Indian Institute of Technology Madras
Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 8
Hamiltonian dynamics (Part IV)

Prof. V. Balakrishnan

Department of Physics
Indian Institute of Technology Madras

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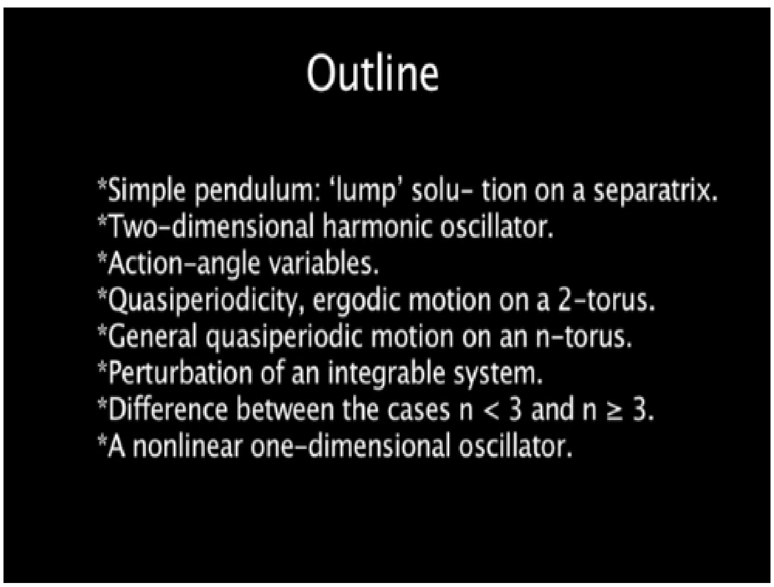

We assume with our discussion of the simple pendulum problem.

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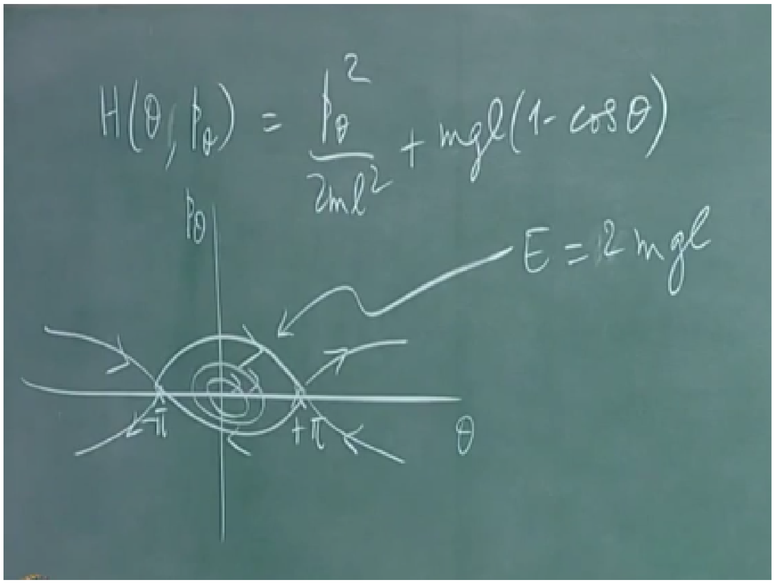

In which recall the Hamiltonian H(θ, Pθ) was $P\theta^2$ / $2ml^2$+ the potential energy which is mgl (1 – cosθ) we were specifically interested in finding out what the nature of the solution was as a function of time on the separaterix which took for instance from the saddle point at - π to the saddle point at +π in.

This fashion and back in this fashion and of course you add small oscillations larger amplitude oscillations in ψ and this picture got repeated as you went along we were interested in outing what the explicit solution was on this trajectory here on that particular trajectory the energy had it is value 2mgl which is the maximum value of the potential energy and we also saw that.

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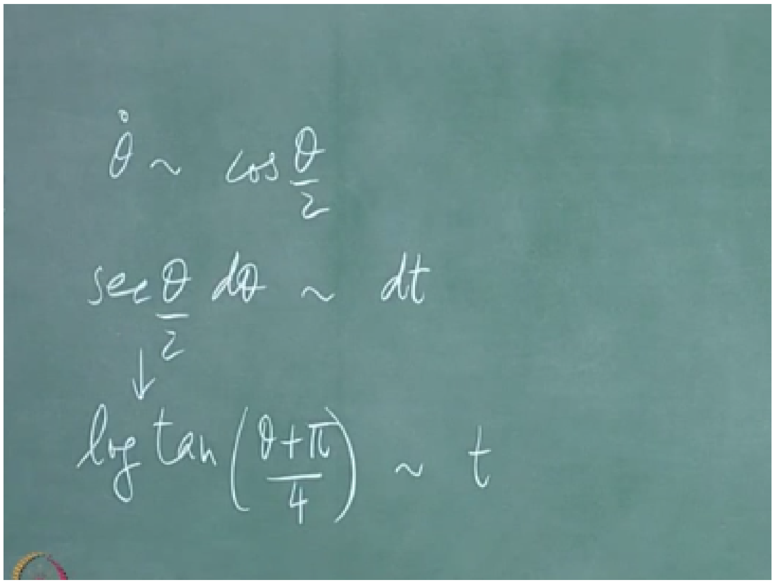

Θ dot is propositional to cos θ / 2 on that trajectory we saw this last time now you all need to do is to integrate this so this gives sec θ/2 d θ is propositional to dt and what is the integral of this leads to something log, tan θ + π/ 4 and that is propositional to t on this side part from some constants therefore θ can be written down explicitly in terms of tan inverse of the exponential of sometime. If you sketch that solution as a function of t then you discover that on that solution θ (0).

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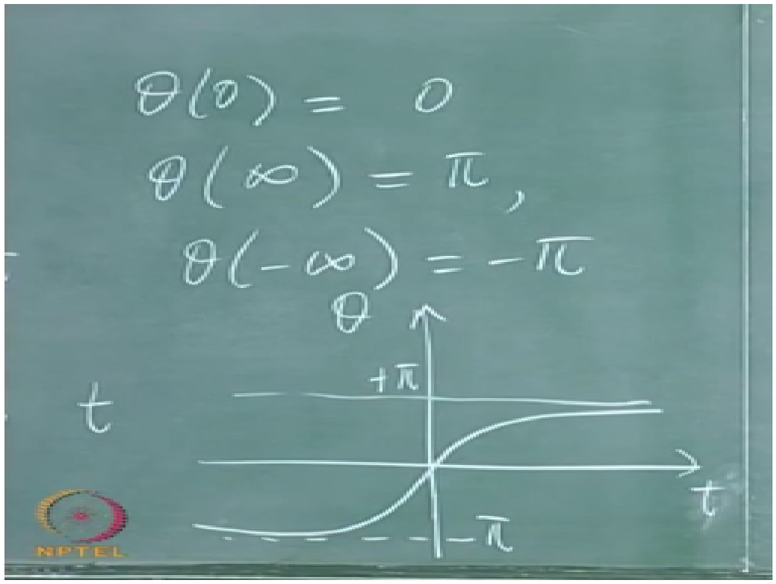

We took this to be equal to 0 $\theta(\infty)$ was = π θ - ∞ = - π starts at this point and ends at that point and we assume that it passes through $\theta = 0$ at $t = 0$ and the solution looks like θt here verses θ started at $-\pi$ passes through 0 and hit + π – so it behave like a kink as if it had a kink and if you compute the angular velocity corresponding to the same solution.

