Indian Institute of Technology Madras
Present

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NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 7
Hamiltonian dynamics (Part III)

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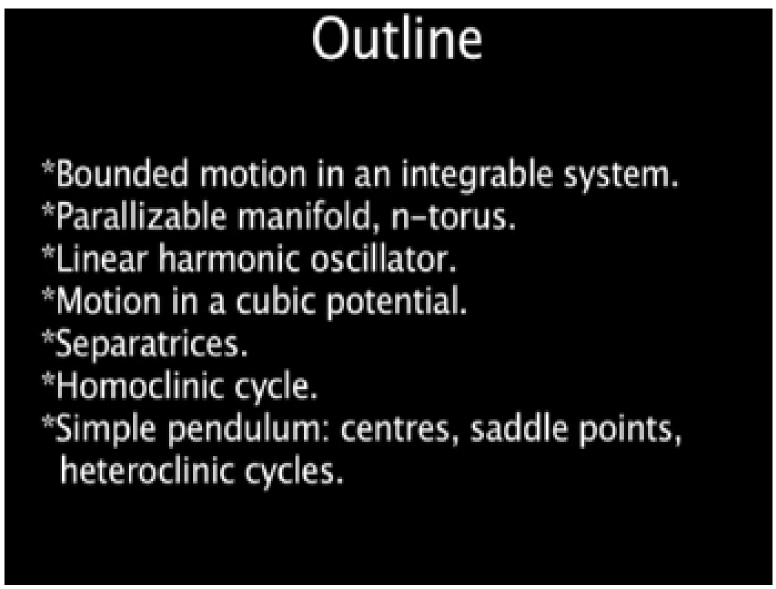

Okay we continue with Hamiltonian dynamics from where we left off.

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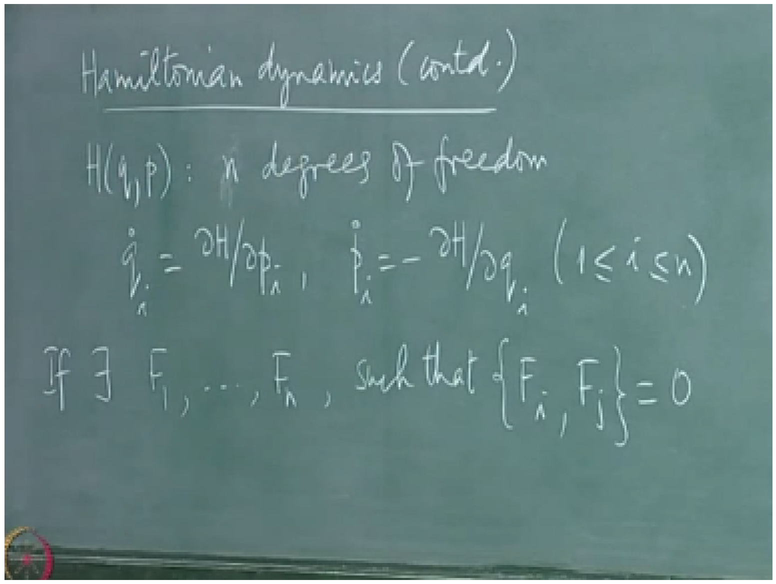

Now recall the conditions we had for the inerrability of a Hamiltonian system once again given a Hamiltonian q,p with f degrees of freedom I think I used n degrees of freedom we have Hamilton's equations qi dot is δ H / δ Pi and Pi dot - δ H / δ qi for I running from 1 to n and I pointed out that this system is integral in the sense that you can explicitly write down time dependent solutions for all the q's and all the P’s given any set of initial conditions provided there exist n constants of the motion F1 to Fn.

In involution with each other such that the Poisson bracket of any Fi with any other Fj vanished identically if there exists these constants of the motion then the system is integrable and the problem is completely solved and it was done by going to what are called action angle variables where the Hamiltonian H of q, p.

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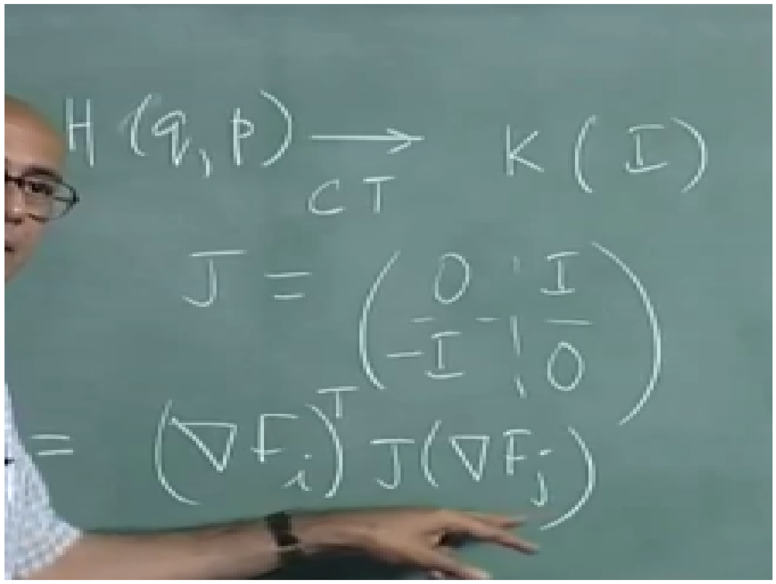

Was transformed by a canonical transformation a transformation with +1 and the Poisson bracket structure preserved to a new Hamiltonian in variables called action angle variables and this became a function of just the action variables and then one discovered that the problem was completely integrable completely solvable now what does this imply in geometrical terms this is what we were trying to understand and I pointed out that the Poisson bracket condition this quantity vanishing here can be rewritten in terms of this matrix J we introduced which was a 2n/ 2n matrix with 0 here 0 here the unit n / n matrix here - the unit n / n matrix here.

And this condition simply became this quantity equal to the gradient of Fi $^{-1}$ J the ∇ of Fj in other words these vector fields formed by the gradients of these constants of the motion of these functions of q1's and p's they are pseudo orthogonal to each other in the sense that the dot product of this row vector with that column vector is 0 this means that if the system is integrable in the liberal sense there exist n vector fields formed by the gradients grad F1 grad F2 through grad Fn which are independent of each other on the space of these eyes and θ completely.

That has a profound implication and it ⇒ that you have in this 2n dimensional phase space a 2 n - 1 dimensional energy hyper surface the motion is restricted to that for any set of initial conditions over and above that the motion is actually restricted to a subspace of this to n - 1 dimensional energy space such that these n quantities are constants completely moreover on this n-dimensional subspace the n vector fields formed by these gradients an independent of each other.

Now it turns out that there is a deep theorem mathematics which says that if you have a space which is compact and everywhere in this space you have n independent vector fields by compact I mean it does not go off to ∞ it is unbounded there is a technical definition which will write down a little later the only such space possible is isomorphic to something called an n-dimensional torus and let me explain that in slow terms a one-dimensional space which is compact and which has at every point a unique tangent vector.

