Indian Institute of Technology Madras
Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 6
Hamiltonian dynamics (Part II)

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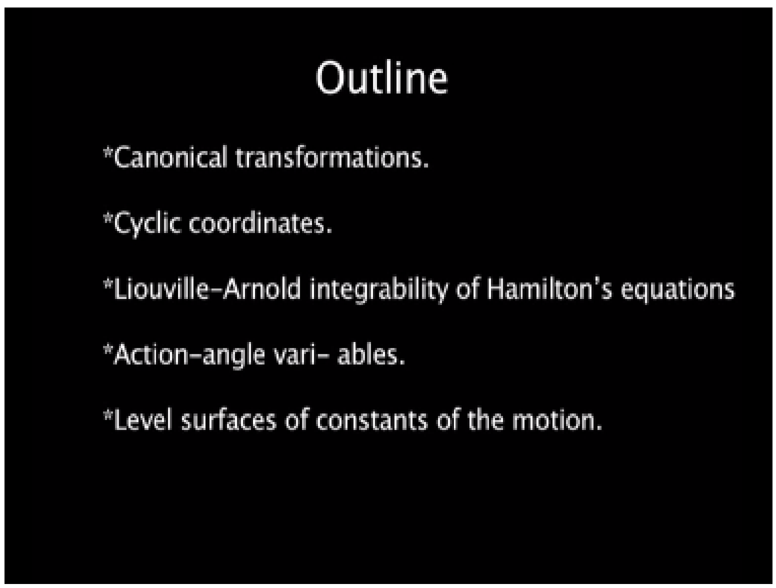

We started our discussion on Hamilton's equations of motion and Hamiltonian systems let me briefly recapitulate the place where we had got to and then go on from there.

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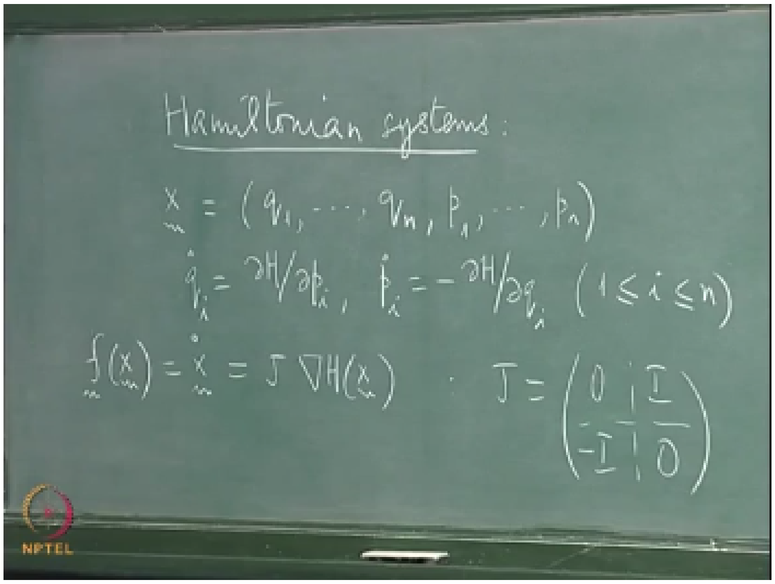

A Hamiltonian system if you recall and a set of dynamical variables called generalized coordinates q 1 through Q an and a set of generalized momenta P 1 2 P n - n of them and all which we have combined together into this vector X which satisfy the equations of motion Q i dot is δ H over δ P I and P AI dot is - δ H over δ Q i for I running over the integers from 1 to N for n degrees of freedom you have to end dynamical variables they are paired together the Q's and PS these are called generalized coordinates and these are called generalized momenta and the equations of motion are these two and coupled first order in general nonlinear partial differential equations to be solved uniquely provided you specify the initial positions the initial coordinates and the initial momentum all to n of them.

We also wrote this by pointing out that the entire vector field on the right hand side governing the time evolution of the vector X the phase space point X is given by the gradient of the Hamiltonian which is a function of all the Q's and PS multiplied by a certain matrix J which we had written down and J was a 2 n / 2 and matrix with zeros in the first block of N / n block the unit matrix here the unit matrix here and zero here once again this is the N / n null matrix the N by n unit matrix - that matrix and again the null matrix.