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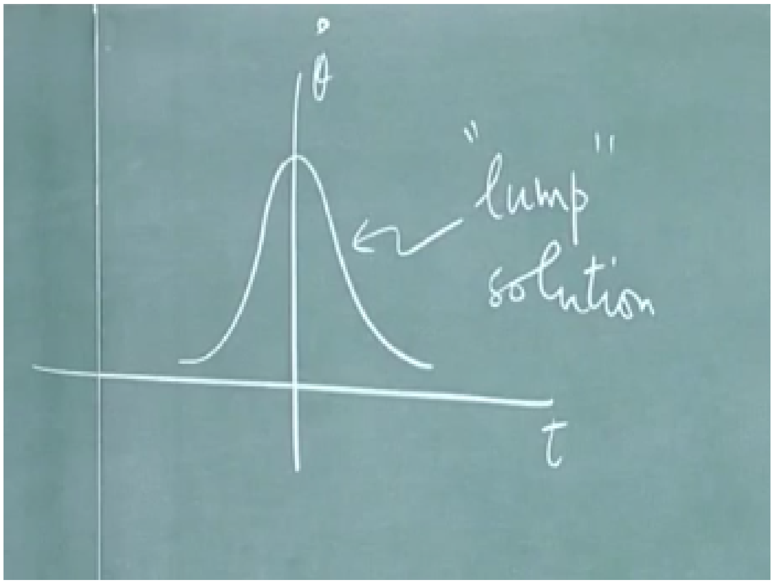

Here is t here is θ dot the angular velocity this quantity is by enlarge 0 except it takes a sharp p cup here and then back to saturation value and therefore looks a little bit like that this fashion and this is like a lump this quantity this solution is related to what is called the saltine solution of the pendulum equation of the sine Gordon equation and this kind of lump solution something which is essentially coherent nothing happens and then suddenly there is a little blip up there and then back down there to coherent this kind of thing is a special kind of saltine it is called an instantaneous.

Since it happens in time but we are not going to get into this business right now except dimension that phenomena like saltines are very characteristic of nonlinear equations and the sine Gordon equation as I mention is a prototypical non linear equation which as a very large number of applications and this is a very well known and famous solution of that equation the point I want to make here is that in as simple a problem as a simple pendulum problem it already makes it is presence.

Oh good now let us get back to our primary discussion which was to understand what kind of phase scape structure does an intangible Hamiltonians system have and for this we go back all the way to a very simple 2^0 of freedom problem so far we have discussed a few 1^0 of freedom problems now let us do something in 2^0 of freedom and we began to see how things can get a little more complicated.

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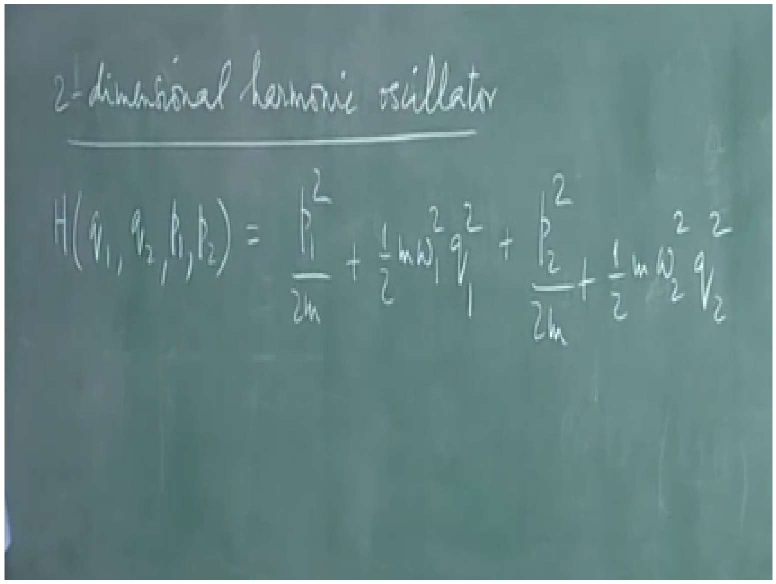

And the model I have in mind is the 2 dimensional harmonic oscillators you could imagine this to be to oscillators simple harmonic oscillators at right angles to each other which you are familiar with from elementary physics I write down now the Hamiltonian and since we have decided to call q and p the coordinates and movement a so let me call this q1, q2, p1 and p2 this = $p1^2$/ 2m.

That is the kinetic energy of the first oscillator + ½ m $\omega 1^2$ $q1^2$ assuming the oscillator to have a mass m and a frequency ω1 + an identical expression over 2m+ 1 ½ m $\omega 2^2 q2^2$ for the second oscillator.

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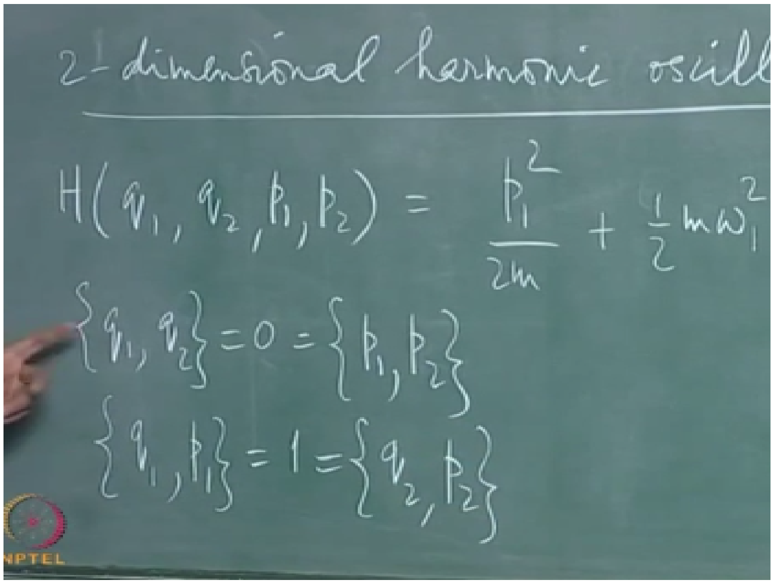

This then is my Hamiltonian and I have the characteristic the typical Poisson brackets relations namely q1 with q2 is 0 p1 with p2 q1 with p1 is 1 as is q2 with p2 so the one variables q1 and p1 have nothing to do with the 2 variables they are completely independent dynamical variables it is 2 uncoupled simple harmonic oscillators with ion general different frequencies added together to form a 2^0 of freedom system.

In 4 dimensional phase space the phase space is the space of the 4 variables q1, q2, p1, p2 each of it regions from - ∞ to ∞ s o that is our set of the Hamiltonian of the system what is the set of equations well it is quite evident that they just two decoupled oscillators.

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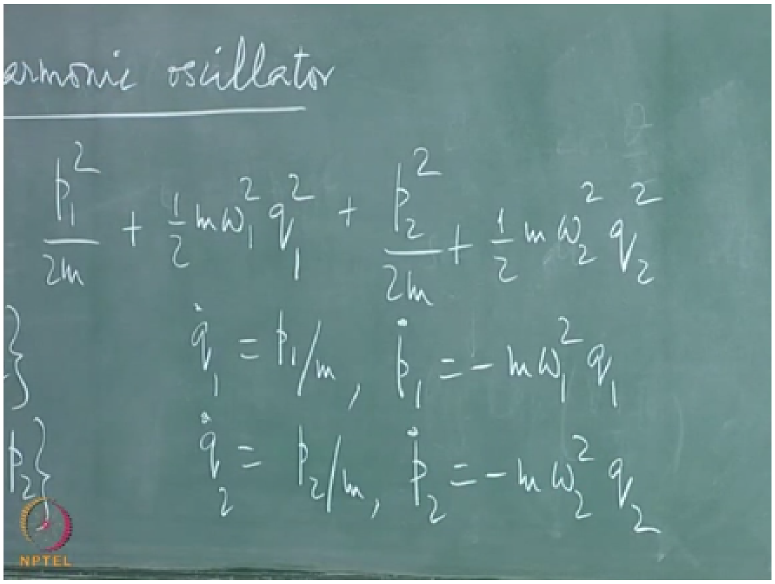

So we can write down the equations instantaneously this is p1 / m and p1 dot – m $\omega 1^2$ q1 and similarly for 2 with the subscript 2 replacing 1 so q2 dot is p2 / m, p2 dot is – m q2 then of course each of these oscillators if you imagine them to be at right angles to each other oscillates may be in a plane the q1, q2 plane completely orthogonal to each other so the problem as it stands is completely intangible it is solvable.

Now let us apply the general results that we know to this case and ask whether we can say something about the nature of the phase space and what kind of motion we really have we need for it is integrality since in thus problem n = 2 the number of degrees of freedom we need to functionally independent analytic constants of the motion and of course it is not very hard to see that this combination here which represents the energy of the first oscillator is a constant of the motion.

