here and in addition I will use this expression for Fourier transform f tilde f 1 tilde of x i will denote this to be the Fourier transform of f 1 k 1.

So, this is integration d cube k 1 over 2 pi cube 2 k 1 0 f 1 of k 1. Similarly, there are there will be Fourier transform of f 2. When I use that and when I use this expression for the delta function. Here, you notice that this term here will give matrix element 1 f 1 tilde star this will give matrix element an f 2 tilde star of x. This will give matrix element f 1 of x this will give matrix element f 2 of x.

There will be an integration over x, the k integration will go away. Because, of this definition of the Fourier transform here, because of the fact that the only k dependence comes through this function f 1 and f 2. I am including the k dependences in all over the terms, I am substituting instead of k 1 k 2 k 1 tilde k 2 tilde. I am substituting the average value of the momentum, which is k 1 bar and k 2 bar. Therefore, the only k dependence comes through this and hence I can simply substitute the Fourier transform here. When I do that what I get for the transition probability is that.

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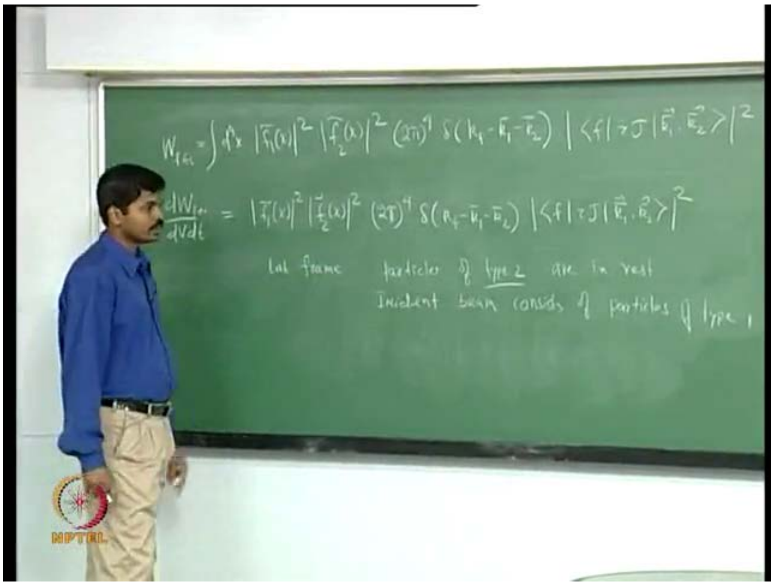

W d 4 x times mode f 1 tilde x square times mode f 2 tilde x square. Then 2 pi to the power four delta of k f minus k 1 bar minus k 2 bar. Then f i star k 1 bar k 2 bar mode square. This is the what is the total transition probability and has the transition probability per unit volume per unit time, which I will denote as d w over d v d T is basically the integrant without this vector d 4 x.

So, this is f 1 tilde of x mode square f 2 tilde x mode square, then 2 pi to the power four delta k f minus k 1 bar minus k 2 bar. This quantity here modulus of f i star k 1 bar k 2 bar mode square. To get the cross section I have to divide this quantity by the number density of the target, which I am considering particle two to with the target and particle one is the incident beam. So, let us consider the laboratory frame, where particle particles of type two are in rest. The incident beam consist of particles of type one.

So, there are two things that we need to determine one is the number density of particles of type two number density of the target. The second thing is the flux of the incident beam. So, both these quantities we can determine from these quantity f 1 tilde and f 2 tilde. Remember this f 1 tilde of X satisfies the Klein Gordon equation for particles of type one. This contains the positive frequency part there is e to the power minus i x dot k 1.

