Indian Institute of Technology Madras
Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 9
Hamiltonian dynamics (Part V)

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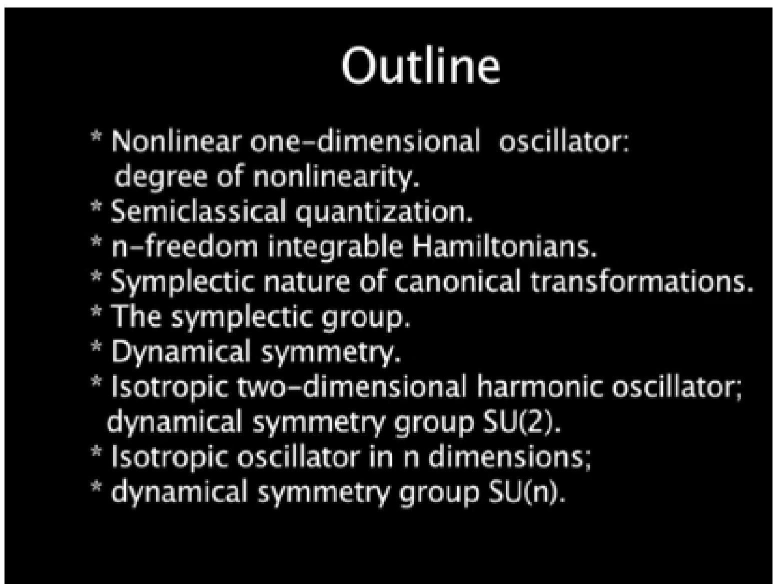

I recall that we were looking at the case of one dimensional motion in a non linear oscillator situation.

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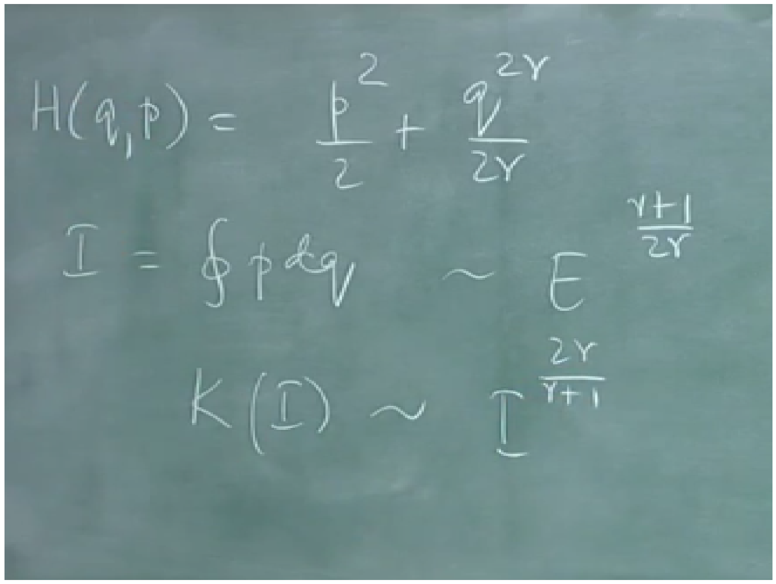

Where the Hamiltonian the function of q and p was something of the form P^2 / 2 + q to the 2 r / 2 r where r is a positive integer and the case r = 1 correspond to the simple harmonic oscillator we had computed what the action was for motion in this potential and we discovered that the action which was an $\int$ or pdq for bounded motion oscillatory motion in this potential was $\propto$ a certain power of the energy of the oscillator itself which is E^{r+1} /r this is the result we obtained of course if you translate this back to action angle variables.

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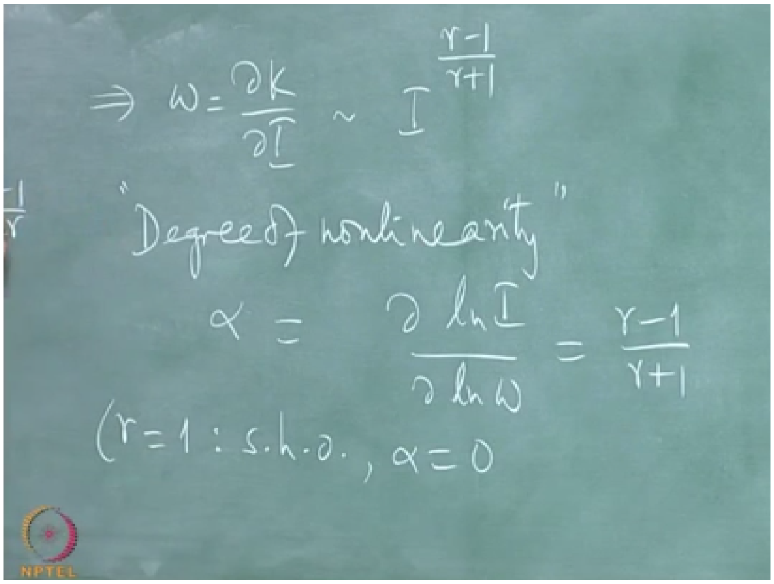

The way we had defined it earlier this ⇒ that K the Hamiltonian as a function of the action is ∝ the action to the power 2 r / r + one which immediately ⇒ that the frequency ω of motion which is defined as $\delta K/\delta I$ is ∝ $I^{r-1/ r+1}$ if you therefore define a degree of non-linearity α say and that is defined as δ log I / δ log ω then this is = r - 1 / r + 1 and notice that r = 1 which is the simple harmonic oscillator α is 0 no non-linearity and as r becomes larger and larger the non-linearity tends as r tends to infinity to the value unity.

So it is a very useful indicator of the degree of non-linearity if you like of an oscillator of this kind not a universal measure of any by any means not a universal measure or anything like that but a very useful one in many contexts we also saw that the semi-classical.

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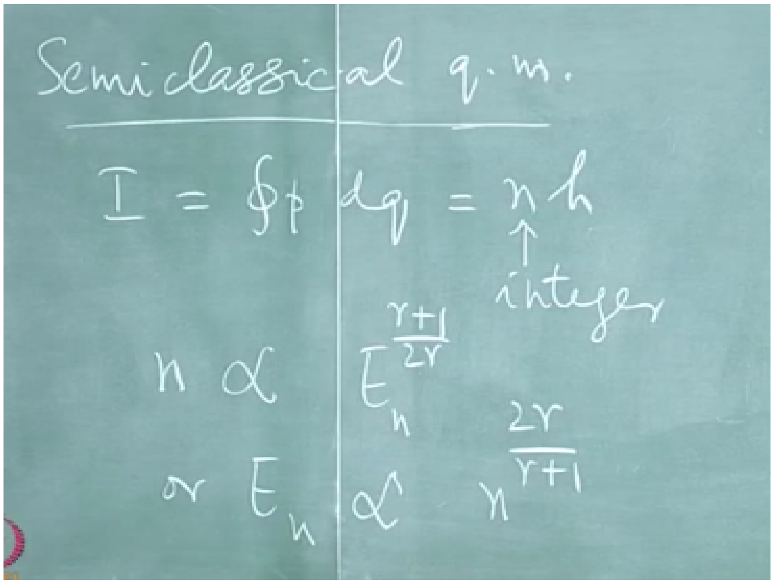

Quantization of this system is immediate it follows at once because if you recall in semi-classical quantum mechanics semi-classical quantization corresponds to writing I which is ∫ pdq = n times Planck's constant where this is an integer and if you take that along with this at once ⇒ that n is ∝ the energy level En $^{r+1}$ / 2 r or En is ∝ n $^{2r/ r+1}$ so tells you something about the level spacing for quantized motion in this potential is it the other way about but oh yes of course this is log ω / log I and that is this degree of non-linearity okay the way the frequency changes as a function of the action thank you.

So semi classically we find that the energy level the nth energy level is dependent on the quantum number n in this one-dimensional problem-- according to this relation here valid for n much bigger than unity which is where the semi classical rule is valid and again you notice that when r is = 1 so here r = 1 which is simple harmonic oscillations we know that En is ∝ n itself.