As is this combination here and since the 1's and 2's are nothing to do with each other this and that are in involution with each other so we could in fact cal this f1 and this f2 and f1, f2 the Poisson bracket vanishes and each of these is nice analytic function of the arguments and therefore by the livable analytical theorem this system is intangible completely we can write it solutions down explicitly.

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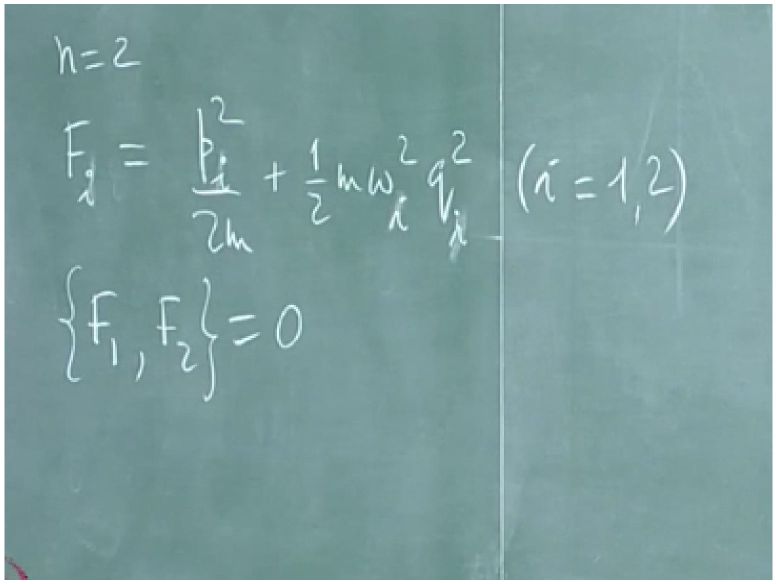

So let us formula is this we have n = 2 and we could take f1 to be = $p1^2$ / 2m + 1 ½ m $\omega 1^2 q1^2$ in fact we could write Fi this Pi i i and i = 1 or 2 we guaranteed that F1, F2 is 0 hence the system is intertribal completely now we ask what are the constants what other constants of the motion could the system have it is evident that the phase trajectories cannot be drawn by means it requires 4 dimensional space but we could draw the trajectories as projections of the trajectories into various sub spaces.

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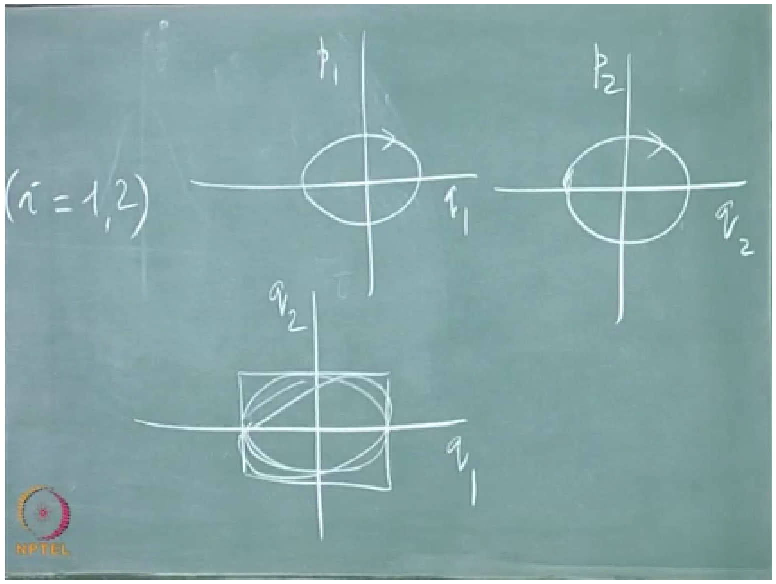

So for instance if I try to draw the projection of the phase trajectory in the q1, p1 plane for example since it as nothing to do with this oscillator it would just correspond to F1 = constant whcuih are ellipse as we know therefore you get phase trajectories of this kind and similarly in the q2 p2 space you get ellipse of this kind the periodicity here is 2π/ ω1 and 2π/ω2 there what would happen if we try to project a trajectory on to the q, q2 plane.

What would you get in general well it is evident that q1 for given initial conditions it is clear that q1 stays within this range and q2 stays within that range so the motion is restricted to some kind of rectangle this being the amplitude in the first oscillator and this being the amplitude of the oscillator but is the representative point is the projection of this trajectory is going to be closed curve or is it going to be open curve.

It is certainly bounded it is not going to go out of this region but it is going to be closed are is it going to be open it depends absolutely right it depends on the ration of ω1 to ω2 if the ratio is rational this means that the 2 periods are commence rate with each other one of them is a rational multiple of the other then it is clear the overall motion is periodic and the system given enough time will come back to it is original point.

Yes in general you get Liza Juice figures now the question is the Liza Juice figure going to close on itself or is it going to be complete open your familiar with from elementary physics with for instance depending on what the initial phase differences between the oscillators you would perhaps get pictures like this you could get something like this or you would get something like this or you would get a pattern which fills up this space and never comes back to itself depending what on what the frequency ratio is.

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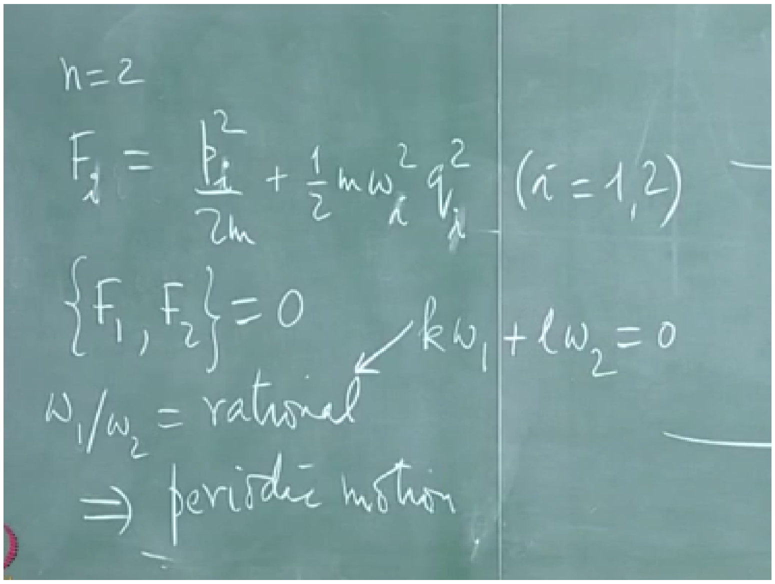

So it is clear we know this from elementary physics that if ω1/ ω2 is rational ⇒periodic motion in other words if we have 2 integers say l and m assume what down want to use m because I have already used that for the mass say k and l such that k times ω1 + l times ω2 = 0 it ⇒ that the ratio is rational where k and l are integers and then it is quite clear that the overall motion is periodic.

But if no such relation exits with k and l being integers when this ratio is irrational in general and the motion is not periodic although it continuous to periodic for the first oscillator and for this subsystems separately we are now interested in whether the motion is periodic for the entire system are not in other words in the 4 dimensional phase space thus the point represectives point come back to it is original point position.

And the answer is no not if the frequency ratio is in commensurate if it is irrational the motion is not periodic such motion where the actual motion is made up of 2 or more mutually incommensurate time periods is called cosi periodic and the motion in general is cosi periodic in this case so let me write this down .

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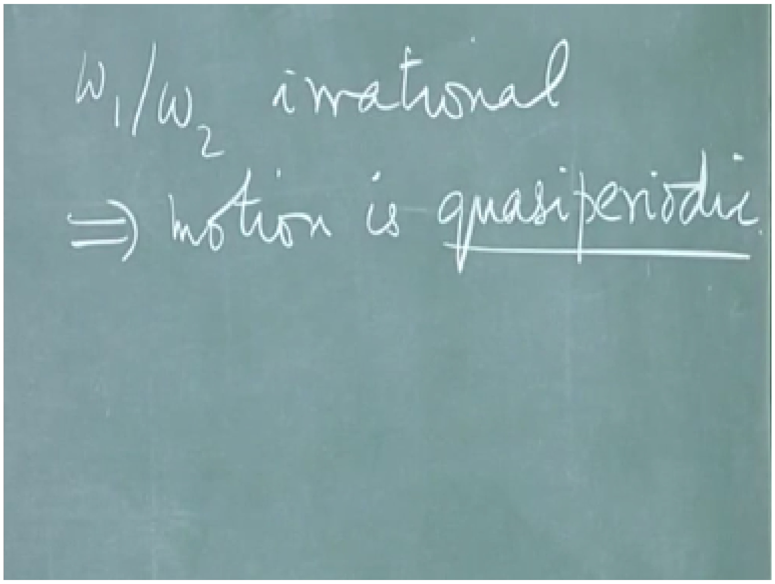

What can we say about the motion in this case well here is where the general theorem I talked about helps us the motion as to occur on a surface on which H is constant since the total energy is constant so Hamiltonian autonomous system it also as to appear on a surface on which F1 is constant and on a surface on which F2 is constant simultaneously and of course the Hamiltonian in this trivial case is just the sum of f1 and f2.