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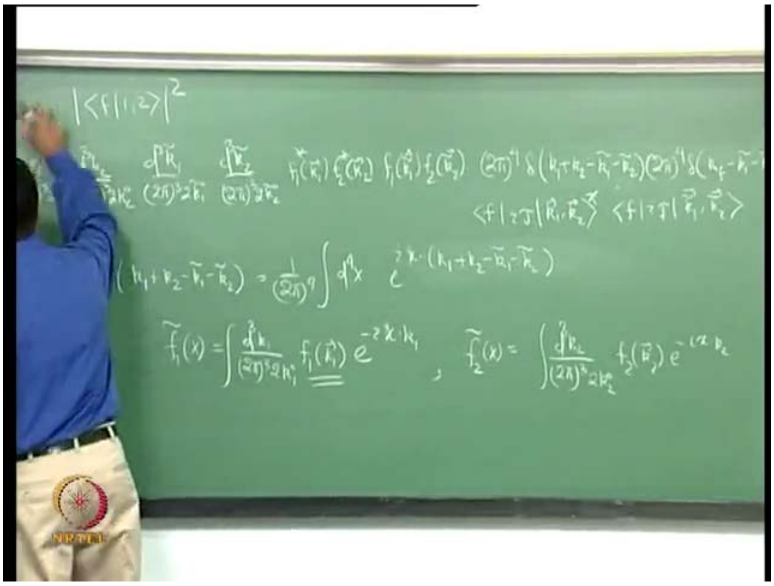

Because, I am considering the Fourier transform here there is an exponential vector also. This quantity f 1 tilde of x is the Fourier transform of f 1 of k 1 as you can see this f 1 of x capture only the positive frequency and this satisfies. This is a complex scalar field you can show that this satisfies the Klein Gordon equation for particles of type one. Similarly, when you consider f 2 tilde of x, which is d cube k 2 over 2 pi cube 2 k 2 0 f 2 of k 2 e to the power minus i x dot k 2.

This will represent the positive frequency part of particles type two. This will satisfy the Klein Gordon equation for particles of type two. We can consider the conserved current for these fields f 1 and f 2 for the field f 1 tilde of x. Let us call the conserve current to be j 1 mu current density, which is basically i times phi f 1 tilde tou of x del mu f 1 tilde of x minus del mu f 1 tilde f star of x times f 1, tilde of x this is the current density for particles of type one.

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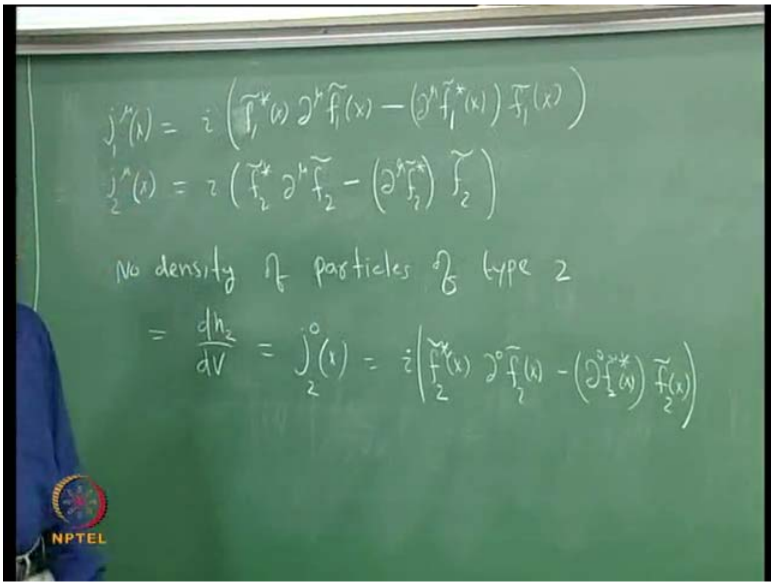

Similarly, you have current density for particles of type two, which I will call as j 2 mu of x. This will have an identical expression with f 1 replace by f 2 i f 2 tilde star del mu f 2 tilde minus del mu f 2 tilde times f 2 tilde. This is the current density for particles of type two. What will be the number density we will quantizing the free field. We have already see that zero, component of the current density gives basically the number density here.

So, the number density is just given by let us say if I consider particles of type two then the number density of particles of type two is given by d n 2 over d v. If I denote the number density of particles of types two to be d n 2 over d v. This is just given by j 0 2 of x, which is nothing but i times f 2 star of x, f 2 tilde star of x del 0 f 2 tilde of x minus del 0 f 2 tilde star times f 2 tilde of x. Now, look at the expression for f 2 tilde of x.