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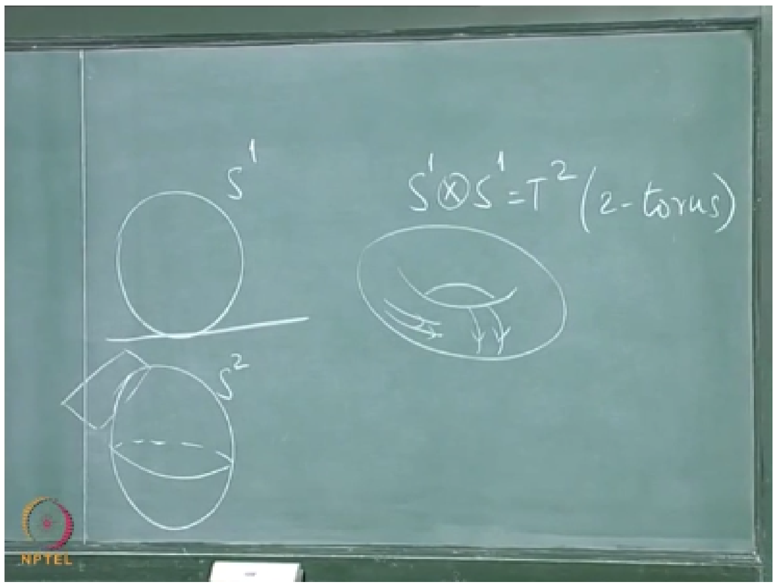

Would be something like a circle and the mathematicians would note this by s1 this space is one-dimensional and at every point on this space there is a unique tangent vector a straight line which runs from - ∞ to ∞ tangent to this point and as you can see if you took this tangent and moved it along this entire circle when you come back here you come right back to the same point the same tangent as before such a space is said to be parallelizable or developable.

In the sense that you can unroll it you can roll it on a sheet of paper and make a one-to-one mapping between a straight line segment and this circle without any kinks without any difficulty you can do exactly the same thing with a two-dimensional donut a torus in which you can take two vector fields one of which is for example directed along these lines and the other one is directed along these lines in this direction form a basis from these two vector fields and move it all over the space and come right back to the starting point to the original configuration itself.

You cannot do that on a sphere this is by the way called the direct product of two circles it is just the Cartesian product of two circles and it is called the two torus a moment's thought shows you that you cannot do this with another two-dimensional object the surface of a sphere embedded in three dimensions denoted by s2 it is not possible to find a unique tangent map which is not singular at any point on this sphere because if I start with some point here and draw the tangent plane.

So that point as I move it around it is clear that there is going to be at least one point where this direction of this tangent plane is undetermined there is a singularity what does it mean you could define this tangent plane by saying imagine like a tennis ball that there are fibers sticking out of this ball and you are trying to comb it and when you comb it is clear that somewhere maybe at the North Pole there is a little cowlick there is a little point that sticks out a singularity of the vector field.

So the technical way of saying it is that there is no non singular global tangent map to s2 like there is to s1 or t2 and the statement being made here is that the most general space which is compact and which is parallelizable in the sense that you can form a basis set of n vector fields and could I explain why this imagine comb being a ball imagine comb being a tennis ball what happens you comb it down everywhere flat.

Tangent what happens can you do this without a cowlick without a parting there is at least one point where there is going to be a singularity and the hair sticks out in the direction in which it is placed is indeterminate there is a singularity of this field they will be a bald spot invariably that is not true for a torus you can comb it down completely.

So this is a basic difference in a property of a Taurus as opposed to a sphere and the statement being made here is that if you can find an independent vector fields on an N dimensional manifold which is globally applicable smooth everywhere then that space has to be an n-dimensional torus it is a generalization of the two-dimensional torus I cannot draw an n-dimensional torus here because I cannot draw anything more than three dimensional where it applies here is the fact that integrable Hamiltonian systems integrable in the sense of Louisville Arnold the phase space on which the action takes place is eventually just n-dimensional not too n-dimensional.

It is reduced from 2n 2 to n - 1 by the constancy of the Hamiltonian itself now it is further reduced from 2 and - 1 to just n by the fact that it is integrable now this is an abstract statement we look at specific examples and see how this works out so we take simple examples and I will take the simplest of them all namely the harmonic oscillator and we see how this thing comes out how the two torus structure comes out for a couple of uncoupled for a pair of uncoupled harmonic oscillators so we will do that step by step. And before I do that let us give a few examples of what Hamiltonian systems look like so this was a bit of a digression but we come back.

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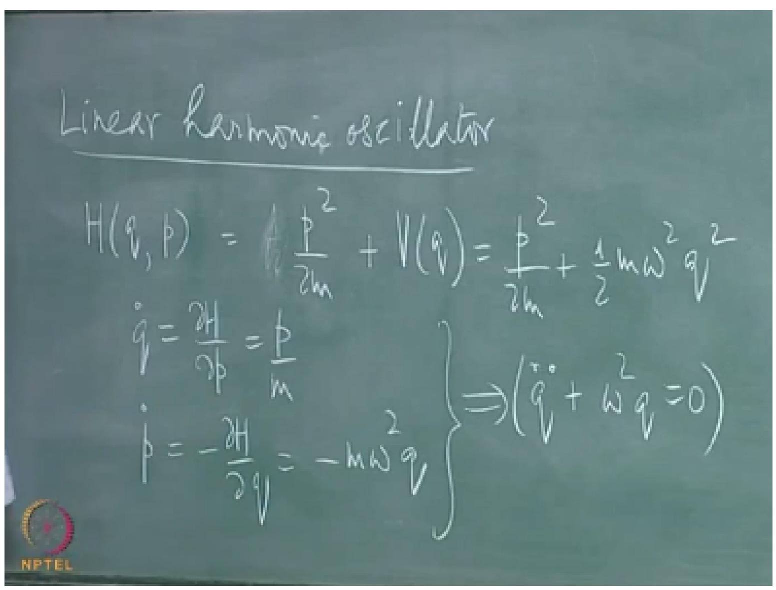

So first let us look at the linear harmonic oscillator this is of course our simplest problem of all it Is got one degree of freedom the Hamiltonian as a function of a single q and the p is 1/2 it is P^2 / 2m the kinetic energy + the potential energy V(q) which in this case is P^2 / 2m + 1/2 $M\omega^2 q^2$

where ω is the natural frequency of the oscillator and M is its mass of course we are going to get equations of motion which is just the simple harmonic oscillator.

The equations of motion but let us go through the steps simply to see how this works out and we know that q dot is δ H / δ P which turns out to be just B over m in this problem and P dot is - δ H / δ q that is equal to - M ω^2 q we have just written Newton's equations down because the conventional Newton's equation would say q double dot the acceleration is equal to 1 over m times the force which would be the rate of change of the moment.

So together it is clear that this ⇒ the usual q double dot + ω^2 q = 0 which is the oscillator equation of motion but like I said we prefer to write everything down in phase space because that is where the dynamics is taking place and we have the set of coupled equations now let us go through the formal analysis of this is a linear set of equations on the right hand side so there is no need to linearize the problem it is already linear where is the critical point of the system at the origin the right hand sides must vanish.