It is not functionally independent of f1 and f2 we have just two functionally independent constant of the motion which are isolating integrals namely f1 and f2. If you now go to action angle variables sand there is a standard method for finding the action variable in this case it turns out that this Hamiltonian which we have written on here could be written in terms of action angle variables as follows.

This Hamiltonian becomes I1 ω1 + I2 ω2, where the two action variables I1 for example is an integral over p1 dq1 over a complete cycle if the oscillator starts at the point say a and moves to b in the first oscillator when it say integral twice the integral from a to b of pdq the corresponding pdq. So it is an integral over the orbit that I mention in the projection in the q1 q2 plane.

And similarly for i2 and if you use those as your variables and they can be shown canonical variables when the Hamiltonian, this is what I call k of I as from is does not depend on the angle variables but only on the action variables and it is linear in the two action variables. You can check that the dimensionality is right because action of the dimension of energy multiplied by time and the frequency is inverse time so the product of the toll has the dimensionality of energy.

No angle variables are present here and indeed you can see that in this simple case δ k / δ I sub I is in fact ω I and the frequency is are independent of the amplitude of the motion because the Hamiltonian expressed in the action variables is linear in the action variables, therefore when you differentiated there is no further dependents on the energy. We could therefore now go to a representation of the phase space in terms of the action angle variables themselves.

And then what is it look like? When I know I1 is constant and I know i2 is constant and this is the combination i1 this is like an ellipse in the q1 p1 plane, and then ellipse apart from units is topologically equivalent to a circle or some kind. And similarly in the q2 p2 plane this is an ellipse and it is this combination that remains constant that is again like an ellipse which is again topologically equivalent to a circle. So I could in fact now identify what the angle variables are, going back to p1 q1.

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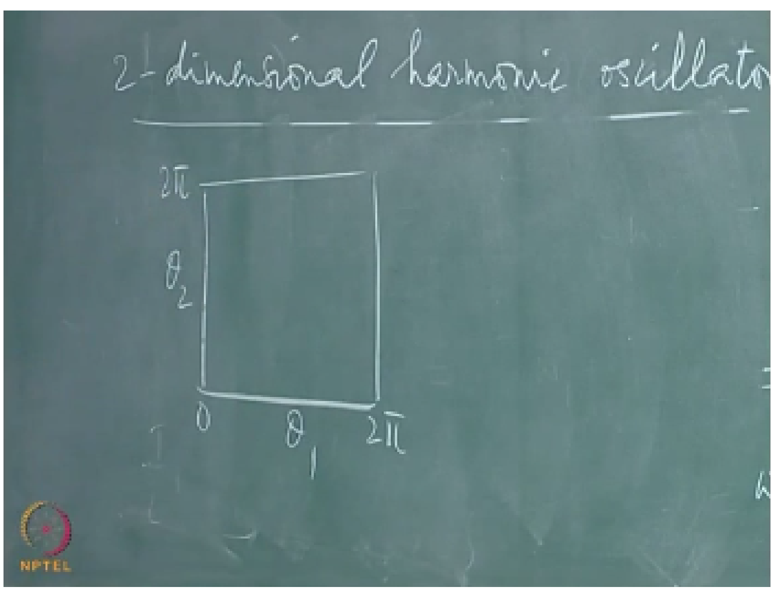

Once I give you f1 in other words once I give you this ellipse when the state of the system of the first oscillator is specified completely by specifying for instance what this angle is as the oscillator moves around. And similarly for the second oscillator, therefore the state of the full system once you give me f1 and f2 which depends on the initial conditions is specified completely by specifying two angle variables which in this problem can actually be identified with the phase of the oscillator and the q1 p1 and q2 p2 planes.

What kind of space is span by two angles each of which runs from 0 to 2 π two independent angles it is a two dimensional space what would that space b, well a rectangle but with the ends identify because 2π should be identify with 0 so it is completely right it is like taking square of length to outside 2 π.

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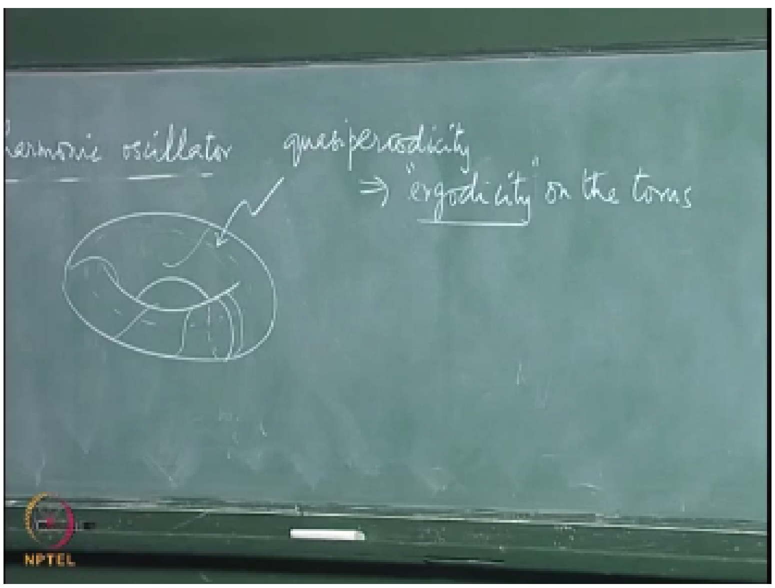

This is the square of sight 2 π so here is 0 to 2 π in the first angle variable θ 1 and 0 to 2 π in the second angle θ 2 but I must identify that point with this and this point with this I must further identified this point with that and this point with this, so I must take this piece of paper and role it around and stick this glue this on to that and glue that on to this what happens if I just glue these two together preserving the orientation I get a cylinder I take the cylinder and I preserve the orientation of the two ends and I glue the mode and I get a two tours.

So this is how the torus structure appears in the action angle space so this phase space is nothing but the surface of a two tours in which I could measure θ1 as the angle in this direction and θ 2 as the angle in transverse direction the cross section of this Torus. And any point on the surface on this two torus represents a state of this oscillator for prescribed initial conditions, in other words the action variables the size of the action variables is prescribed once in for all.

Periodic motion would correspond to the following situation as the represented point reverses in this direction the complete loop it should do so and integral number of time in the other direction that would be 1:2 or 1:3 or even 1:1 for instance more generally k times this period plus l times the other period that should be come and so let with each other k times 1 period should be l times the other period and other words the frequency ratios should be ration.

In that case this represented a point which wonders on the surface of this torus returns to its original point, and the motion is periodic. If the motion is 1:2 or 1 : 3 this simply means we takes if it goes around here once maybe it goes around the other direction twice or thrice it hard from me to draw this and come back to the original point. The trajectory therefore would not fill this torus up it will just will be a curve on this torus folding around of few times like this and then it does this and then it does this etc and eventually comes back to the original point.

On the other hand the frequency ratio is irrational when we are guaranteed that the representative point never comes back to its original value prissily but does so arbitrarily close to the original value and infinite number of times. The motion becomes quasi periodic one way to see this would be to ask let us imagine this frequency to be unity and then we ask how often does this represented a point come and hit a cross section in this direction.

So I take a cross section of this torus and I ask here is the circle which represent the cross section I start at some point here so I am here main while it is going around the other direction when it is comes back it comes back somewhere else, and hits this cross section at some other point, it comes back a second time it hits somewhere else and so on. The question is does it come back to the original value ever or does it keep filing this circle completely.