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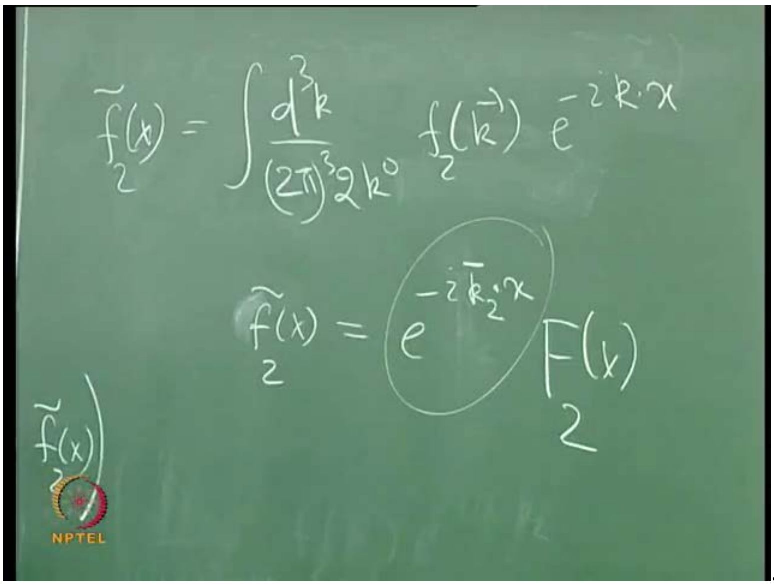

This is basically the Fourier transform of f 2 d cube k over 2 pi cube to k 0 f 2 of k e to the power minus i k dot x, but this f 2 has a. It has a distribution, which has a p kit k 2 bar. Therefore, the f 2 tilde here will roughly be f 2 tilde of x will be given by some e to the power minus i k 2 bar dot x times some f 2 of x. Where, f 2 of x will be some slowly varying function of x that is because of this distribution. Here, it has a peak at the value k 2 of x if it was a delta function then you would simply get e to the power minus i k 2 dot x, but the there is finite distribution in the momentum here. So, therefore it will have a part which is e to the power minus i k 2 dot x times some function f 2, which is slowly varying. So, when I put this expression here what I get is…

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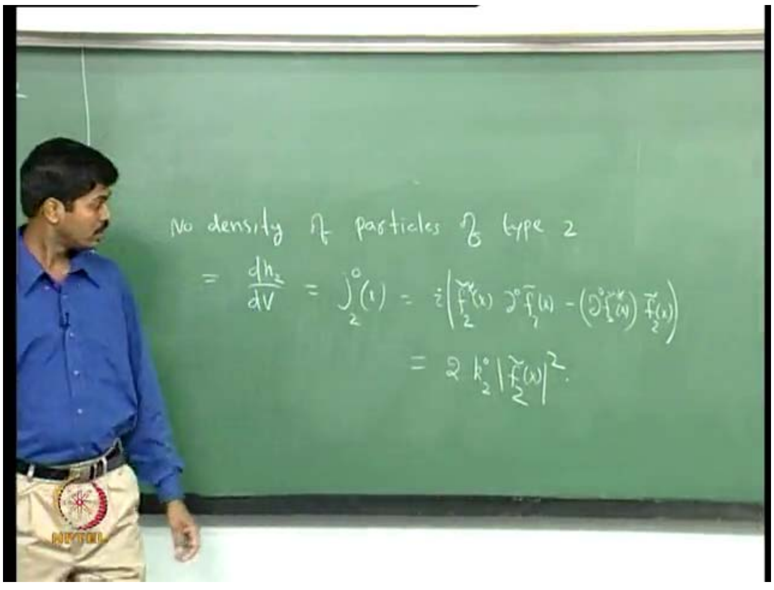

This will be matrix element twice k 2 0 times mode f 2 tilde of f square. So, therefore the number density of particles of type two is basically given by this what about particles of type one. What will be the flux of these objects the flux is going to be basically the velocity of the particles velocity of the beam times, the number density of particles of type one. So, what is the velocity of particles of type one the velocity will simply be given by, so let us summarize this for k 2 0 bar. This the same as this f, because of the fears vector here, but here I will have a k 2 0 bar.