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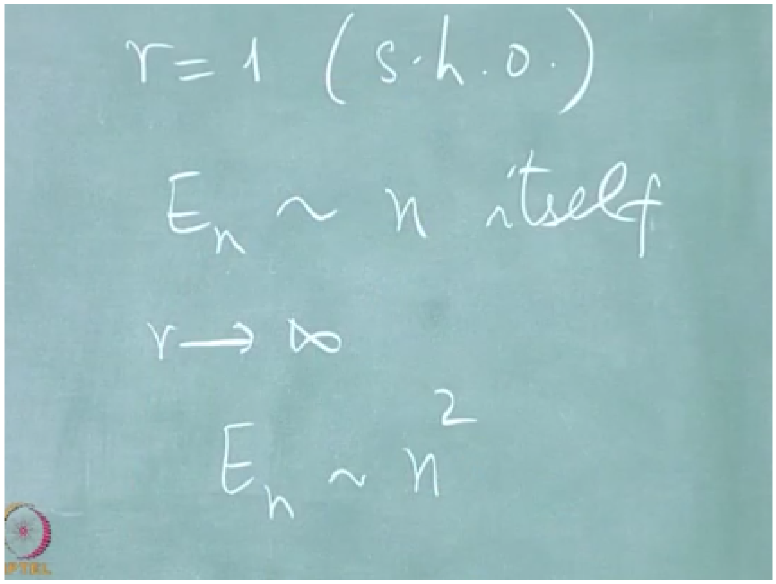

That is a level spacing which is equally spaced equally spaced energy levels and this is exactly what the harmonic oscillator does in quantum mechanics of course the exact relation for En is n + 1/2 times H cross ω and the half arises from so-called zero point motion it is the ground represents the ground state energy of the oscillator but we're not going to get into quantum mechanics here just to point out that this semi classical argument is immediately leads to this result that En is ∝ this power of n for this whole family of potentials notice also that as r tends to infinity we end up with E and going like n^2

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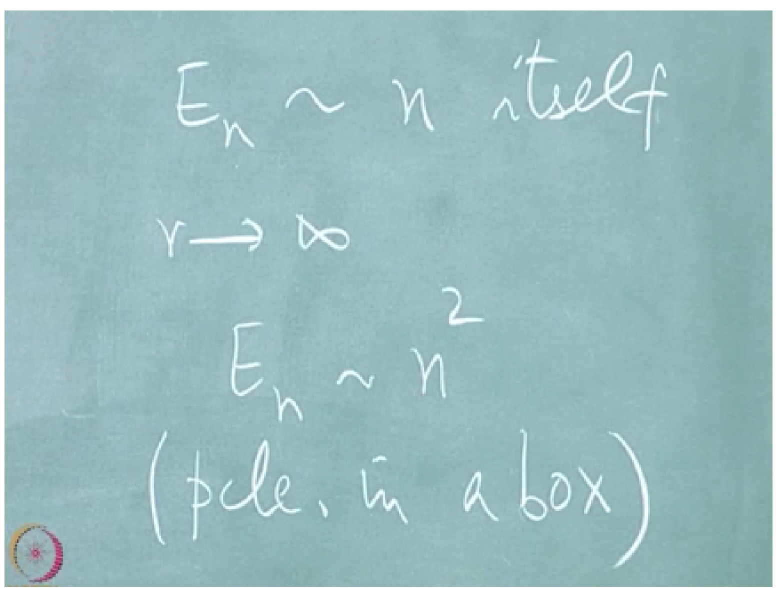

And that is precisely the level spacing for a potential which rises more and more steeply infinitely steeply namely a particle in a box so this is exactly the same as the level spacing for a particle in a box in a one dimensional box so that limit too is correctly obtained from this semi-classical formula one can go further and actually try to find the correction to this end Corrections to other this quantity here etcetera.

But we are not going to get into that right now so much for a little digression on semi-classical quantization which follows from the arguments we have been giving here. Now let us go back a few steps and ask what is the reason for having integrability in a Hamiltonian system.

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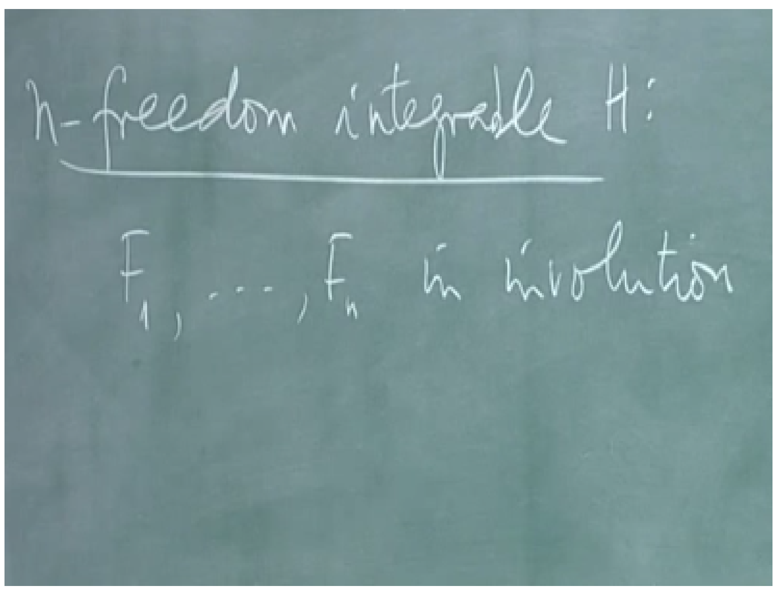

So we go back to n degrees of freedom and freedom integrable Hamiltonian and ask what is the underlying physical reason for the existence of n constants of the motion f1 through FN in involution with each other yes the way we have talked about it so far is a Hamiltonian system the phase space variables come in pairs so we identify n degrees of freedom and independent degrees of freedom which means that you must specify n numbers q1 through q and for me to tell me completely the configuration of the system in real space.

If you like and in addition you need n conjugate momentum to complete the description of the system in phase space in other words to describe the dynamics of the system so I call number of degrees of freedom the same as the number of generalized coordinates that I have number of independent degrees of freedom in the case of more general dynamical systems first of all the phase space does not have to be even dimensional this pair wise structure the Poisson bracket structure is not necessary at all.

And I do not distinguish between different kinds of variables I just call the whole set dynamical variables 1 to n or as many as there are yes absolutely yes absolutely it is the same it is the same as the degrees of freedom if I give you for instance in statistical mechanics when you discuss monatomic gases diatomic gases and so on for a diatomic gas you ask how many degrees of freedom does a diatomic molecule have yes ah the question is whether in chaotic motion the concept of degrees of freedom appears or not of course it does it has absolutely nothing to do with the kind of motion identification of the number of degrees of freedom yes.

I believe so I believe so unless there are other reasons to believe that external forces are present or there is a time-dependent perturbation acting on the system or anything like that but if I give you a collection of molecules and you assume Newtonian mechanics to hold good and the system is isolated they act with forces acting upon each other caused by themselves then I do not see why it is not a Hamiltonian system why do you say I cannot formulate the system I assume let us assume put a model on it let us assume that Newton's equations are valid.

Let us assume that there is a certain potential energy between two molecules a certain distance apart once I have a model of that kind I have a Hamiltonian system with a very large number of degrees of freedom no doubt the motion is in general chaotic it is very irregular it is not integrable yes it is assumed to be Hamiltonian yes indeed of course in real gases you have many other complications.

For instance you might have to bring in quantum mechanics you might have to solve the entire problem quantum mechanically that is a separate subject in itself you are not going to get into that here but otherwise yes it is a Hamiltonian system the kind of motion that a system has whether it is regular or integrable or regarded chaotic this has nothing to do with the identification of the number of degrees of freedom.

You have absolutely nothing to do with it so let us look at an N freedom Hamiltonian system for which the Louisville Arnold criterion tells us that the system is integrable if you have n constants of the motion F1 to Fn in involution and then we saw that a transformation to action angle variables is possible and once you make this transformation the Hamiltonian becomes independent of the angle variables it is a function of the action variables alone and then you can integrate the entire two and set of equations that you have.

All to n equations for prescribed initial conditions at least in principle you can do this now you could ask what is the physical reason why this system is integrable why these systems are integrable what is the physical significance of these F1 through Fn of course the action variables which we talked about which lead to the natural frequencies of the system the ωi are certain combinations of these EPS so in that sense there is already some physical interpretation for these EPS but can we think of this in a slightly more physical fashion is there something much more immediate and the answer is yes the existence of these constants of the motion is related to some hidden symmetry in the problem a certain dynamical symmetry in the problem.