It again depends on the frequency ratio and we can prove the following theorem quite regress given a circle of unit circumference if I add that each time step I add the rational number you are guaranteed that wherever you start that point we will be written to in a finite amount of time after a finite number of iterations. Every point on this circle whatever be the initial condition is going to be revisited over a lower again all points are periodic points and the motion is periodic.

On the other hand if the number you add is irrational modular one because you are on a circle then one can show regress that no starting point is going to be revisited, there are no periodic orbits at all. In fact you can go further you can show that the iterates of any starting point are going to eventually densely fill up this entire circle and uniformly so. So it is not only important that it fills it up densely but uniformly there is bios there is no particular part of a phase space which is preferred over other parts of phase space on this Torus.

So the iterates the trajectory any trajectory which starts here will wind around this tightly till eventually it densely fills up it is like a space fill in curve it densely fills up the torus but never comes back and closes on itself. Comes arbitrarily close to its initial point many times in infinite number of times but never recurs and the motion is quasi periodic. So this is characteristic of quasi periodic motion that the system densely fills up in certain sub space of the phase space and it set to be argotic on this Torus.

So quasi periodicity implies we use a technical term argotic on the torus, I repeat by argotic I mean that any initial set of any set of initial conditions I take a little patch of initial conditions here as time goes on each of the points in this patch evolves in time wonders around this torus this little patch visits every neighborhood on this torus and infinite number of times comes arbitrarily close to every starting point and eventually fills up this entire torus uniformly and densely with no exceptions.

It set to be argotic on the torus what that implies is if you want to compute long time averages of any physical quantity you can replace the long time average by a statistical average with the uniform measure on this torus a uniform probability distribution on this torus and that is the implication of what this word argot city means.

And fact you know that in statistical mechanics you normally replace time averages by ensemble averages by statistical averages and the distribution function has to be prescribe to you by the conditions under the which the system is kept and that implies the certain kind of argot city that assumes the time averages can be replace by statistical average.

This exactly what happens here in this dynamical system which displace quasi periodicity here we can go further our phase space is four dimensional any motion for given initial conditions is restricted to a three dimensional sub space of this four dimensional space, namely the energy surface. Now in a three dimensional space the motion is further restricted in each case to a two dimensional surface namely some kind of torus.

And it is evident if I try to draw this fins perhaps there is a torus here and if I draw cross section of it is like this a different initial condition would correspond to a different torus of this kind be like that for instance schematically. So the whole space the energy surface is simply filled up by the successive nested torrid. And moreover since phase trajectory is cannot cross themselves it is evident that if you start on one torus you never gone to escape you going to be on that torus.

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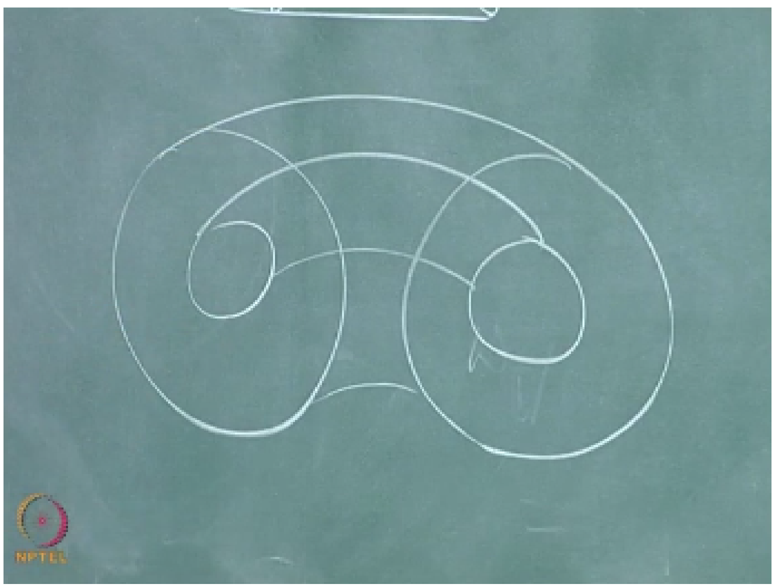

If you start on the different torus you remain on that torus the gaps between these tarried are all also filled up completely by other torrid and that is the structure of the phase space for an integral system like a two dimensional harmonic oscillator. But the matter is more general than that ever more generally than this if you have a two degree of freedom system which is completely integral when the motion in general is restricted always to some dimensional torus.

What would be the major difference between an arbitrary two degree of freedom enterable system with bounded motion and harmonic oscillator what special about the harmonic oscillator, what is the feature that is absolutely special about the harmonic oscillator you cannot jump between tarried but that is true even for a non linear system even in general it should be true. And that is simply a consequent of the fact that once your torus that represents the constant of the motion and you are on that surface.

What is very special about the harmonic oscillator, fact that the cross section is the circle between we talk about topological aspects here so the fact that the cross section is the circle and set up in ellipse and so on, and this makes no difference that is simply choice of units. What is particularly special about the harmonic oscillator? The equations of motion are linear that certainly true it is not a non linear system but what is very special about it.

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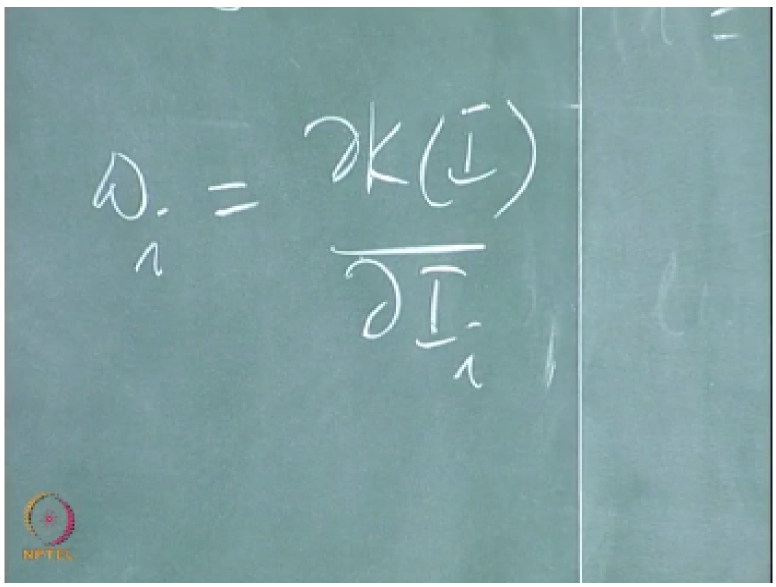

Go back and recall that ω I was in general equal to δ k of I over δ Ii what happen in the case of harmonic oscillator it was constant the frequency is your constant independent of the initial conditions independent of the energy of the oscillator independent of the amplitude of oscillation this feature is not true in general In general this is some function this is = some function of all the action variables and that would mean that the frequencies on different torrid would be different in general. It also immediately implies that if you have a non linear system of that kind on certain.

Depending on what the values of action variables because that is what determines the frequencies, frequency ratio could become commence rate but you move to another Torres it could be in commence rate in general. So the motion could be actually periodic on some torrid and quasi periodic on another torrid, generically of course it will be quasi periodic. But there could be accidental cases where you have resonances and the motion reduces to the periodic motion on some torrid.

What happens further is even more interesting in general 2^0 freedom system if the system starts out as integral and you add to Hamiltonian in a small species which takes you away from inerrability and loose the property then what happens is more interesting in general the set of torrid will no longer exist because it is no longer an integral system but in general it does not happen globally.

In another word first for certain sets of initial conditions for certain delicately balanced frequency ratios the torrid get destroyed, the rational torrid get destroyed first and gaps form between torrid and in those gaps the motion will actually becomes irregular it is no longer periodic or quasi periodic becomes chaotic and this happens when you have a not a integral system but to an integral system you add a small perturbation, which destroys the property of integral.

And then typically you have chaotic behavior in the gaps between these torrid as you jack up the strength of this perturbation and make it go further and further away from a system in other words certain symmetries are destroyed more and more, then successive torrid go on getting destroyed and still certain torrid persist on which the frequency ratio cote and cote extremely irrational.