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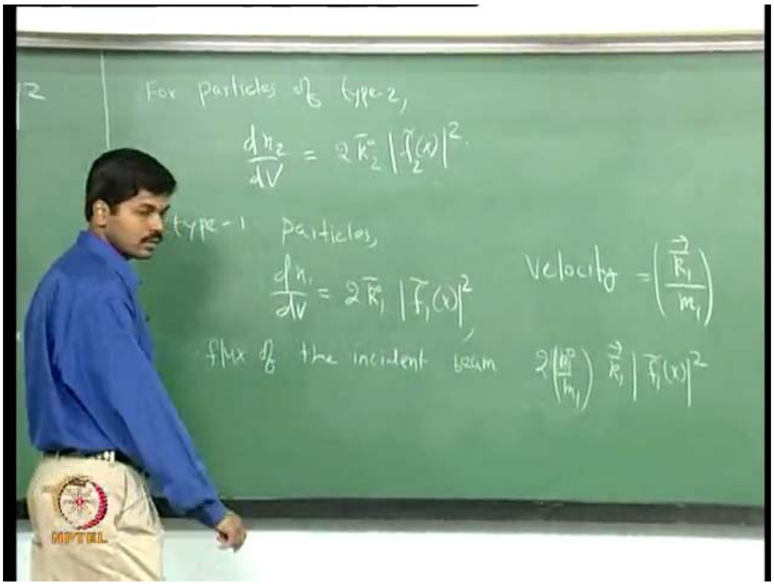

So, for particles of type 2 d n 2 over d v is given by 2 k 2 0 bar mode f 2 tilde of x square. Similarly, for type one particles again the number density the expression for the number density will be similar. So, for type one particles d n 1 over d v you can call this it will have the expression, which is 2 k 1 0 bar times mode f 1 tilde of x square. However, the velocity of particles of type one is will be given by the momentum p 1 k 1 bar divided by m 1, where m 1 is the mass of particles of type one.

So, this will give matrix element deflux of the incident beam to be twice k 1 0 divided by m 1 times k 1 bar times mode f 1 tilde of x square. You can either computed this way or you can directly compute this expression for the i th component of the current density j 1 i of x. You can consider that and you can evaluate this expression directly from there. Now, that we have all these thing, we can compute this scattering cross section.

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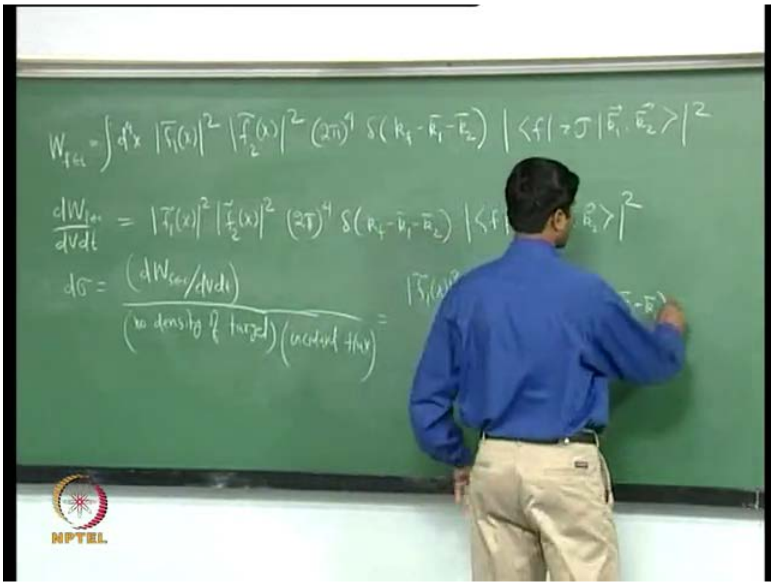

So, the cross section d sigma will basically be the transition probability per unit volume per unit time divided by the number density of target times. The incident flux the magnitude of the incident flux, which is basically these quantities. Here, f 1 tilde of x modes square f 2 tilde of x mode square times, 2 pi to the power fourth delta k f minus k 1 bar minus k 2 bar times mode f i T k 1 bar k 2 bar modes square.