For instance if I look at two harmonic oscillators and the spring constants are the same in the two perpendicular directions then you would immediately tell me that the Hamiltonian is rotationally invariant you make any rotation in the xy plane and the Hamiltonian does not change at all so there is a certain symmetry in the problem and the existence of symmetry is linked to integrability so whenever a system is integrable whenever you have these constants of the motion there exists a certain dynamical symmetry in the problem.

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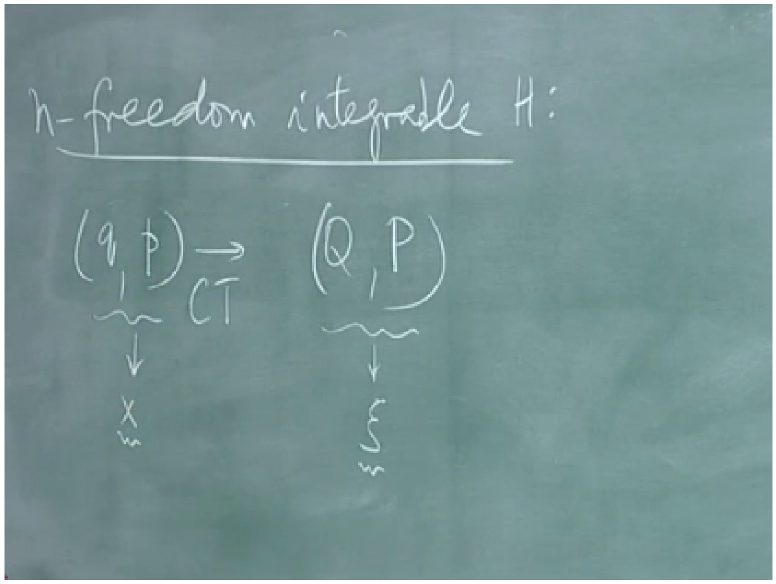

Now let us try to understand what this dynamical symmetry is at least go a little bit into this it is a vast subject by itself but let me at least give the rudiments of this subject let us go back and ask what does a canonical transformation actually do to a system so we start with the system with variables QP and you make a canonical transformation to a new set of variables Q and P what does this do what kind of transformation is it or if you like if these q's and PS are combined into a phase space variable x 2n dimensional vector.

Then these could similarly be combined into a - n dimensional vector these quantities are functions of these quantities which preserve the Poisson bracket structure.

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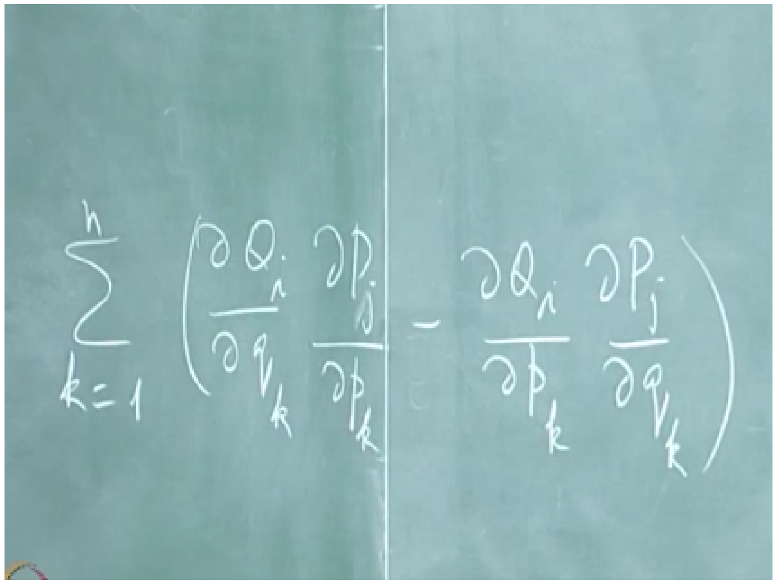

In other words we know that qi qj = pi pj is = 0 and we know that qi pj is = 𝛿ij now what is meant by a statement like this what is meant for example by this statement here what I mean by this is this ⇒ if I write out this Poisson bracket explicitly it ⇒ that a Σ from k = 1 to n / all degrees of freedom 𝛿 Qi / 𝛿 QK 𝛿 Pi Pj / 𝛿 Pk - 𝛿Q i / PK 𝛿 PJ / 𝛿qk this quantity is = the 𝛿 function that is the definition of the Poisson bracket in the original coordinates and if the Poisson bracket structure is preserved it means you have a relation of this kind for every value of i and j similarly we could write these down as well.

Now is it possible to use this matrix J which we had introduced and write this in simpler form well yes indeed because you already know that I can write any Poisson bracket I could write any Poisson bracket in more compact form if I use the J matrix and what would that be it would say take this quantity Qi and take its derivatives take the ^{T} of this constructed into a row vector put the J in between and then the column vector from the derivatives of this and that would be = 𝛿 ij.

So can we combine all these three relations into a single relation and the answer is yes not surprisingly it turns out that all these relations can be combined let me do that here.

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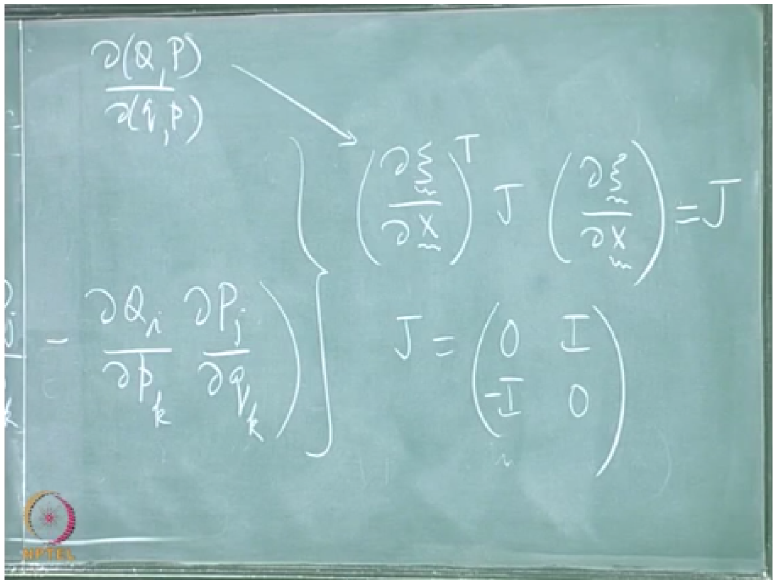

This whole set of relations is equivalent to saying that the Jacobian matrix of the transformation which let me denote symbolically as $\delta\xi$ / this δ X by the way this stands for the matrix the cube P / δ P this matrix^{T} J times δ Z / δ X this quantity is = J itself it is easy to check that all these relations are summarized in the single line by this equation this matrix equation it is not very difficult to verify that this is so I have used this symbolically it just to tell us immediately that it is just the Jacobian matrix corresponding to the canonical transformation.

Now what does this tell us this quantity is a 2n / 2n matrix its ^{T} with a J in between times the matrix must be = J itself the same matrix J and recall that this J was = 0 the unit n / n matrix - that and 0 it was a 2n / 2n matrix of this kind what would you call a matrix such that its ^{T} times the matrix itself is = the unit matrix what do you call a matrix which obeys this condition.

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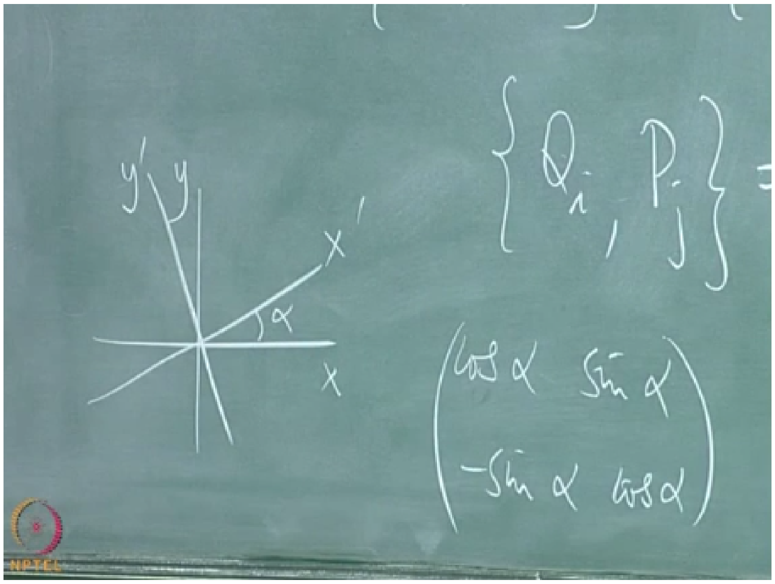

Matrix ^{T} times the matrix square matrix = I itself well if it is real then it is an orthogonal matrix this is the definition of an orthogonal matrix the $^{-1}$ of the matrix is = its ^{T} that is an orthogonal matrix if the matrix is restricted to real entries then of course the unitary matrix is an orthogonal matrix because there is nothing to complex conjugate but in general the matrix could have complex elements in all cases here all these are real variables therefore we do not have any complex elements.