So now we are getting into the number theoretic considerations and we have to discuss we have to decide what is meant by a number being more irrational than the other number we will come to that in a second and eventually all the torrid get destroyed when the motions become completely chaotic and we will talk about chaotic motion in much greater detail later. But this is the mechanism by which this is the rough idea of what happens in systems.

In the same breath let me mention that if you took n degree of freedom system for example n harmonic oscillators independent of each other what would the motion be on. Well the individual energy of these oscillators is n constant of motion in evolution with each other. So the system is integral in the same sense, what would then decide where the represented point is just the angle variables.

The individual phase angle of all the oscillators in the corresponding Qp spaces what would you represented that kind of phase space by? By an n Torres the generalization of a true torres to n dimensions in which n angles would specify the state of the system and then you could ask is the motion quasi periodic on n Torres or it is periodic. In general of course the answer it would quasi periodic.

Unless you had n – 1 relations of the form k1 ω1 + l1 + ω2 + etc something times ωn = 0 and then you had n – 1 such relations between the frequencies, if you get than all the frequency ratio are with each other and you have periodic motion but if even one of them is lost we do not have that relation then you have some degree of quasi period and if you have no such relation at all with the integers value of these parameters here then the motion is completely quasi periodic on a n Torres.

I cannot draw a picture of a n Torres what we have done here is to draw a section of 2 torus that is the best I can do on the black board but you can easily conceive of the fact that you have motion on n torus if the system is integral, the phase space dimensionality is 2n. The energy surface dimensionality is 2n – 1 and the torus is a n dimensional object.

Now in 3 dimensional spaces it is clear that 2 dimensional objects such as the rubber tubes they try it the entire space you can put one inside the other and fill up the entire space, if you had instead of 3 dimensions if you had 4 dimension space and you took the 2 dimensional objects here it clearly do not fill up the space, there is no way in which the entire space is laminated by them. So it is evident immediately that if you took 3 or 4 degrees of freedom.

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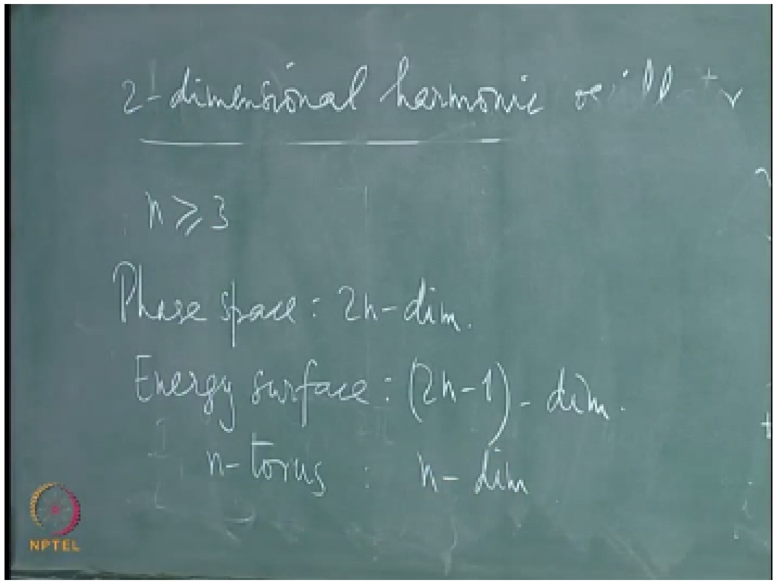

n≥3 the phase space is 2n dimension the energy surface is 12 – 1 dimension if the system is integral the motion occurs on n torus in a certain variables and this is the n dimension. If n is 3 then this is the 3 dimensional object this is the 5 dimensional object. The difference in dimensionality is 2 not 1 so it is clear 2 3 dimensional objects cannot separate this 5 dimensional space to put it another way in 3 dimensional space if I have 2 dimensional surface of kind this torus there is a inside and outside.

It splits up the whole space into a inside and outside that is not possible if the dimensionality between the object that you are considering and the dimensionality of the space is > 1. Is this clear if I take 3 dimensional space and draw I infinite plane in it that is the 2 dimensional object there is up and down, it splits the space into the two parts but if I took a straight line which is 1 dimensional object it does not split the 3 dimensional space into 3 parts.

And exactly the same way the 2n – 1 dimensional energy surface cannot be split up into disjoint parts with the n dimensional objects if n ≥ 3.w hen n is 2 it happens because this is 2 and that is 3 and this is exactly the situation we looked at there. The implication is deep the implication is these gaps here would all be connected to each other, there is no way you can have a inside and outside, all such gaps you must imagine would be collected to each other.

There is enough room to go from one gap to another gap and I already mentioned that once you have the perturbation which destroys then some torrid get destroyed and part of the motion for some initial conditions becomes chaotic and if the motion is chaotic in this region and this region is connected to all other chaotic regions this means that the chaos can spread but all random motion all chaotic motion with 3 or more degrees of freedom all such regions of chaotic motion are general connected to each other.

And a kind of diffusion will occur over long time periods from one region to another and this has performed implications in a dynamically systems especially in application such as accelerated physics this immediately makes a huge difference because in some sense it says that regions where the dynamic is unstable exponentially so that as actually spread out all over the phase space not confined to certain regions of it.

We will try to come back when we talk about chaos we will come back with some implications but the reason that is have find out that a very simple argument based on dimensionality can actually leads to these conclusions fairly general conclusion in this case. So having talked little bit quasi periodicity, let us now take up a specific example we will go back little and take up a 1 degree of freedom system where I show you explicitly how the non linearity plays a role and how time period could actually depend.

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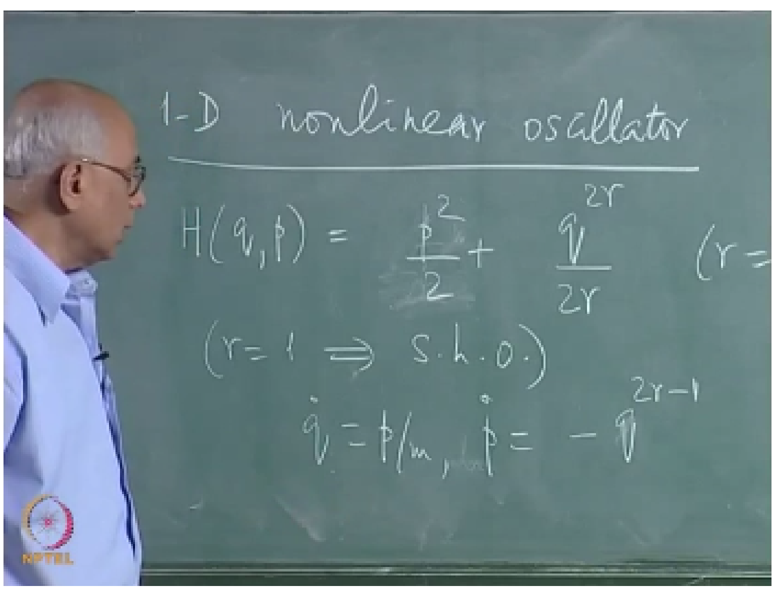

So let us look once again at 1 dimensional 1D non linear oscillator I have a single degree of freedom let us call it q and momentum p and the Hamiltonian and H (q, p) and = 1 ½ and let us write this as p^2/ 2m + a potential energy let us fix the frequency of the oscillator the natural frequency and the constant = 1so in suitable dimension these thing in here becomes q not 2 would be harmonic oscillator but in general have ^{2r} = 1.

So let us start with let say r is the positive integer, so r = 1 → simple harmonic oscillator, the linear harmonic oscillator, well we could write the equations of motion down immediately but would they be as before q. p/m and p. - Δ h/Δq = - 2r cancels. If r > 1 it is clear this is a non linear system and the badly non linear 1 because of r is 2 for example this is a cube term and you cannot linearize about the origin. Where are the critical points? The origin is the only critical points.