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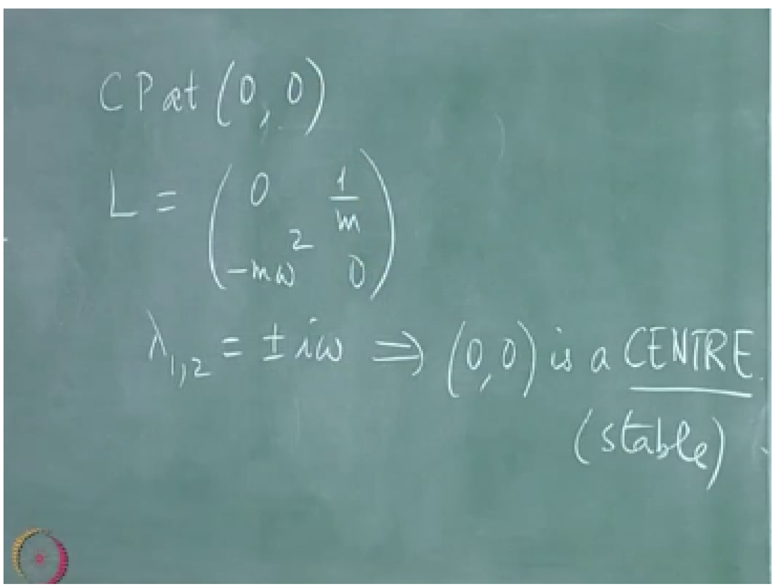

So the only critical point is that 0, 0 in the qp plane what about the matrix L which acts on the right hand side it is 0 1 / M - M ω^2 and 0 and what are the Eigen values of this matrix + or - I ω so it immediately says λ 1, 2 is + or -iω this ⇒ 0, 0 is a center is that stable unstable or asymptotically stable its stable it is not as importantly stable it is just a stable Center and what do the phase trajectories look like.

In general they are ellipses depending on the units you choose because there is just a single constant of the motion in this problem since n is 1 the set f1 through fn becomes just f1 and you need to find just one constant of the motion to integrate the system that constant of the motion is already given to you it is the Hamiltonian remember that for any Hamiltonian system which is autonomous the Hamiltonian is always a constant of the motion.

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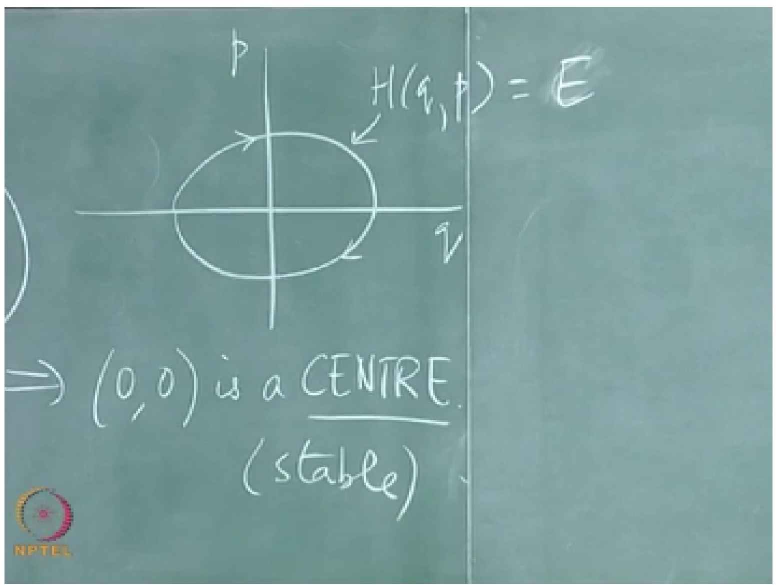

So we have a phase portrait in this case which is just a set of ellipses H (q, P) to a constant and in this case the constant is simply the total energy of the system which direction is the phase trajectory traversed would this be in the counterclockwise or clockwise direction why do you say that exactly if you pull this oscillator and let go from the rightmost point it moves back towards the left so P becomes negative at that point and therefore if you start here the next instant it is here and that fixes the direction in which this thing is traversed therefore clockwise the critical point at the center at the origin is a stable center all motion is periodic no matter what the initial conditions are and every point in this plane lies on one and only one ellipse.

Phase trajectories do not intersect themselves for autonomous systems and the entire plane is laminated by these concentric ellipses what is the time period of motion it is $2\pi/\omega$ and it happens to be independent of the energy in this problem because it is a linear harmonic oscillator it turns out that this is one of the unique properties of the linear hum of the harmonic oscillator that the time period is independent of the amplitude of motion or of the energy of the motion.

There are other oscillators which are not linear the equations of motion of which are not linear for which this phenomenon occurs and will come up with an example very shortly a little later but this is a distinguishing feature of harmonic oscillators unless of course you look at a very special class of oscillators which are nonlinear but also isochronous anything where the time period is independent of the energy is called isochronous.

So in this problem the motion does occur as you can see on one-dimensional tour I on a torus which is essentially one-dimensional namely this curve itself or on this curve and this magic happened simply because this problem had a potential which was quadratic and therefore it led to an equation of motion which was linear on the right hand side so the problem is exceedingly simple as you can see now let us take this a little more general again with one degree of freedom and see what we can say before we go on to two degrees of freedom.

Suppose I have a general potential of this kind what would this equation of motion become on this side yes.

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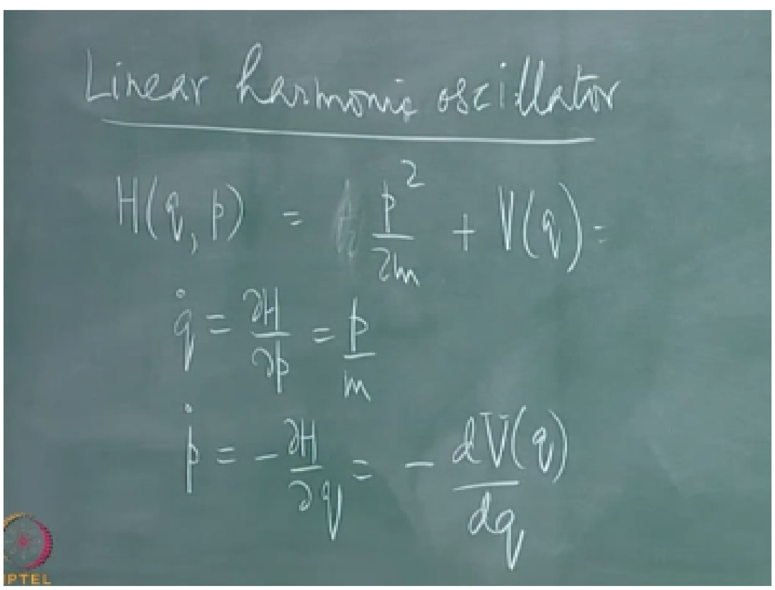

Just the derivative - dVq / dq and of course that's the force - the gradient of the potential with respect to the coordinate so we recovered Newton's equations of motion except that now the critical points of the system would be given by the vanishing of P and the vanishing of V'(q) in other words the extreme of the potential this could be Maxima this could be minimum they could be inflection points at which the slope is zero and then of course you would have to further examine the stability or otherwise of these critical points.