So a matrix of this kind is an orthogonal matrix can you give me an example of a transformation of coordinates a which is represented by an orthogonal matrix every rotation absolutely right every rotation in physical space is an order represented by an orthogonal matrix in 3-dimensional Euclidean space any 3x3 orthogonal matrix of unit determinant represents a physical rotation of the coordinate system the simplest of these of course as you are well aware is if you took the xy axes and you went off to x` and y` at an angle α then this rotation in the xy plane about the origin is represented by a two-by-two orthogonal matrix.

Whose structure is what is the matrix representing this rotation absolutely it is just cos α sin α - sin α and so on in three dimensions you can generalize this to higher dimensions so our problem matrices with unit determinant with determinant + 1 represent physical rotations that matrix there which represents a canonical transformation in the 2n dimensional phase space is not our problem because there is a J sitting here and a J sitting there but by now we have got used to this

J appearing everywhere in Hamiltonian dynamics this is pseudo orthogonal in a certain sense such a matrix is called a symplectic matrix.

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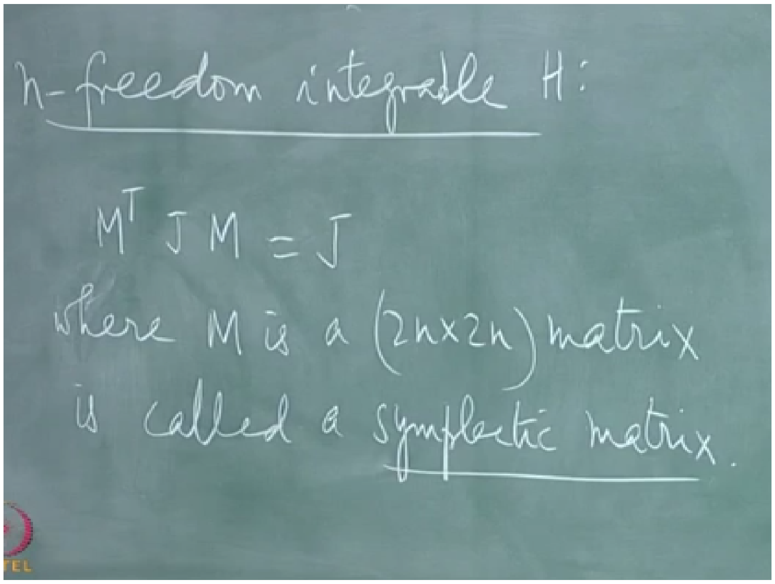

Let me write that down a matrix $M^T J M = J$ so this is some where m is a 2n / 2n matrix it is called the symplectic matrix so what is the lesson we learned from that equation there canonical transformations are represented by symplectic matrices just as rotations are represented by orthogonal matrices of unit determinant canonical transformations are represented by symplectic matrices therefore in exactly the same way that the study of orthogonal matrices tells you everything you need to know about rotations in exactly the same way the study of symplectic matrices tells you everything you need to know about canonical transformation.

So this is an algebraic approach to the study of canonical transformations now this kind of equation has remarkable properties we will see in a second the first of which is the following if I took the determinant on both sides what is the determinant of J j was defined by that matrix the determinant of J is + 1 we check this out.

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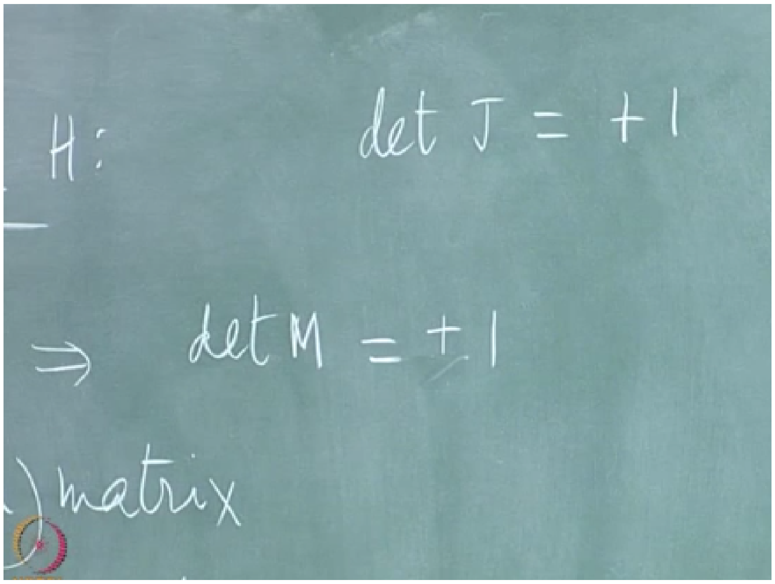

So determinant J + 1 it turns out from this equation it follows if I took the determinant of the left-hand side the determinant of a product of matrices is just the product of determinants and the determinant of M ^{T} is the same as the determinant of M so this tells you at once that this ⇒ determinant M is = + or - 1 because the square is = 1 it turns out that you can show without too much difficulty although it is a non-trivial exercise that the determinant of a symplectic matrix is in fact +1 this is why I mentioned earlier that canonical transformations also preserve orientation.

They do not just preserve they preserve a number of things among other things preserve orientation but this is reflected in the fact that the determinant of a MIT symplectic matrix is +1 just as orthogonal matrices form a group the product of two orthogonal matrices is also an orthogonal matrix every orthogonal matrix has an $^{-1}$ and they form a group in exactly the same way the symplectic matrices of a given order from a group of matrices this is a subgroup of the set of all 2n / 2n matrices which are non singular.

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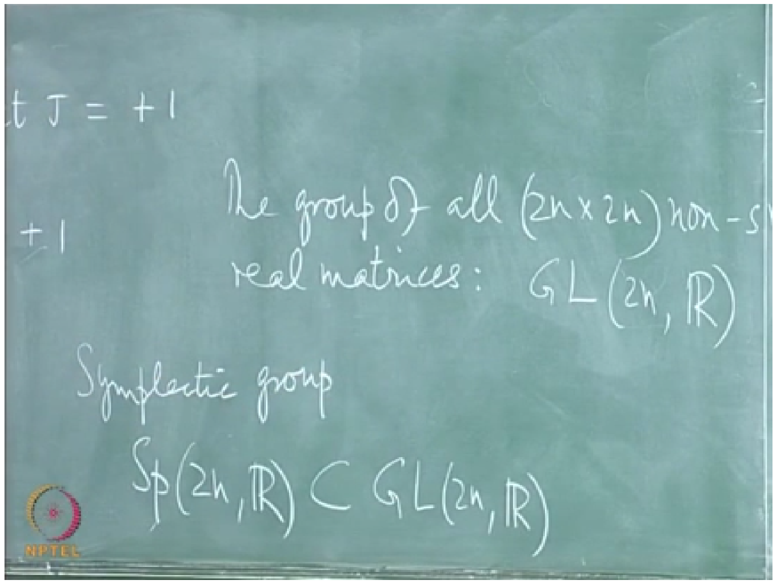

So let me write that down the group of all 2n / 2n real matrices so we are restricting ourselves to matrices which are real with real entries 2n / 2 and non singular what is a non singular matrix the determinant is not = 0 in other words there is an $^{-1}$ for the matrix so the group of all 2n / 2 and non singular real matrices is denoted by GL the general linear group of order 2n / the real's.

So this stands for the general linear group among such matrices the symplectic matrices form a subgroup of such matrices in other words the product of two symplectic matrices is again a sympathic matrix all 2n / 2n for a given order so it is a subgroup of this thing here and this group is called the symplectic group and it is denoted by SP 2 n and it is a subgroup of GL.