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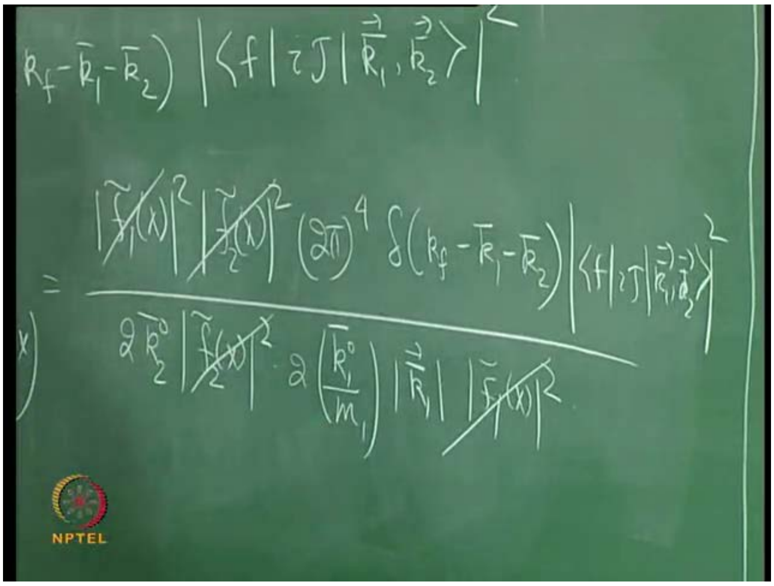

This quantity divided by the number density. There is twice k 2 0 bar mode f 2 tilde of x square times twice k 1 0 over m 1 mode k 1 bar mode f 1 tilde of x square. So, as you can see this will cancel with this will also go away what will remain here is this is 4 k 1 0 k 2 0 bar k 1 0 bar and mode k 1.

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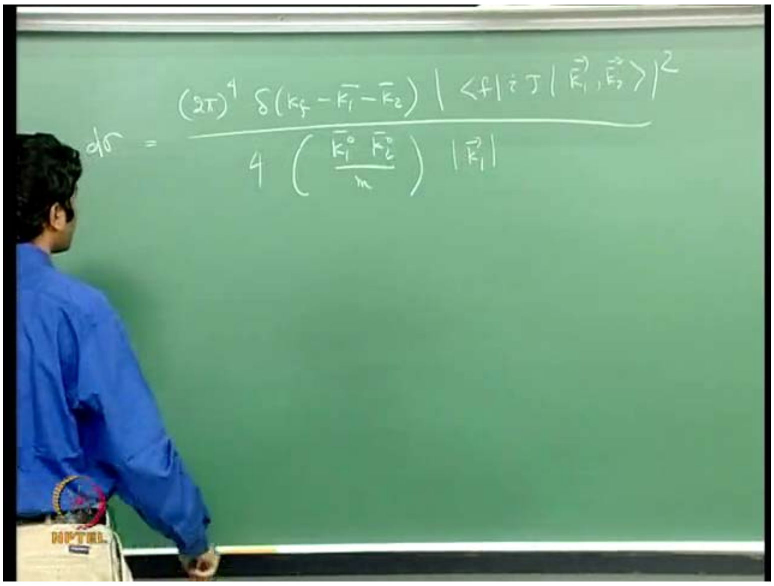

So, therefore, so the cross section is a given by d sigma equal to 2 pi to power of four delta of k f minus k 1 bar minus k 2 bar times mode f i T k 1 bar k 2 bar mode square divided by 4 into k 1 bar zero k 2 0 bar divided by m mode k 1. So, actually for the particle of type one the flux, which is given by the velocity times number density, which is d n 1 over d v.