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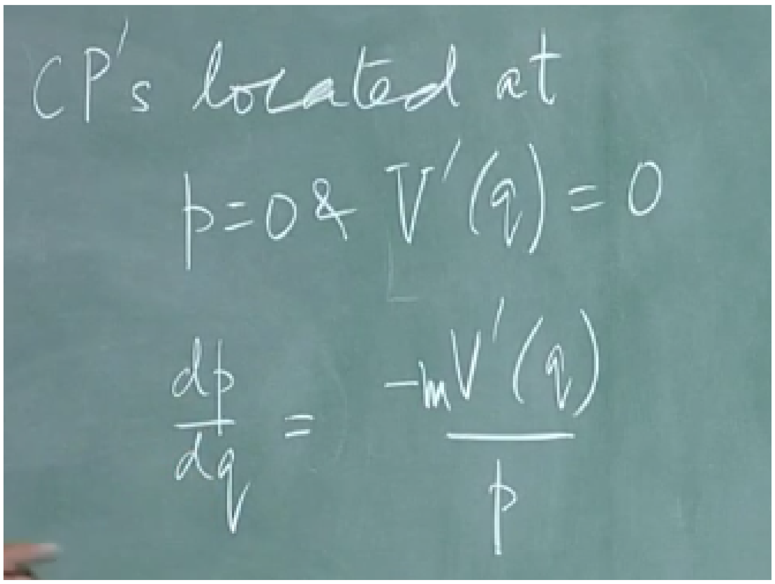

And you could in principle write the entire phase trajectory down the phase portrait down simply because there is just one constant of the motion but notice one interesting fact right away so let us say where the CP's is located at P = 0 and the roots of V' q = 0 which corresponds as we have said just the extreme of the potential but notice an interesting fact right away that the phase trajectories are actually already known to you they are simply given by $P^2 / + V(q) =$ constant.

Since one equation between two variables P and q on a plane specifies a curve the phase trajectories are completely specified even without solving the equations of motion solving the equations of motion for a specific set of initial conditions will of course tell you how P and q change as a function of time explicitly but to write the phase trajectories down you do not need that notice something else notice also that the phase curves are given by dividing this by this equation and you get dP / dq = - V' (q) / Pm times that.

Therefore whenever the phase trajectory intersects the horizontal axis the q axis it will generally do so at right angles because this quantity vanishes on the x axis on the q axis but this may not if it does then you have to examine the problem further and take limits but otherwise phase trajectories would intersect the q axis at right angles and that is indeed what happened in the harmonic oscillator example.

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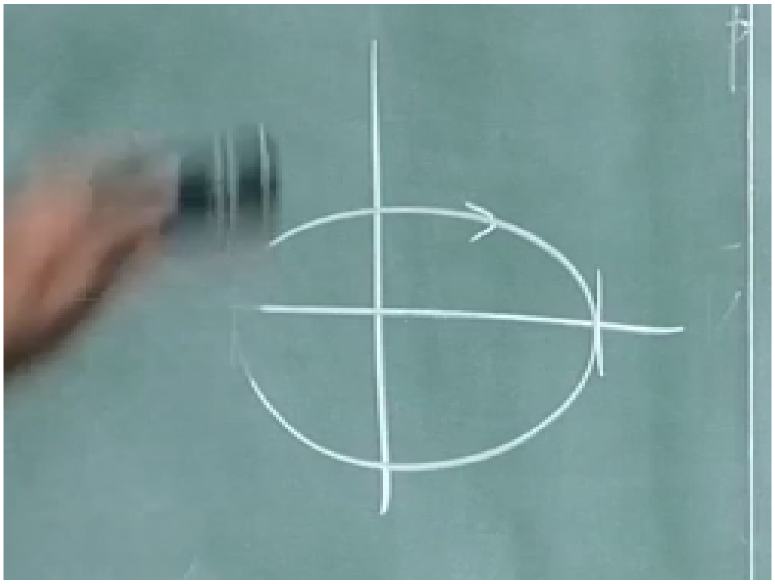

Where you had this kind of behavior and these were at right angles that happened because the restoring force is not zero at those turning points at the endpoints of the motion but the momentum vanished at those points you could integrate this equation you get $Pdp + mV'(q)\,dq = 0$ and if you integrated it what would you get you would simply get $P^2 / 2m + V(q) =$ constant which we already know.

So in principle a one degree of freedom Hamiltonian system is always integrable you do not need any further conditions let us look at a potential which is a little more complicated than a linear one so let us suppose we have q and let us choose units conveniently so that I do not have to run into problems with writing these constants down.

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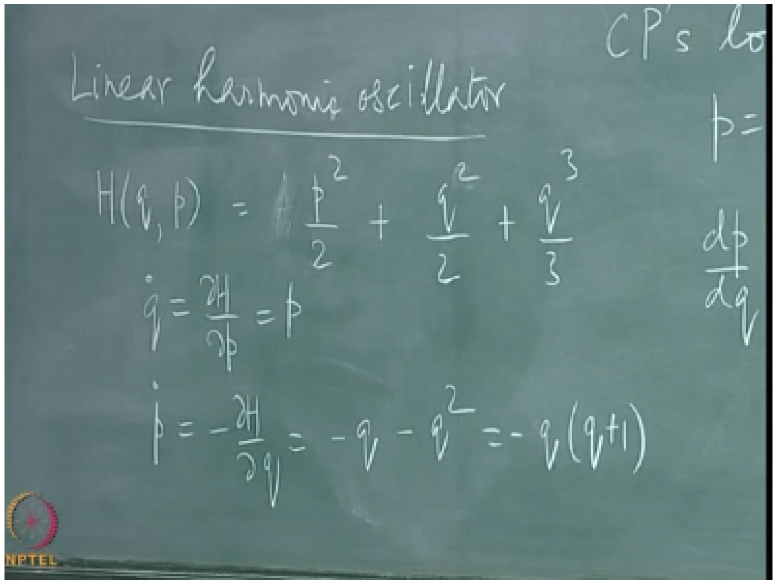

So let us simply write $P^2 / 2$ + perhaps q^2 over 2 so it is a harmonic oscillator without any extra terms but then I include a non-linearity and make it $q^3 / 3$ in suitable units what happens to the right-hand side here this becomes P what happens here - q because you need a - sign there - q^2 that becomes = - q times q + 1 where are the critical points of this system well 0 is still a critical point but - 1 0 is also a critical point and we can easily write down what the solutions are .