Therefore for a given dynamical system with a given number of degrees of freedom n the study of its canonical transformations amounts to the study of the symplectic group of the same order of order to it now a great deal is known about the properties of such matrices so great deal is known about the symplectic group and what it ⇒ and what is generators are and so on and so forth therefore we have a fairly good idea of what the symmetry possessed by a system should be a Hamiltonian system should be under a canonical transformation a Hamiltonian flow goes to a Hamiltonian flow.

Measure is preserved what you need for a symmetry of the system though what do I mean by the dynamical symmetry of a system what would you say is given a dynamical system what would you say is a dynamical symmetry of the system one possibility is to look at its Hamiltonian if it is a Hamiltonian system and ask what kind of transformations leaves the Hamiltonian unchanged that is one possibility but of course in the case of a Hamiltonian system you read more than that you actually must make sure that the transformations of coordinates of the variables is also canonical.

So that the structure of Hamilton's equations is unchanged the most general way of defining what the dynamical symmetry group of a dynamical system is to say a set of transformation which leaves the solutions unchanged maybe takes one solution to another and it mixes up different solutions but the solution space the set of solutions of the dynamical equations should remain unchanged that is what I would call a dynamical symmetry of a system the full set of solutions must remain of the dynamical equations must remain unchanged.

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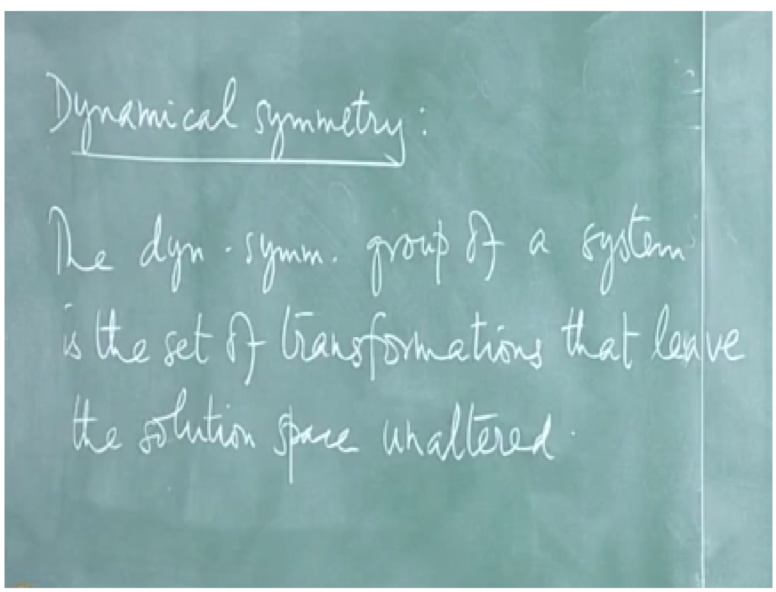

If I accept that then it is easy to see what I mean by the dynamical symmetry group of a Hamiltonian system and let us write that down like this I mean the dynamical symmetry group of a system is a set of transformations of its phase space variables set of transformations that leave the solution space but me that is sorry I did not get the question what is F no not necessarily not necessarily there is nothing true with the Hamiltonian system right.

Now we are talking about a very general statement I am simply saying when I have a dynamical system described by a set of differential equations and I ask what is meant by a symmetry group of the system the most obvious statement is that it is a set of change of variables of the system such that the solutions do not get changed the set of solutions does not get changed that is what I would mean by dynamical symmetry group now let us come to a Hamiltonian system and ask what is the dynamical symmetry group of a Hamiltonian system what can it possibly be well what do you need for a Hamiltonian systems solutions to remain unchanged first of all I certainly need the following.

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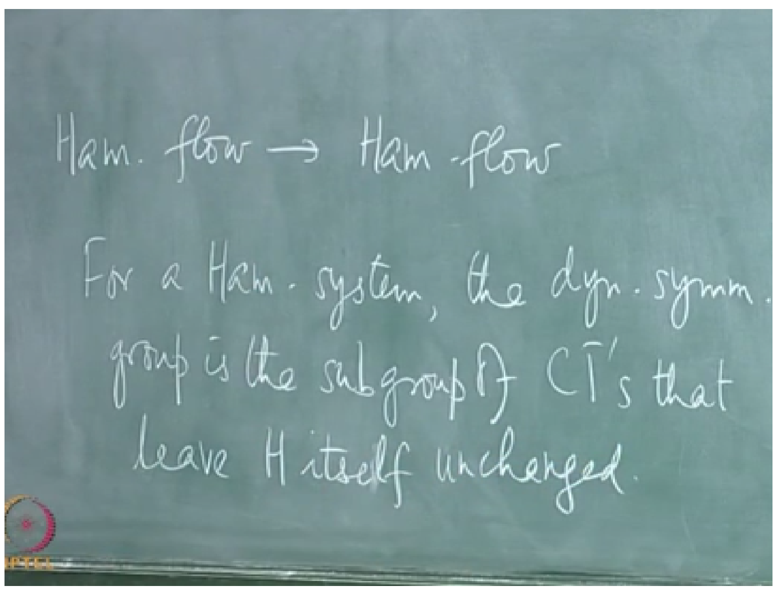

I certainly need the Hamiltonian system the Hamiltonian flow to go to a Hamiltonian flow certainly need that this is done by the set of canonical transformations but I need more than that I would like to leave the solution space unchanged in other words the Hamiltonian itself should not change so that the equations of motion are exactly the same in the new coordinates as they were in the old coordinates right so not only do I need the set of canonical transformations but I need a subset of this group of canonical transformations which leave the Hamiltonian itself unchanged.

In other words I should not change from H to KH must remain unchanged that does not always happen with all canonical transformations therefore the dynamical symmetry group of a Hamiltonian system for a Hamiltonian system the dynamical symmetry group is the subgroup of canonical transformations that leave H itself unchanged so that not only do you go from a Hamiltonian flow to a Hamiltonian flow but more over it is the same Hamiltonian and then you are guaranteed that the dynamical equations do not change and therefore the space of solutions does not change either.

In general therefore this dynamical symmetry group of a Hamiltonian system is a certain subgroup of SP 2 and r whatever be the number of degrees of freedom that also could be a group it may not exist you may not have much symmetry in the problem at all and that is in fact what happens in general but then in those cases where the Hamiltonian is integrable it turns out that you have a dynamical symmetry group and in fact that group is generated the infinitesimal generators of these transformations that belong to the dynamical symmetry group are related to the constants of motion f1 through Fn ye.

Of course yes in a group every element has an $^{-1}$ so this is certainly true yeah canonical transformations have $^{-1}$s that was our first premise that going from the small cues and piece to the Q and p's was actually invertible this was our promise that the canonical transformations we have talked about our global canonical transformations namely they apply in all of the phase space concerned.

Of course you could have a local canonical transformation which is not applicable in all of phase space we have not looked at that possibility at all we are all only talking about global canonical transformations and they are certainly invertible in fact since you raised the point the canonical transformation is a point a symplectic transformation.

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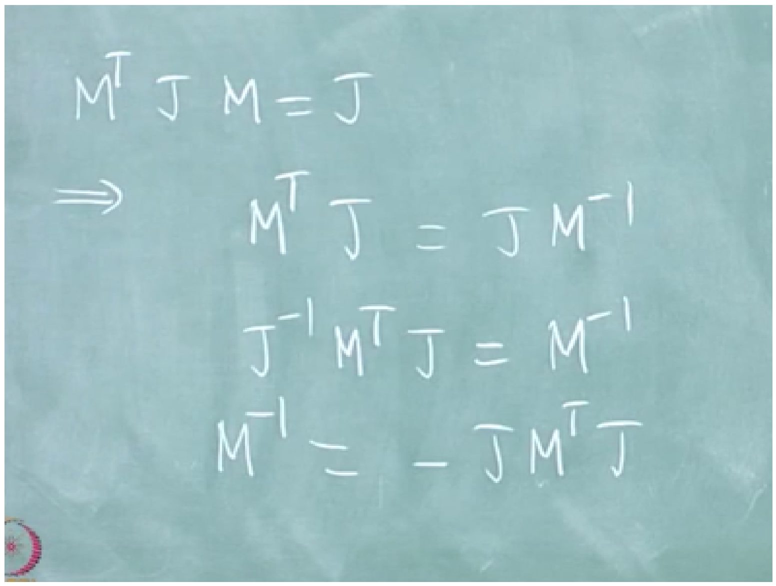

So it satisfies this relation here and of course what does this imply at once let us take this M across to the other side by applying the $^{-1}$ operator so it immediately says M^T J is = $JM^{-1\ -1}$ and let us bring the J to this side by applying the $^{-1}$ of J and therefore I get J^{-1} M^T J is = M^{-1} but I know that J^{-1} is - J J^{-1} is = J^T is = - J so in fact I know that M^{-1} is = - J^T G for a symplectic matrix therefore the $^{-1}$ exists no reason.