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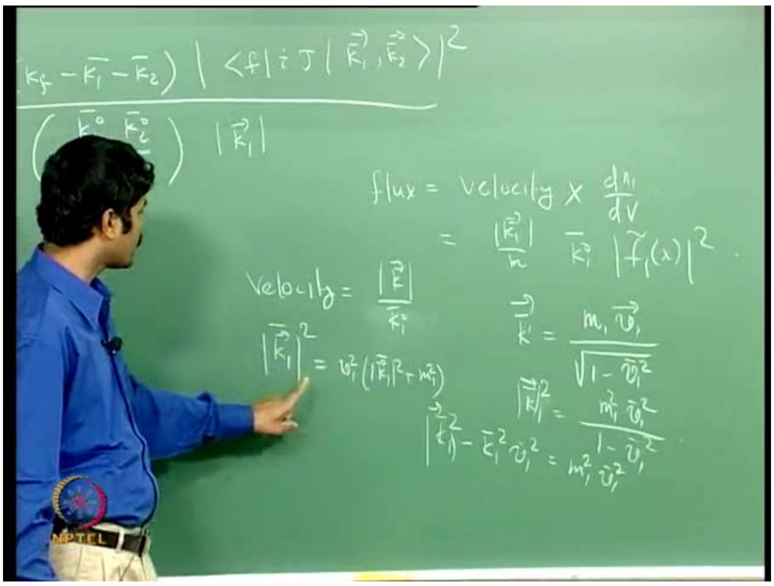

What we have it is a velocity is a mode k 1 divided by m the number density is a k 1 0 bar mode f 1 tilde of x mode square. The velocity is naught k 1 bar divided by m, but it is actually the k 1 bar divided by k 1 0 bar this is because for relativistic particle. You know the momentum k 1 bar is just a mass m 1 divided by square root of 1 minus v 1 bar square times v 1. We have to solve this equation for the velocity.

So, this simply implies k 1 square k 1 bar square h m 1 square v 1 square divided by 1 minus v 1 bar square. In other words k 1 bar square minus k 1 bar square v 1 bar square equal to m 1 square v 1 bar square. If I take it this side, then what I get is k 1, this is mode k 1 bar square. So, this is mode k 1 bar square of mode k 1 bar square is equal to v 1 square times mode k 1 bar square plus m 1 square, which is nothing but k 1 0 bar square. Therefore, the velocity is simply mode k 1 bar divided by k 1 0. So, here therefore the velocity is not k 1 bar divided by m, but its k 1 bar divided by k 1 0 bar.

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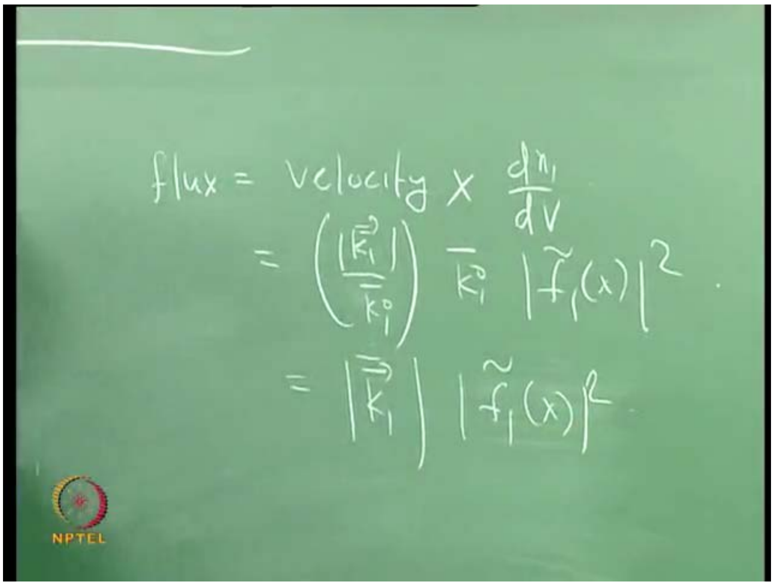

So, I have the flux for particles of type one is simply k 1 bar modulus times mode f 1 tilde x square. Therefore, the cross section here is instead of a instead of dividing by m here, what I will have is I will have k 1 bar 0.