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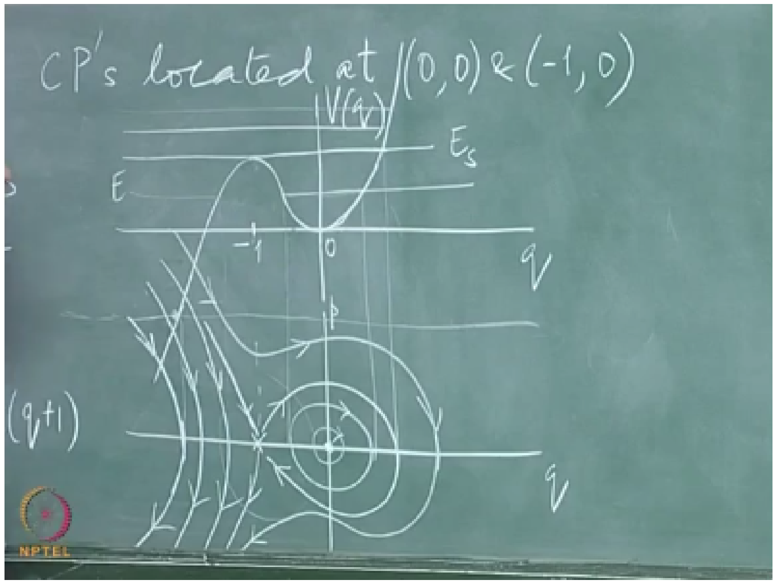

0, 0 and – 1 we should draw the phase trajectory or phase portrait but before we do that let us draw the potential so we get some physical idea of what it looks like so here is the q axis here is V(q) now what does this potential look like just this alone sufficiently close to the origin the q^2 dominates over the q^3 and therefore it looks like a parabola so we certainly guaranteed that the potential looks like this here and then for large positive q it shoots off like cubed goes off to ∞.

But then for large negative q this term dominates over this no matter how large you get once q becomes sufficiently large this becomes much bigger than this and then this curve has to come down in this fashion and not surprisingly this extreme is at - 1 this is at 0 which is a minimum there and a maximum at - 1 what do the phase trajectories then look like so if you permit me to draw it on the same curve on the same vertical axis but.

Now I draw P here versus q this would be a critical point and this point here would be a critical point it is quite apparent that this is a center about which you have stable oscillations small oscillations and what kind of critical point would this be it would certainly be unstable you would have to linearize the equations of motion about the point q = - 1.

So you might want to set say U = q + 1 and then shift the origin to q right to q = 1 and see what happens in the vicinity of this point but we have already seen that for Hamiltonian systems there is no dissipation and the only critical points possible are centers and saddle points and this is a saddle point it is unstable.

And that is the center and that is stable what do the phase trajectories look like what would the phase portrait look like in this problem you would have to specify now the initial conditions in other words you have to tell me the initial q and P or better still tell me the initial value of the energy and that remains constant because you are on curves in which this quantity is constant.

So what we are really doing is plotting the curve $P^2 / 2 + q^2 /2 + q^3 / 3$ = a constant that constant could be positive or negative in this problem because this term could take on large negative values as well so what would these phase trajectories look like in general suppose I started with a value of the total energy that correspond it to some level like this at this level on this figure this is the 0 of the energy of V(q) suppose I had a total energy equal to this much where would the motion be this is my total available energy it is clear I cannot go into this region because if I did so then this quantity $V(q) + P^2$ should be equal to this number but V(q) is already larger than this number which ⇒ P^2 should be negative that is not possible with real P.

So if this is the total energy the system does not have enough energy to get into this region its restricted to this region and therefore it can never move to the right of this point now imagine you start with a little ball bearing here in this potential Hill and let go from rest what would it do it would move away to $q = -\infty$ with increasing acceleration in which direction would this acceleration be to the left or to the right in which direction would the momentum be it would be to the left to move further and further to the left.

So P would get more and more negative and q would get more and more negative and therefore this is what the trajectory would look like on the other hand imagine starting at - ∞ in queue and shooting a ball up this potential hill with a fixed amount of energy equal to this much it is clear it can crawl up this hill this barrier up to this point where it is energy where it is kinetic energy goes to 0 and then it rolls back what would that trajectory look like that half trajectory.

But it starts there and moves to the right but with smaller and smaller values of P of momentum till it reaches this point with 0 momentum it would therefore be the other ½ of this curve and in principle if you shot something here you started off with something here with this much total energy at - ∞ it would crawl up this hill and fall down corresponding to this phase trajectory and as we know already.

Since the restoring force at this point is not 0 the slope is not zero there V'(q) is non zero at that point therefore it must intersect this line at right angles what happens if I have a little higher energy nothing much nothing much happens it follows another trajectory which does this what happens if I have a total energy equal to this much it is clear that the particle could move up to this point and this would be a trajectory but it is also clear that if the initial conditions permitted it to be inside this region to start with it would simply oscillate about that origin therefore for the same value of the total energy there exists another trajectory which would correspond to oscillations.

As I said a closed phase trajectory ⇒ periodic motion and vice-versa so for the same total energy there are two regions in configuration space where the particle could find itself one would be to the left of this point and the other would be in this well motion here would correspond to periodic motion here would correspond to open or unbounded motion but both these phase trajectories correspond to the same total energy same value of h equal to constant same constant.

These oscillations here would for sufficiently small energies above zero be essentially ellipses because you could neglect the effect of the q^3 term and then you have a harmonic oscillator but it is quite evident that as the amplitude increases this is no longer a parabola but it flattens out on this side and becomes cubic on that side and therefore it is non harmonic it is some kind of oval but it is still periodic motion the time period in general would depend on the energy except for very small amplitude oscillations.

When their system looks like a simple harmonic oscillator what happens if you have an energy which is higher than the height of this barrier it is clear that the barrier no longer can tap this particle into oscillatory motion therefore this would be open trajectory of some kind and open trajectory imagine shooting the particle up it comes up here it certainly slows down because you have very high potential energy here but then it crosses this barrier falls into this well climbs up till that point and then goes right back and falls off in this fashion therefore I would expect this thing to come down to go around and go off escape to ∞.

Again crossing this at right angles that is what would correspond to an energy which is higher than the height of the barrier so now you begin to see that there is one very special value of the energy where these two possibilities namely periodic open motion versus periodic motion they merge the boundary between the two which would correspond to a total value of the energy a value of the total energy which is exactly equal let me call that Es which would correspond to two different kinds of motion.

One of which would be remember this point by itself this point by itself is an unstable critical point it is a phase trajectory by itself if therefore you shoot a particle from here up this hill with just this critical value of energy so that it can barely reach it out there it is going to take an infinite amount of time to do so it would eventually as T tends to + ∞ go and stop there in this fashion that would correspond to a trajectory which comes along like this and tends to this point as T tends to + ∞

Had we started with a particle there and displaced it infinitesimally to the left it would fall off and go off to - ∞ here which would correspond to this had we started on this side up here and pushed it slightly to the right it would go up the barrier and go down this well go up to this point turn back and come back and crawl back to this point and the reason it would crawl back is because the slope is getting flatter and flatter the restoring force is getting smaller and smaller and therefore it is barely able to reach this top.

It would therefore do the following oh yeah go around and come back and this point of course would correspond to that so it is quite clear that a lot of interesting things happen in this region and let us magnify that region and see what it looks like that region near.