Why it should not it is determinant we already saw was + 1 so these transformations indeed form a group the point I am making is that the dynamical symmetry group could be much smaller than this group than the group of canonical transformations could be much smaller in general because I also pointed out that in cases where the system is not integrable you do not have any symmetry at all but when you do have some special symmetry the system could become integrable this is the idea let me for example go back to the 2-dimensional harmonic oscillator and ask what the symmetry group could be this is not easy to identify it is not a trivial task to identify the dynamical symmetry group of a system which we know to be integrable in general one has to do a little bit of work to do this.

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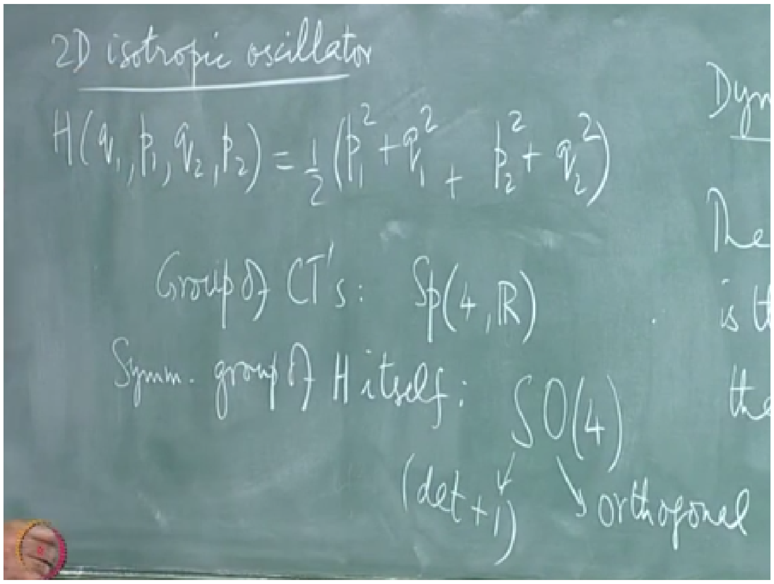

So let us look at the two dimensional oscillator which is H(q1, P1, q2 and P2 and this was $P1^2 + q1^2 / 2 + P2^2 + q2^2 / 20$ if I took an oscillator which has exactly the same frequency in the 1 and 2 directions this is the isotropic oscillator this Hamiltonian has a great deal of symmetry the symmetric group the group of canonical transformations is the symplectic group in two degrees of freedom so this is sp 4 / the real's what is the group of transformations that leaves this Hamiltonian unchanged.

What would you say is a group of transformations that leaves this Hamiltonian itself unchanged notice that I can just take out the factor ½ and write this in this fashion so I should write here 2D isotropic to mean that it is got exactly the same properties in all directions as you can see the potential energy is $q1^2 + q2^2$ which is invariant under rotations in the q1 q2 plane so it has circular symmetry in this case what would you say is the symmetry or the group of transformations of these 4 variables each of which runs from $-\infty$ to ∞.

That leaves H unchanged we can switch variables but there is a big huge continuous group of transformations what would leave this unchanged this combination unchanged if I have $x^2 + y^2 + z^2$ in three real variables x y z what group of transformations leaves is unchanged all rotations about the origin leave it unchanged in the three dimensions x y and z what leaves this unchanged all rotations in phase space in this four dimensional phase space all possible rotations about the origin would leave this unchanged what would that group be the symmetry group of it of H itself.

What would this group be it is the group of rotations in four dimensions for you plead in dimensions what would that group be it is a group of 4x4 matrices but what sort of matrices would be is B they have to be orthogonal what should the determinant of the matrix B + 1 so volumes are left unchanged and that group is a group of orthogonal transformations in 4 variables called So4 but because the determinant is 1 you write an S here to write special or unimodular determinant + 1.

On the other hand this stands for orthogonal so on the one hand the group of canonical transformations in two degrees of freedom is SP 4 are the set of all 4x4 matrices with real elements which are symplectic which satisfy that condition on the other hand this particular Hamiltonian is left unchanged by the group of all 4x4 matrices with real entries which are orthogonal and which have unit determinant it is clear that for the dynamical symmetry group of this problem.

The set of transformations in phase space that leaves this Hamiltonian unchanged and takes the Hamiltonian floor to a Hamiltonian flow does not change the Poisson bracket structure is that set of transformations which belongs to both this group as well as this group in other words the intersection of these two groups so all matrices which are both symplectic as well as belong to this so4 that set of transformations that special subset of canonical transformations is in fact the symmetry group of the system.

So as you can see it is not easy to identify the symmetry group of a dynamical system because not only must the transformation be canonical but it should also preserve the form of the Hamiltonian in this case since the Hamiltonian was so simple I could do this directly without too much trouble I could just identify it by inspection this looks like the surface of a sphere in four dimensions.

And therefore I could immediately write down a so forth now that is group which is the intersection of these two groups is a different group altogether it is smaller than either of these groups and it is the group of turns out to be the group of all 2 / 2 unitary matrices with determinant + 1. So let me not write that down in explicit terms and get into Group theory but let me just write the result down and say.

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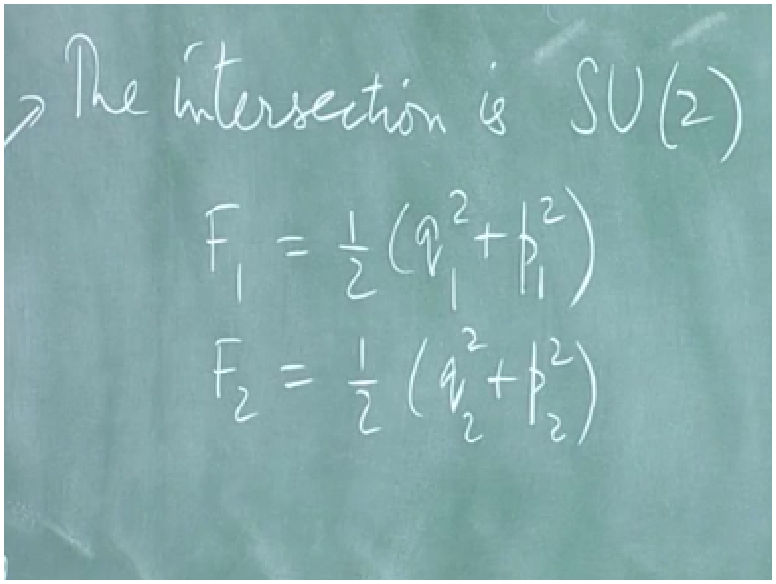

The intersection SU(2) which stands for the group of all 2/2 unitary matrices with determinant + 1 so it is a set of transformations which can be put into 1,2,1 correspondence with this set of matrices and this requires some mathematics to prove which we would not prove but I am going to simply assert that this is so the reason I do so is because I did like to identify the generators of this group and show you what the physical meaning of these generators is you see what are the constants of the motion here first of all we know it is integral we had two constants of the motion.

Which were in involution with each other and what were those two constants one of them was f1 which we could take to be just this the energy of the first oscillator and f2 was 1/2 - 2 ^{2}+ b^2 what else do you think is a constant of the motion here it is clear the system is integral and we know that the frequency ratio is unity in this case the same frequency for both these oscillators so on the surface of this torus we talked about the last time when you go around the torus in this way once you also go around this way once exactly.

Once so the motion is periodic completely periodic there is just one period unit frequency these two are also an involution with each other now to describe the trajectory you need one more constant of the motion which is an isolating integral so that the trajectory doesn't wind itself around on this torus completely but rather is a discrete curve is a curve it is isolated curve for each set of initial conditions what else do you think is constant in this problem let us think a little physically q1 let us say is the x-axis and q2 is the y-axis and you got a problem.