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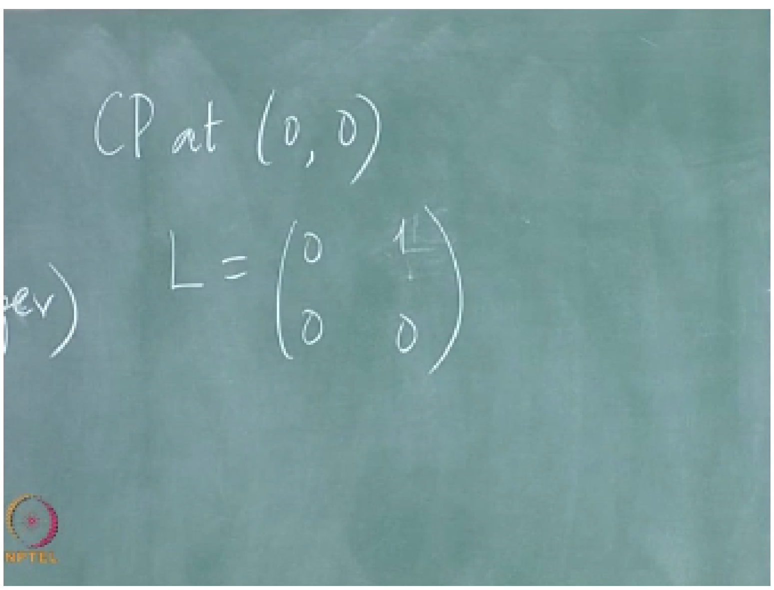

So CP at 0, 0 what is the linear matrix say if you linearize this what would this be, this becomes 0 and then what would happen here you will get a matrix a linear matrix by linearizing I will get L = 0 as I have chosen units such that the mass is 1 so this is 0 and this is 1 and out here I get 0 and 0 because this is already a higher order term for r > 1.

What would you say the Eigen values of that? 0 and 0 any triangular matrix the Eigen values set on the diagonals, so this is infact a degenerate case. We are not able to decide if the origin is center or it is some other kind of critical point based on linearization, simply because this system is degenerated. But let us take recourse to physics and ask what kind of system we have and then I will point out a method by which we can decide that the origin is a fact a stable center.

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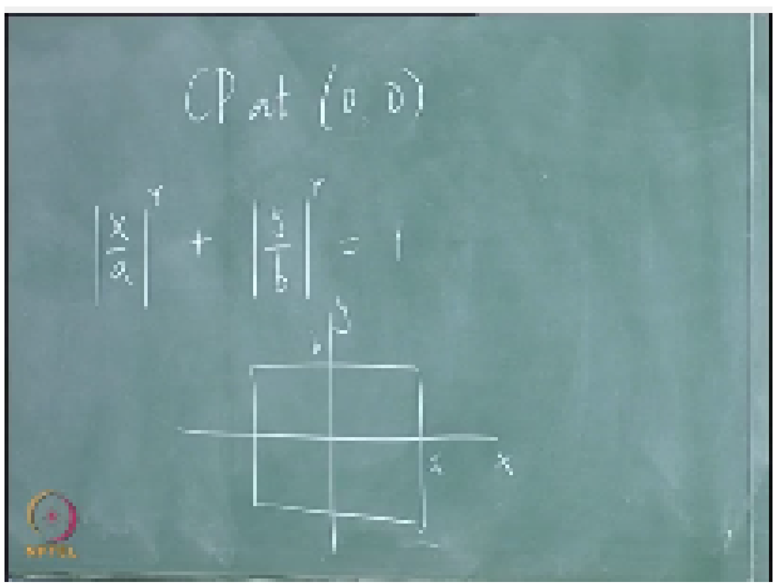

By simply drawing the phase p versus q yeah yes I am taking a special case by the question is why I am restricting myself r = positive integer I am going to relax this condition subsequently, so let us first take r = positive integers see what happens and then we relax this condition see what happens okay.

What kind of phase do you get for this system? The way I have written it, it is clear that whatever phase the close curve for all positive e because these are all positive terms each of these it is completely symmetric about the q axis as well as the p axis. The motion is evidently restricted to some kind of rectangle, so we are going to get some kind of oval.

This being the amplitude of the motion r is 1 and of course it becomes 1^{th} and when r = 2 then you get p^2 + q 4 = constant and that is sort of squashed oval, by the way what do you think will happen if r becomes larger and larger what do you think will happen this curve? We are plotting a curve you are plotting a graph of this quantity = some constant e which is positive and what happens to this graph as r becomes larger and larger.

In which direction it is going to get squashed what is going to happen? It is evident that this amplitude restricts you're the total energy restricts this amplitude, this amplitude infact is given by if this is a and then it is quite clear that a=2r is in fact the energy because the kinetic energy is 0 suppose what kind of graph would you get in r becomes larger and larger I fix the energy then I just change r what happens would this ellipse becomes factor like that or it would become thinner what will you do clear no I fix the amplitude I fix this fix their energy.

And then ask what is this graph look like I fix the amplitude and ask what is this look like I am talking about let me be specific I fix the amplitude and then ask I change this r I increase it I do the same problem for different r as r increases what happens to this graph to this you start doing this as r increases eventually if r tends to ∞ it will be compared at that time suppose nr will take what he said is seriously.

Now suppose r becomes smaller and smaller if r =1 what would happen then off course you could say this term changes sign if r becomes less than one for example this term changes keep everything positive so let us take modules what happens if r =1m it is just an ellipse what happens if power =1/2 what is parabola so let us go back let us go a little bit of elementary calculus.

I ask the question x over a^r +yb^r=1 and I want to plot this on xy plane what is this graph look like a and b are positive numbers since these are positive numbers it is quite clear that the motion is this some kind of rectangle of this kind is size a and this is b when r is 2 this is an ellipse so let me draw that what if r=1 what kind of slide this is the module angle absolutely so this will be some curve like this what if r=4 it would be at fact and fact.

It will start doing this and finally this rectangle is limiting case when a r tends to ∞ what happens if r becomes less than 1 and absolute in the curves start getting concave it starts doing this so this would correspond to r<1 what happens if r tends to 0 either if x =a and y is equal to b and rest are the other variables must be 0 so it would be the axis this sense so it is really whole family of crops the ellsip0se is just one such member of this family so coming back what we were doing.

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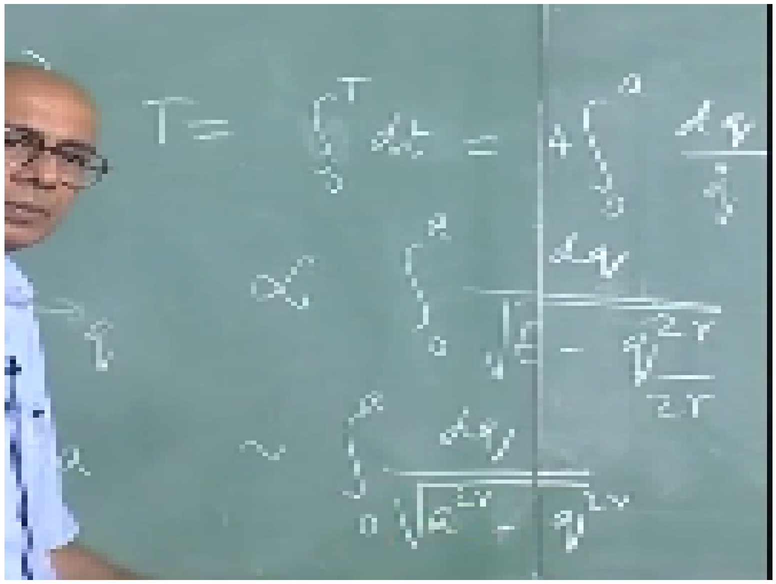

Here is q and p of this side the phase refectory are some kind of walls from pp symmetrical this fashion and for every set of initial conditions you have a concentric over of something and these are the phase the very non linear oscillator and off course the system is in take as it scans simply because there is a constant of promotion just the unit self one degree of freedom that is the 5 or 6 some algebraic function of e and q.

So that is the constant the motion is bounded it is periodic motion it is completely clear what can we say about the time period of motion how does it depend on the energy how we determine this off course in the case r=1 the simple harmonic oscillatory case we could solve for p and q explicitly because the equation of the motion are linear and we have got trigonometric function and we know the time period of the trigonometric function we know the periodicity.

But what happens if it is non linear what would happen if r=2 or 3 or 4 what would happen how it will determine the time period of this oscillator specifically I would like to know how it depends on the energy of the oscillation on the total energy what would one do in such a case while you agree that the time period of motion by symmetric is four times the time it takes to go from there to there and by symmetry just four times the rime taken go from here to here or the oscillator started the origin.