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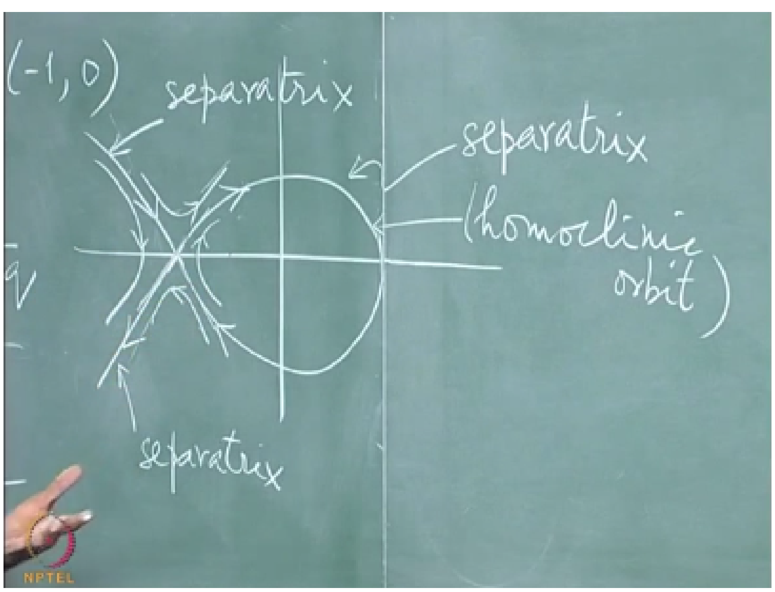

The separatrix is a saddle point here there is an unstable orbit coming out of it which eventually falls back tends back towards it and then there is a separate rays which is flowing in and something which is flowing out I should really let these things tend to that point as emphatically but instead of that let me just draw it in this fashion so you can see that this is a limiting point this saddle point if you linearized about the saddle point you discover that the system has two Eigen values one of which is positive and the other is negative and the two Eigen directions or eigenvectors of the linearized matrix l would correspond to these directions.

This is called the stable manifold of this critical point and that is called the unstable manifold of this critical point and as is typical of a saddle point two lines come in and two lines go out near the saddle point the system is hyperbolic and the whole thing looks like hyperbolas the phase trajectories look like hyperbolas so you have behavior of this kind of course these trajectories would eventually flow off and this would go around and join up there and similarly inside here these would be parts of periodic orbits and these would be parts of open orbits but locally it looks like a saddle point should.

Notice also that this tangency here is not at right angles this is the one case where this intersection at right angles does not happen and the reason is $V'(q)$ also vanishes at this point and therefore you have to take the limit $V'(q)/ P$ as you approach the critical point both numerator and denominator vanish at that point and you have this typical saddle structure I leave it to you as an exercise to find out in this problem what this angle is what the angle subtended by the two separatrix are.

This trajectory which separates open motion of this kind from open motion of this kind is called as separatrix corresponding to the energy E_s and that is the reason I used a subscript s there to show that it is power energy corresponding to a separatrix this trajectory is a separatrix and this trajectory which separates open motion from periodic motion inside this closed loop is also part of the same separatrix this particular trajectory has an even greater significance there is a special name given to it.

Because it is starting off from a saddle point moves off in the unstable part of the unstable manifold and it looks back and comes back to the same saddle point as part of the stable manifold such an orbit is called a homo clinic homo clinic orbit play a crucial role in the behavior of nonlinear dynamical systems as you can see small changes in initial conditions around these separate races around this point can cause very different futures altogether.

This is a lesson of some generality if I started here slightly above the separatrix I would move down this way and move off there but if I started here I move off somewhere else similarly if I am here I move off altogether to ∞ but if I am here just inside this loop I keep going around so it is clear that separate races play a very crucial role in the behavior of nonlinear systems this is a nonlinear system it is very clear because of this the equation of motion has become nonlinear here and that is responsible for many of the things that we see here having seen what a typical phase portrait would look like for such a one-dimensional problem a simple problem let us look at the very model of one-dimensional problems of this kind the simple pendulum in the absence of dissipation once we do that we are set to look at higher degrees of freedom,.

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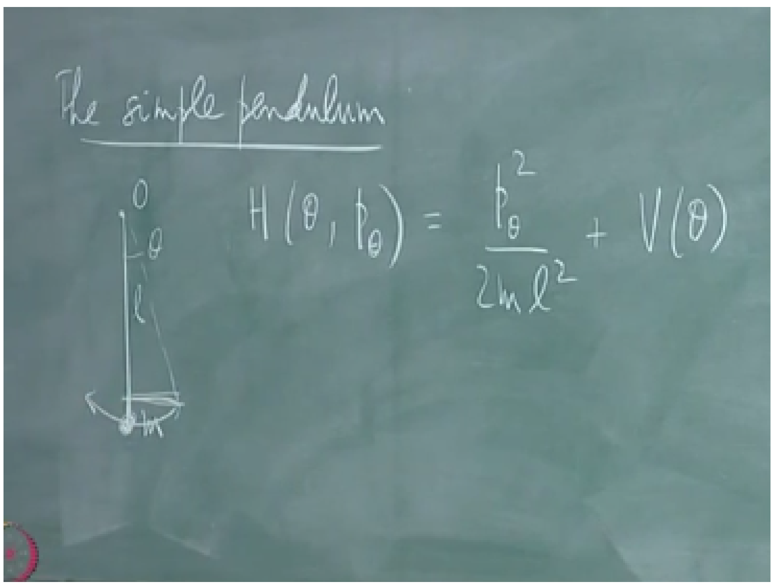

What I mean by a simple pendulum is a mathematical pendulum it corresponds to a bob of some mass M suspended without friction from by a light rod of some length L so this is the point of suspension which I take to be the origin and from that you have light mass less rod of length L and a heavy bob of mass m and the motion of this pendulum is in a specified plane say the plane of the blackboard and the angular displacement about the vertical I call θ and the pendulum moves back and forth in this position.

Now the question is what sort of Hamiltonian does it have once again because we know there is no friction in this problem the degree of freedom that we have is 1 the dynamical variable which specifies the position of the pendulum at any point at any time is in fact the angular coordinate θ about the vertical so it is a function of θ and a conjugate momentum P θ which is nothing but the angular momentum of this pendulum the orbital angular momentum of this Bob about the origin and this is P θ ^{2} 2 ml ^{2} since ml ^{2} is the moment of inertia of this Bob about the origin.

So it is the square of the angular momentum divided by twice should have switched it off earlier so it is the square of the angular momentum divided by twice the moment of inertia + the potential energy which is a function of the angular displacement θ alone now let us assume that the potential energy is zero when the Bob is at its lowest position then when it is at an angle θ about the vertical the potential energy corresponds to raising the Bob by this height here and therefore it is nothing but 2 ml ^{2} + mg L times 1 - cos θ.

So you have to subtract this distance from that distance multiplied by mg and that gives you the potential energy now remember this is a light mass less rod a rigid rod and therefore two kinds of motion are possible either the pendulum oscillates about its lowest point or else it rotates completely and both possibilities are included in this expression for the potential energy so all we have to do is to plot this potential energy find out where the maxima and minima of the potential are and we have our phase portrait.