Where you have an isotropic oscillator the same spring constant in both directions and the potential is circularly symmetric if I write this down in circle up in plane polar coordinates this is just 1/2 r^2.

So what do you think is constant in set emotional well here is this particle in the q1 q2 plane well the actual shape of the trajectory will depend on initial conditions it will depend on what the phases are what the initial values of q1 and q2 are that is certainly true but what do you think is a symmetry. I mean work what else is constant what sort of force if the particles here in is attached by a spring to the center what kind of force is exerted by the spring in what direction is this force it is always radial therefore this is a central force problem.

It is certainly a central force problem what do you expect is constant in a central force problem the angular momentum that is there is no torque on this particle the angular momentum is therefore a constant of the motion we guaranteed that now what is the angular momentum in this planar problem you just have two variables X and Y and this is Ps of X and Ps Y what is the angular momentum well if you use this formula R cross P since you have reduced to a plane everything is in a plane this cross product is essentially a single number.

So what is r cross p there is only one component to it and what is that = P 1 P 2 q 1 q2 what is think in 3 dimensions if you have a hard cross p in three dimensions what is the Z component of r cross p what is the third component of the angular momentum which I will call L or since I know that it is the third component if ,I am looking at a three-dimensional problem let me just call it j-3 what is this = = yeah it = q 1 p2 -let me put 1/2 here for a reason it should become clear in a second I am guaranteed that this quantity is a constant of the motion in other words.

The Poisson bracket of j3 with h 0 that is not hard to verify so indeed this is a constant of the motion and now it turns out that the following quantities.

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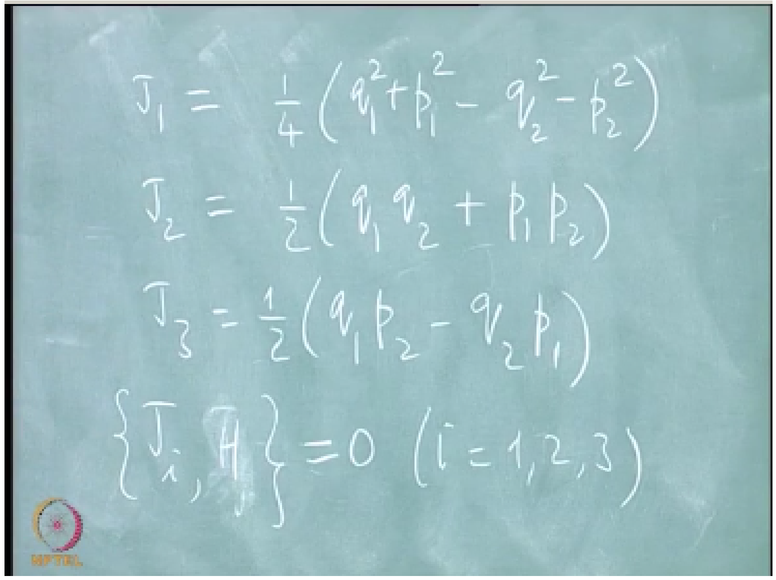

j 1 j 2 and j 3 which we have already written as q 1 p 2 - q 2 p 1 j 1 is 1/4 (q 1 ^{2}+ p 1^2- q ^{2} -P $^{2)}$ and this is 1/2 (q 1 q 2 + p 1 p 2) it turns out that each of these quantities is a constant of the motion so indeed it turns out that{ J i J k} sorry Ji with the Hamiltonian is 0 (i= 1 , 2 , 3) so we have other constants of the motion these two together with this third one the isolating integral actually specifies the trajectory completely the moment you put these three quantity is equal to constant.

Then in this four dimensional space you found the trajectory completely because the intersection the mutual intersection of surfaces on which this and this are constant specifies a curve and that is indeed the phase trajectory for any given initial set pardon me J one is a yes I am not saying they are independent I am going to come to the significance of the J's a little later but I am saying f1 f1 + f2 is the Hamiltonian itself of course for inerrability you just need two constants of the motion which are independent of each other and which are an involution they are represented for instance by f1 and f2.

But I am actually finding a whole lot of other constants of the motion for describing the motion you actually need three isolating integrals in this problem and they are represented for example by f1 f2 and f3 and j-3 but in addition these combinations are also constants of the motion sure this is just f1 - f2 apart from some constant factor this is something else altogether and this quantity here is the angular momentum about the origin but now you cannot have more than two constants of the motion in involution with each other that is for sure.

So in fact that is the maximum number that could be independent and in involution you can choose them as you please with various linear combinations I did like to choose this because this completely separates degree of freedom one from two altogether the others mix it up in some fashion or the other of significance of this j1 j2 j-3 is that they obey very interesting Poisson bracket relations and they obey the following relation among themselves.

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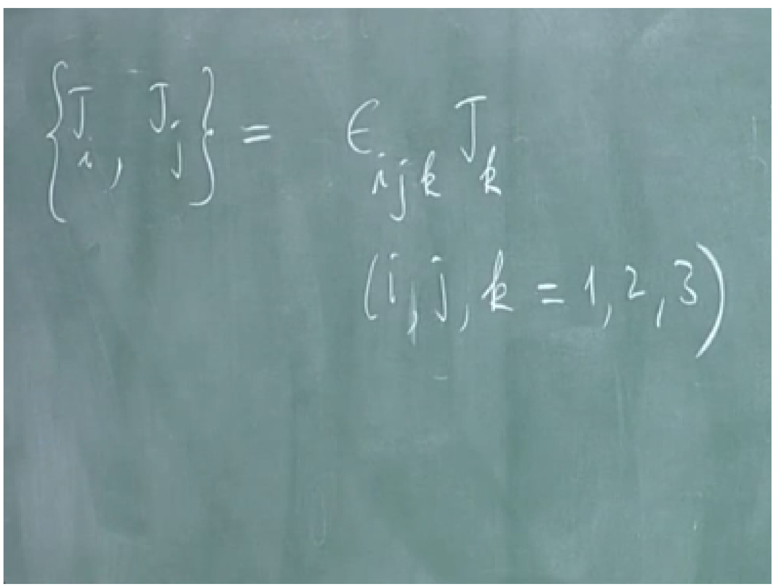

You have {J i Jj} = ε, ijk this is the totally anti-symmetric symbol in three dimensions it is =+1 if i jk on a natural permutation of even permutation of 1 2 3 - 1 if it is an odd permutation and 0 in all other cases this is the Levite root R symbol in 3 dimensions so this stands for a set of 3 relations J1 , J2, J3, J2, J3 is J1 and so on and cyclic permutation what does this remind you of this is exactly relations obeyed by angular momentum components if you write R cross P in 3 dimensions and ask what is the Poisson bracket of LX with ly or ly with lz or lz with lx you get exactly this set of relations so in some funny fashion angular momentum.

Algebra in 3 dimensions is related to the dynamical symmetry group of the 2 dimensional isotropic oscillator and the reason is group theoretical it is purely algebraic it turns out that this group is generated by three combinations of Q's and Ps which are precisely these three quantities where if you like the generators of this group SU 2 so you see what is happening is to summarize things here is a system which is completely integral it has two degrees of freedom.

So we have two constants of the motion in involution with each other which are functionally independent of each other and those are f1 and f2 since the motion is periodic rather than quasi periodic we need a third isolating integral another algebraic function of the Q's and Ps that is provided by this quantity J 3 for instance which has the significance now of being the angular momentum about the origin. So we have the energy of the first oscillator the energy of the second oscillator and then the angular momentum about the origin in addition there are these combinations J 1 and J 2 such that J 1 J 2 and J 3 are all constants of the motion

But they are not an involution with each other they cannot be too many of them instead they obey a certain algebra the Poisson bracket of any two is a linear combination of the same quantities in this case just the third one with the appropriate sign that algebra represents something much deeper this algebra the existence of this set of relations implies that there is a certain dynamical symmetry group in the problem which happens to be the intersection of the simplistic group SP for R of canonical transformations.

In this problem with the symmetry group of the Hamiltonian itself which happens to be a so forth and that group can be put into one-to-one correspondence the set of transformations with the group of 2/2 unitary matrices which is this group here and this group has three generators which happen to be the same as that of angular momentum in three dimensions and that's the reason why you have these combinations J 1 J 2 J 3 so as you can see dynamics and the algebraic structure underlying integral equations they are very closely linked with each other.