And go to the end of its amplitude so suppose the amplitude is a and i look at this portion of the directory and I write the time period as equal to over the full orbit dt 0 to t but I could write this as equal to 0 to a dq over dt which is q. but I must make sure sign of this quantity could either be + or – it is positive in the upper half part of the diagram and negative here and by symmetry since we said it takes the same time to go from here to there as we here to here this is equal to 4 times on this n set to go from here to there from 0 to the amplitude where by q.

I mean the positive √e- this one part of constant factor so this is proportional to this apart from constant to this metrical constant it is proportional to 0 to a to k dq/q. but $q.^2$ p^2 by the way is just same as $q.^2$ since I have set the mass equal to 1 is $e\text{-}q^{2r}$ so this is $\sqrt{e\text{-}q^{2r}}$ apart from numerical constants 2r over 2 but what is e for a given amplitude a so it is evident that this whole thing keeping all factors of a as part from some numerical constants this goes like integral 0 to a dq over √a2r-q2r how are we going to evaluate this for general positive integer r.

This is not easy to evaluate it is some problem to get in a2 we cannot write this in simple form by any biometric substitute as you could in the case one r = 1 then off course you have sin-1to the integral but we do not need to do that we only need to extract the dependents on the amplitude a what should I do to extract this dependencies.

I could expand it by normally integrate term by and then re constitute over thing but there is a much simpler method by which you can extract in fact a indecencies of the b one short q is bounded between o and a so what is that suggests while only the length scale in the problem is a we like to know how the t depends on a so I should yes I should scale out this quantity a so I could

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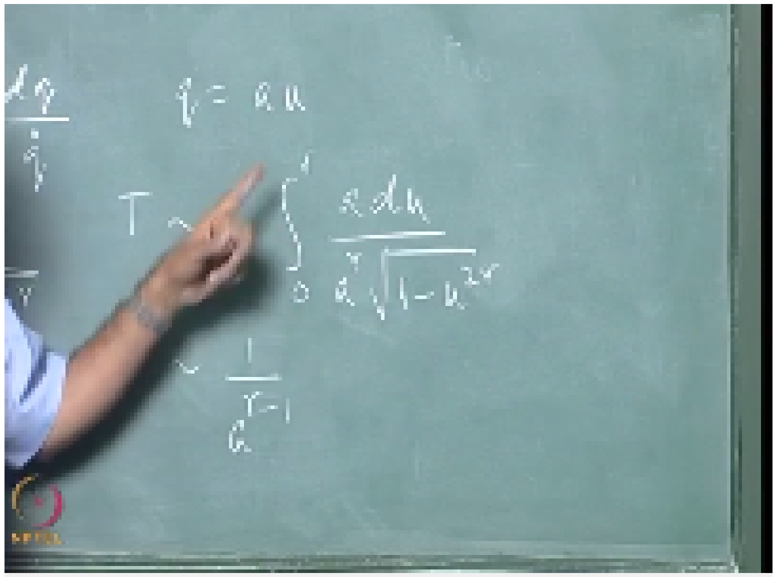

Q=au change variable of integration and this implies that t α integral 0 to 1 numerical integral nm dq that give a factor adu/ I get an a2 here which comes out of this square root and gives me an a^r multiplied by a numerical factor that integral is guaranteed to some finite number because the only similarity which you have is that u=1 for in sense you have u^4 here then you have to worry about whether this integral exist or not but you have 0 to 1.

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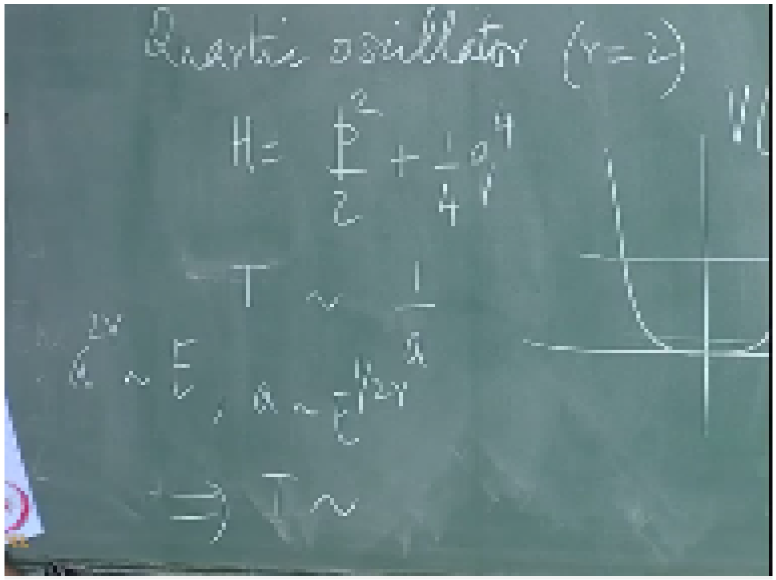

Since you have 0 to 1 du/1-u,1+u and then you immediately see that this is the part that processes double at 1 not case this is quite finite and this is like dx over square root x when you integrate the square root of x goes on top and the constitutional term has vanished if you have a higher power by $1-u^4$ for instances then that could be written has $1-u^2n$ times $1+u^2$ and once again it is only square root of 1-u from and this is two for every positive integer so the integer actually finite it has no difficulty at all.

And it is some γ function written down in terms of all that both are interested in physically is the dependences of the time period on the amplitude that is like 1 over a^{r-1} what happens when r is 1 it is independent of that amplitude that is exactly what the harmonic oscillators is for all other r does it depends of r on the syntax r in fact if you set r=2 and then we have a oscillator which is the quadratic oscillator you have $p^2/2+1/4^4$ the time period of oscillation in this oscillator is proportional to one over the amplitude in another words.

The smaller the amplitude the longer the time period seems counter but why does it happening why is this happening why is the amplitude increasing as the amplitude decreases and in fact as the amplitude goes to 0 the time period goes to ∞ what is the reason this is happening what is the shape of this potential energy it is very flat at the origin absolutely right so if I plot vq versus q this potential as third order minimum this is extremely flat so as you come smaller and smaller as the total energy increases the storing pores becomes smaller and smaller but it is extremely flat if you started with this much as a total energy.

And then by the time it gets here as quite a lot of kinetic energy this right passes but if you start here it does have enough time it rolls down slowly because it is not a simple minimum of the potential it is not parabolic at that point to have a diverges time period as a function of ampere this is character tics of non linear behavior it is extremely flat higher order minimum immediately quadratic we are actually interested in finding out how the time period or the frequency changes as a function of the total energy of the system.

How to do that we already have this results t goes likes 1 over a to the r-1 so how does it depend on the energy to the power 2^r is proportional to the energy which implies the a is proportional to e1 over 2r that implies that t goes like a^{1-r} that is the dependency of the energy we could translate this to the frequency will do this see what is the action variable is in this case.

And we see how the fluoride actually depend on the conditions so it is a simple illustration for non linear oscillator which we have almost the same kind of structure as in the simple harmonic oscillator but because it is non linear to have natural dependents of frequency of oscillation on the energy of the system and we will stop here.

Online Video Editing /Post Production

K.R.Mahendra Babu
Soju Francis
S.Pradeepa
S.Subash

Camera

Selvam
Robert Joseph
Karthikeyan
Ram Kumar
Ramganesh
Sathiaraj

Studio Assistants

Krishankumar
Linuselvan
Saranraj

Animations

Anushree Santhosh
Pradeep Valan .S.L

NPTEL Web &Faculty Assistance Team

Allen Jacob Dinesh
Bharathi Balaji
Deepa Venkatraman
Dianis Bertin
Gayathri
Gurumoorthi
Jason Prasad
Jayanthi
Kamala Ramakrishnan
Lakshmi Priya
Malarvizhi
Manikandasivam
Mohana Sundari
Muthu Kumaran
Naveen Kumar
Palani
Salomi
Senthil
Sridharan
Suriyakumari

Administrative Assistant

Janakiraman.K.S

Video Producers

K.R. Ravindranath
Kannan Krishnamurty

IIT Madras Production

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