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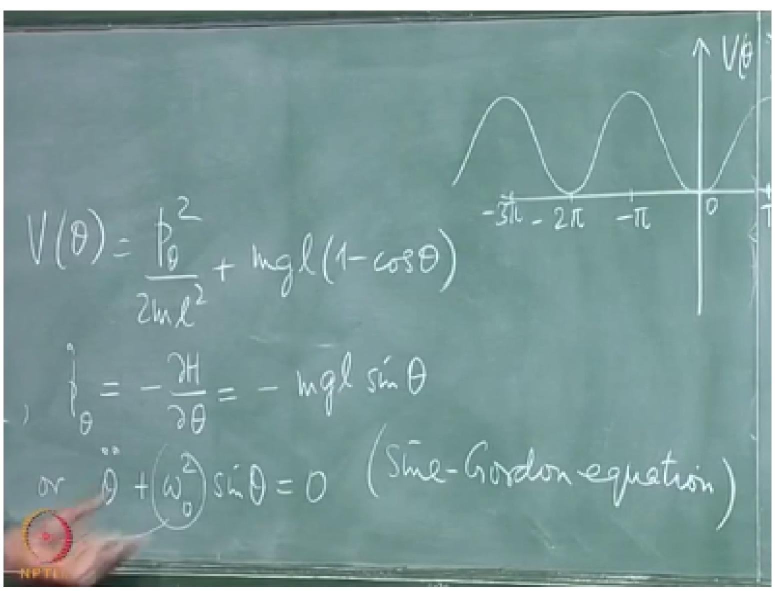

So let us do that let us write down what $V(\theta)$ looks like we have to plot mgl times 1 - cos θ and that is simple it has a bunch of maxima and minima of this is at 0 this is at - 2π this is a + 2π and so on this is at π this is at - π this is at - 3π and so on where are the critical points of this system let us write the equations of motion down state a dot is δ H / δ $P\theta$ which is equal to $P\theta$ over ml^2 just corroborates the fact that the angular momentum is the moment of inertia multiplied by the angular velocity.

The dynamics is buried here P θ dot =o - δ H / δ θ what is that equal to is equal to - I differentiate this mgl I - sin θ this - goes against this - and cancels is this a linear system on a non linear system a highly nonlinear highly nonlinear because of this sin θ it is got all powers of θ in it all odd powers of course you can eliminate V θ completely by differentiating this a second time and substituting for P θ dot here and what would you get it a double dot + g / l sin θ =0.

That is the famous pendulum equation and if I call g/l this quantity the square of the natural frequency for small oscillations then this simply says or θ double dot + $\omega 0^2$ sin θ = 0 where I have set $\omega 0^2$ equal to g/l this equation is very famous there is a long history it is a non linear second order ordinary differential equation it is called the sine garden for reasons we would not go into right here this started off this name was a joke to start with but then it stuck completely it is similar and formed to an equation which is known in other contexts for instance in relativistic quantum mechanics called the Sine - Gordon equation.

And this non-linear equation has is related to the Sine - Gordon equation and because it has a sine here it was as a joke initially called the sine garden equation and that name is stuck completely it is got a long and distinguished history very interesting properties it is a very nonlinear equation but it has some very special solutions as we will see nonlinear because of all powers of θ sitting here I might mention here that you can actually solve this equation in general and the solution is in terms of elliptic functions.

And elliptic integrals which are not elementary functions they are not ordinary trigonometric functions they are a little more complicated than that but we are not going to do that we are not going to write the solution down we are going to look at the phase trajectories and see what the phase portrait looks like I remind you that in the small amplitude approximation where you can replay sin θ / θ this becomes the harmonic oscillator equation and then of course the time period of oscillation is just 2 π / ω0.

But that is only true for small oscillations the moment the oscillations become reasonably large in amplitude then the time period depends on the amplitude and in fact increases with the amplitude in a fairly complicated fashion now what do the phase trajectories look like where are the critical points.

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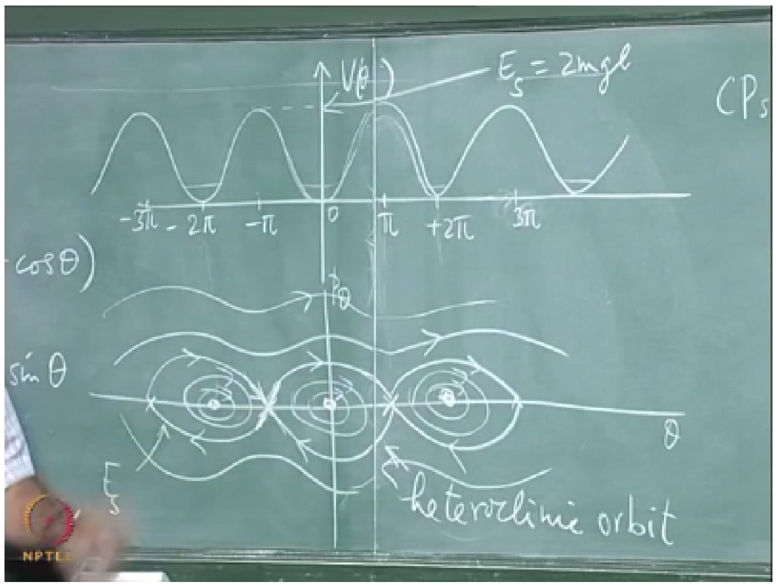

So CPs at Pθ = 0 and θ =0 of sin θ which happens at all integer multiples of π. Now of course to cut a long story short we pretty much know what these critical points are going to be like so if I plot Pθ here versus θ I know that this is a minimum of the potential and therefore there is a center here so is this and there is a center here and so is this centers here these points so centers occur at all at zero and all even multiples of π what sort of critical points do you have at odd multiples.

The maximum of the potential they are unstable and in this Hamiltonian system the only possibility is saddle points once again so you have a saddle point here a saddle point at this point a saddle point here a saddle point here and so on what would the phase trajectories look like well it is quite clear that in this problem the only allowed values of the energy are non-negative of the total energy the moment you have a small positive energy the system could find itself trapped in either this well all this well or this well or this well and in each of those it would execute small oscillations.

Looking like that little higher energy and these oscillations were slightly bigger ovals these are not ellipses except for extremely small amplitude oscillations because this θ is not approximated the $\sin \theta$ is not approximated by θ except for θ is sufficiently small sufficiently close to a muilt even multiple of 2π and then of course these are ellipses as you come closer and closer but after that they are ovals given by this quantity equal to a constant the entire Hamiltonian equal to some constant.

What would happen if the energy were larger than the maximum value here and the maximum of the potential this thing here corresponds to by the way all these Maxima at exactly the same point the same value this maximum corresponds to the separate X energy which is 2 mgl because that corresponds to $\theta = \pi$ in which case the potential energy becomes 2mg R so if E is greater than 2mgl I would expect the motion becomes unbounded.

Because instead of oscillating this way the amplitude keeps increasing and finally it is got enough energy to overcome the barrier to go all the way around and then it would be open motion to go this way or the other way and this would correspond to open trajectories right here or here but the interesting thing happens when you have an energy equal to the separatrix energy.

Then of course you could for example start here at $-\pi$ crawl down extremely slowly accelerate as you come down and go up and crawl up all the way to $+\pi$ that would correspond to a trajectory which starts here and ends there and vice versa which would correspond to something doing this as importantly similarly you could start here and go there which would correspond to a loop like that.

So now you have saddle points in which this is the unstable manifold and this is the stable manifold the tangents there if you linearize about these points but now you have a situation by these loops go from one saddle point to the other and back from the next back to this such a loop is called a hetero clinic orbit and of course they correspond to on this trajectory the energy is a service and what about the open trajectories it is clear that you would have an infinite family of open trajectories it should look like this and on the other side as the energy increases these things would get flatter and flatter.