And this two dimensional oscillator gives us a simple model in which to understand the origin of these symmetries in this case of course if you go to three dimensions and write the oscillator down three dimensions that is a much bigger group the symmetry group is much bigger the canonical transformations would be SP 6,r and then you would have to look at the intersection of that or the subgroup of that which also leaves the Hamiltonian unchanged and that is a much more complicated group turns out that happens to be SU3 in that case.

And so on in fact the N dimensional isotropic oscillator has a symmetry group which is the same as a set of transformations the set of transformations it is a symmetry group is the same as the group of n/ n unitary matrices with determinant + 1 a very useful relationship in many applications but I do not want to get into that right now. I did like to go back and give you another problem.

Which also you are familiar with where there is an extra symmetry we will see where this comes from yes if the generators is three it is exactly the angular momentum algebra this thing here no it will not be for example the 3D isotropic oscillator the dynamical symmetry group is SU(n) SU (3) in fact the general statement is the n-dimensional isotropic oscillator in dimensional isotropic.

I should not call it n because this number n has been used for degrees of freedom okay it is the same number actually in dimensional is SU (n) no it does not obey this algebra it obeys a more complicated algebra but the question you could ask is how many of these are there that is something we can directly answer how many to admit how many generators do you think there are in n SU n well let us start from we could start from one by one matrices then we would go on to 2/ 2 / 3 etcetera. The least trivial case is the simplest non-trivial case is SU (2) so let us look at that we want to look at all.

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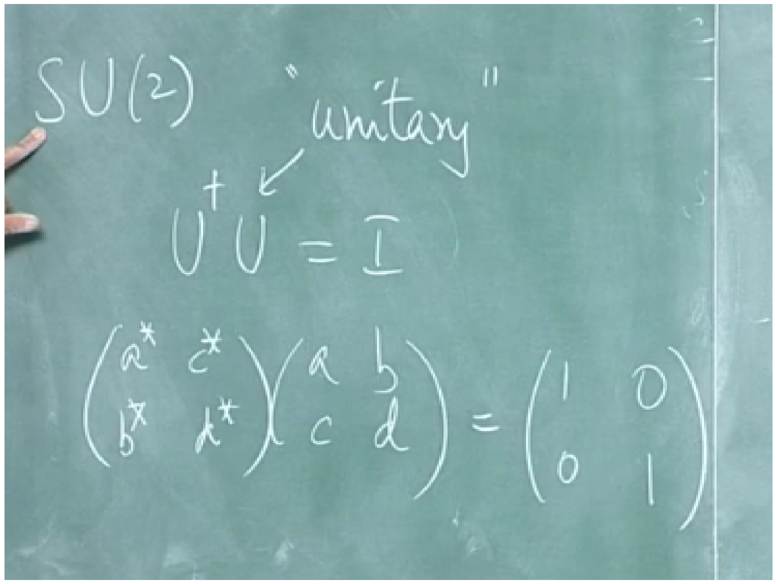

Unitary matrices 2/2 matrices which have unit determinants .I am going to look at the set of all these matrices and I permit complex entries ok we will come back to what a generator is a group since .I have to tell you how these groups are generated what I mean by it is let us come back let us come back and tell you what a generator is I do not want to have a digression within a digression so what is the number of parameters that you need to specify a 2 / 2 matrix it is not with the 2 / 2 matrix with possibly complex entries.

We count the number of real parameters always eight parameters because if the matrix is (a b c d) and a and b are all (a b c d) are complex numbers there is a real part and an imaginary part of these so in general eight real parameters are needed now. I start putting conditions on these matrices what is the unitary matrix a unitary matrix is 1 u $^{+}$u =I so this is unitary the dagger stands for the complex conjugate transpose the hermitian conjugate of this matrix

So I take the transpose and then do a complex conjugate of this matrix and I insist that this be true so what, I am insisting upon is that (a b c d) multiplied by the complex conjugate transpose so this becomes c *b * a * and b *B = 1 0 0 1 I insist upon this how many parameters are left now how many conditions does this give me gives you four conditions.

So how many conditions are left how many parameters are left four parameters are left. I now insist that the determinant of the matrix be = +1 that's what makes it s how many parameters are left 3 and therefore the number of generators is 3 in other words any element of this group is found by taking certain special matrices multiplying them by parameters and exponentiation these matrices and that gives me a finite element that is called a generator the group so there are three generators and that is exactly the number.

We found I will illustrate what is meant by generator in a minute but now what do you think it is for SU (n) what do you think it is going to be for n ^{2} n ^{2} - 1 therefore in 3 dimensions its 9 - 1 which is 8 generators and therefore there are 8 constants of the motion which obey a certain algebra among themselves more complicated than this considerably more complicated than this but that provides you with a symmetric group now let us come to the question of what I mean by a generator and let us go back and do a little bit of elementary algebra here.

So I start with the simplest example of a rotation in two dimensions in a plane and let us write the equations down directly.

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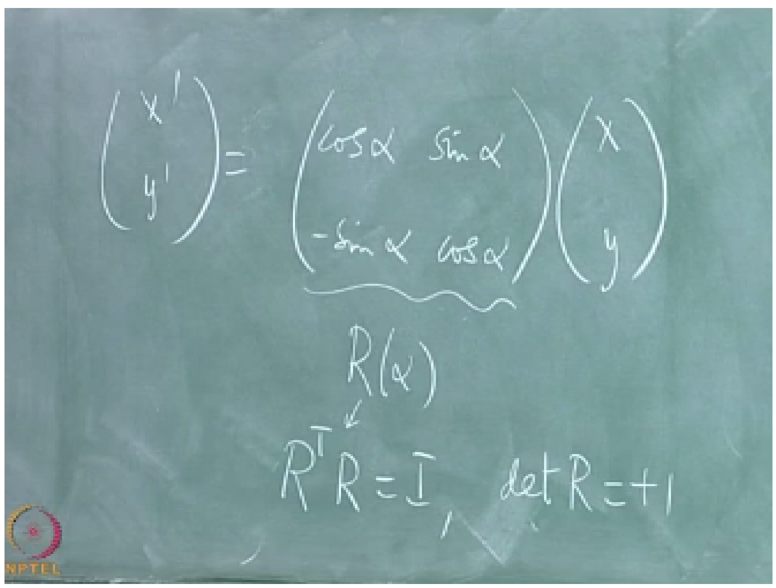

So I start with the XY plane and I make a rotation about an angle α right new coordinates x' and y ' and of course I have x' y ' = ( ) acting on (x y) this matrix here depends on the parameter α and it tells me x' is a certain linear combination of x and y and y ' is a certain linear combination of X and Y and it is well known of course that we have its let us call this matrix which represents rotation by an angle α let us call it R (α) now what are the properties of R (α) what does a rotation do it keeps the origin

Unchanged it is linear it is homogeneous in other words 0 remains 0 the origin remains unchanged and distances do not change nor does this coordinate system become a left-handed coordinate system what is right-handed remains right-handed so this means that this matrix R satisfies $R^T R = \pi$ it is orthogonal which keeps distances unchanged and the determinant of R =+ 1 therefore this R of α is an element of a group of matrices which are orthogonal two by two matrices with real entries in this case and with determinant + how many parameters are needed to specify a rotation one.

So this is a one parameter group and there is one generator in three dimensions you have three Euler angles in general therefore you have three generators for so 3 in n dimensions Euclidean space how many generators do you need for specifying rotations why in our there are n axis or n angles and this is precisely the point where I want to stop and we will take this up that answer is not right because that tells you something about the nature of rotations itself.

Let me redefine a rotation a rotation is a linear homogeneous transformation which keeps the origin unchanged for instance which is orthogonal distances are not changed and the determinant is +1 so right-handed system remains a right-handed system the reason you say n is because you assume that every rotation is a transformation about some axis and there are in axis but this is not true because if I think of 4 dimensions then of course I could have a rotation which changes in the xy plane but leaves both the other two coordinates unchanged or if I look at two dimensions there is no third axis it is about a point.

So an axis need not be identified with the rotation that is an accident of three dimensions it is an accident of odd dimensionality we will come to that so the number of generators is the number of independent orthogonal planes you can find how many planes can you find in n dimensional space orthogonal planes like the xy, yz etcetera n c2 which is n times n - 1 / 2 and that is the number of generators.

So I will explain this is a good example to explain what is meant by generators for some elementary group theory and then we will take it from this point so let me write that down SO(n) we will explain what is meant by generator next time using this example okay.

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