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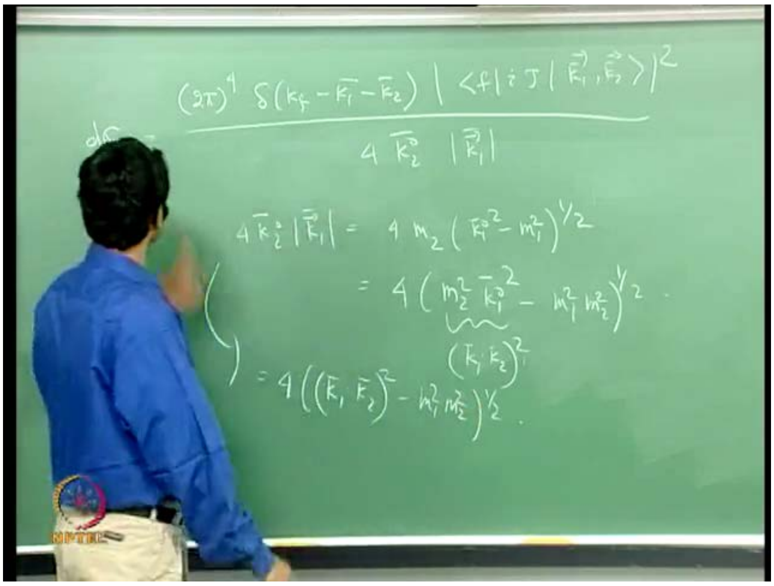

So, k 1 bar 0 is cancel by this. Therefore, the cross section is given by four k 2 0 bar k 1 modulus k 1 bar modulus. I would like to write it in a Lawrence invariant session. So, then denominator the numerator everything is fine, but in the denominator you know that 4 k 2 0 bar k 1 bar is nothing but four times k 2 0 k 2 0 bar is nothing but m 2. Because, we are working in the rest frame of particles of type two. This is simply m 2 and this one is k 1 0 bar square minus m 1 square to the power half. If I take this m 2 inside, then I have 4 m 2 square k 1 0 bar square minus m 1 square m 2 square again. Because, we are working in the rest frame of particle of type two. So, this is simple equal to k 1 bar dot k 2 bar square. Therefore, the denominator I can rewrite it as 4 k 1 bar dot k 2 bar square minus m 1 square m 2 square to the power half.

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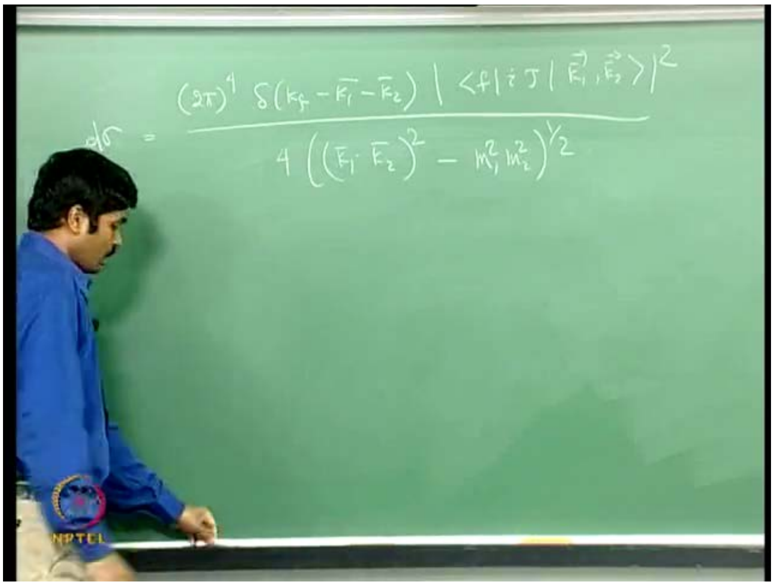

Therefore, the cross section is now given by 4 k 1 bar dot k 2 bar square minus m 1 square m 2 square whole to the power half. Now, here the cross section, where this quantity is actually Lawrence invariant.