So such a trajectory would correspond to counterclockwise rotation in which θ is going on increasing monotonically and the other one corresponds to clockwise rotation where θ becomes more and more negative monotonically the separatrix as before separates rotational motion from oscillatory motion the interesting point and that is it this is the phase diagram of the undamped simple pendulum the moment you put in damping the moment you have a first-order term here a θ dot.

Term which would correspond to a system which is not Hamiltonian then this entire picture changes and it is clear no matter where you start maybe it rotates a few times but eventually it comes to a halt it would oscillate and then damp out so the trajectories would look very different altogether and the fate of any point on the phase plane wherever you start would depend on where you started which of these it goes gets attracted to because all these points would become stable spiral points as importantly stable spiral points.

And which one it goes to depends on where you start what is interesting is that for very small amplitude oscillations the solutions are trigonometric functions the general solution of an equation with the θ here is simply cos or sine ω 0T the solutions for larger amplitude oscillations are elliptic integrals as I mentioned but the solution for this critical value of the energy on the separatrix is again expressible in terms of elementary functions once again it turns out that you do not need any elliptic integrals or anything like that if you set the total energy to be equal to twice mg/l.

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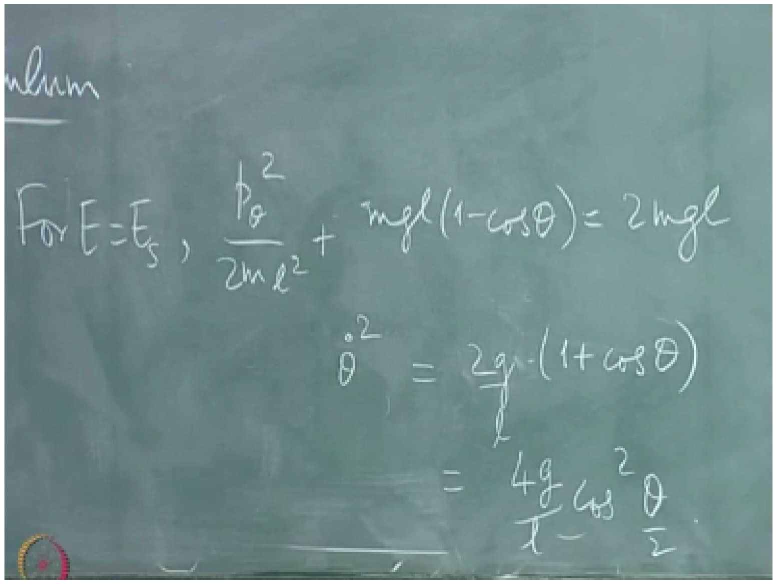

Then those trajectories are actually simple to write down and the reason is on those trajectories I would have H(P) θ and θ = P θ ^{2} / 2 ml ^{2} + mgl times 1 - cos θ =2 mgl and of course we know what this term is it is nothing but 1/2 ml ^{2} θ dot ^{2} and if I bring that down to this side or take this over to the other side what happens comes 1 + cos θ on the other side the m gets removed to take the 2 there and the right hand side gets simplified what's this equal to this is twice cos ^{2} so this becomes 4 g/l cos^2 θ /2 if you took this trajectory for example in which θ dot is positive then corresponding to that you have θ dot = 2 √ g /l cos θ /2.

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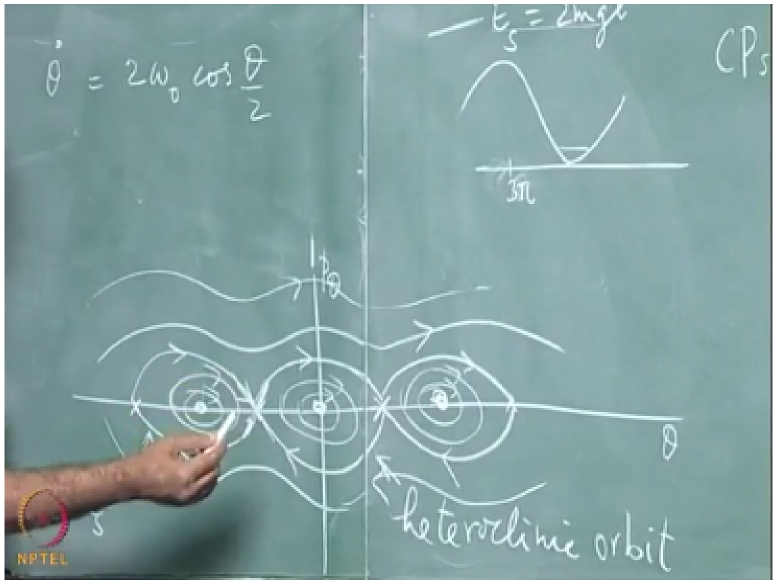

So on that trajectory θ dot = 2^2 √g/l / l but that is ω0 and this can be integrated because all you have to do is rewrite this as a θ x seck θ /2 and integrate it and it can be done in terms of elementary functions so I leave it to you to write down the explicit solution for an initial condition where at T = 0 you are at θ =0 and as T tends to + ∞ you are going to approach θ = π and at e = - ∞ you start off from here at this point so I leave you to write this solution down and then we look at it special features.

And once we are done with this we can move on to understanding how to dimensional and higher degrees of freedom integrable systems would lead to the torus structure I mentioned earlier so we will do that next time any questions yeah it is not with respect to time has been eliminated completely so the point the reason you draw an arrow on phase trajectories is to tell you in which direction the phase space point the representative point representing the system moves as time increases.

But time itself does not appear here it is clear that time has gone has been eliminated and what you have done is simply to say where does the point which is represented in the system which is represented by a point in phase space in the space of all its coordinates and its moment a where is it located and how does it move as a function of time yes one of these Lopes okay that is a phase trajectory.

So actually what is happening here yes absolutely this is a different phase trajectory from this is a different phase trajectory from this point is a phase trajectory by itself so the statement is if you are here then as T tends to + ∞ you are going to flow towards this point if you started here and let T go backwards you would flow towards this point so that is the implication of what is meant by an unstable manifold and a stable manifold to a saddle point because that point alone is a solution to the system's equations of motion in which all the left hand sides vanish and since these are first hand first-order differential equations if all the initial conditions are 0 if all the derivatives are 0 to start with then the system never takes off.

And it remains there so that corresponds to taking this Bob and balancing it at π vertically up of course an infinitesimal displacement would cause it to move so it is an unstable equilibrium but it is an equilibrium point nevertheless the crucial thing to note is that these separate races they actually qualitatively different kinds of motion are separated infinitesimally to the inside of it the motion is periodic infinite is money to the outside of it the motion is completely open its rotational motion as opposed to oscillatory motion.

So they play clearly a very special role and what really happens is that in a system which is perturbed and non integrable unlike this system the separate races would really determine the fate of the system dynamical system in some sense yeah a homo clinic orbit was one where you started at a saddle point made a loop and came back to the same saddle point a hetero clinic orbit consists of more than one separatrix where you start at one saddle point flow into another you started that saddle point and flow back to the original one could be more than two saddle points involved in this loop but it is still a loop and these loops get perturbed very easily and that is how chaos appears in Hamiltonian systems.

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