The introduction of this J ensures that these derivatives get twisted in the sense that the rate of change of the first n components of this vector depend on partial derivatives with respect to the remaining N and vice versa with the appropriate - sign this - sign is taken care of by putting a – here. So a remarkable fact has been achieved for Hamiltonian systems namely that the entire vector field which normally would have been some vector field f of X is now specified by a single scalar function.

But this matrix J has remarkable properties and these are responsible to some extent for the wonderful properties that Hamiltonian systems display the first of these is the following we see immediately.

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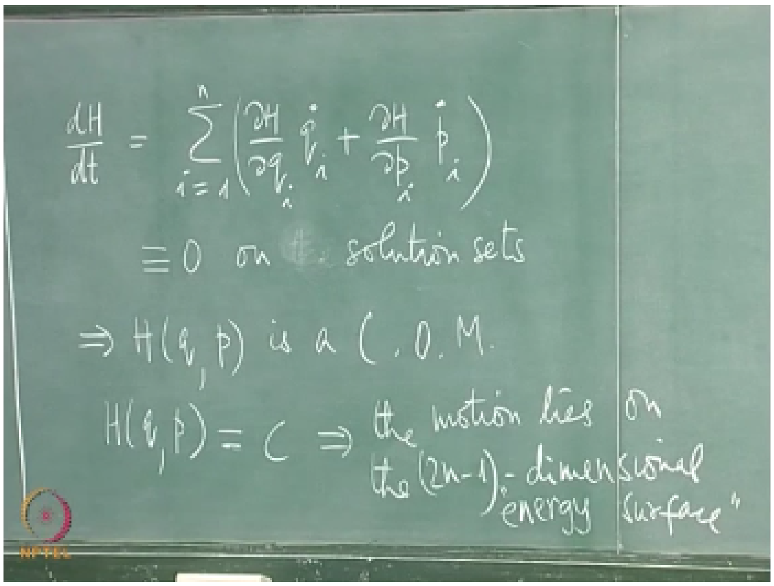

That the quantity H itself for compute its time derivative this is equal to a summation from I equal to 1 to N δ H over δ Q i Q i dot + δ H over δ P I VI dot but then if I substitute for Qi dot and k i dot from the equations of motion it implies that D H over DT vanishes identically on the solution set on solution sets, in other words whenever the equations of motion are obeyed on those curves d H over DT is identically 0 which implies that the Hamiltonian H of Q , P is a constant of the motion and remains unchanged as time evolution occurs.

We have assumed that the system is autonomous so H itself has a dependence on the Q's and PS but not explicitly on time what happens when H has got explicit time dependence we will come back to a little while later but we are right now looking at autonomous Hamiltonians in other words H is a function of all the generalized coordinates and the generalized momentum, mechanical systems which are conservative systems in the absence of dissipation particles moving under the action of forces which they exert on each other and so on these would be examples of such Hamiltonian systems there are many many other examples of Hamiltonian systems.

But right now we would like to develop the theory a little bit more before we look at special cases of Hamiltonian systems; this implies also that the trajectories lie on the surfaces on which H is a constant. So you recall we are now working in a 2n dimensional phase space and any statement of the form H of Q, P is equal to a constant defines a 2n - 1 dimensional hyper surface in this 2n dimensional phase space and you're guaranteed that for any set of initial conditions the motion or the phase trajectory lies on this hyper surface.

So this implies the motion lies on the 2n - 1 dimensional energy surface or hyper surface I call it energy surface because as I mentioned the Hamiltonian generally its numerical values have the connotation of being the total energy of a conservative system this is not always true but we'll use this terminology because that's the most commonly occurring case that the Hamiltonian essentially is the total energy of the system expressed in terms of the coordinates and the momentum of the system.

Normally one would say that the gradient vector of a scalar is directed normal to the level surfaces of the scalar function, because that is the rate of maximum rate of change of this quantity of the scalar quantity occurs in a direction normal to the level surface on which the scalar function is a constant. But the moment you put a J here this immediately implies that this velocity vector in phase space is actually directed on the level surface itself rather than normal to it and that is a major difference between ordinary gradient systems where you don't have this J here and X dot is given by this gradient of some scalar field and Hamiltonian systems for which the motion actually occurs on a surface on which H is constant for each set of initial conditions my statement was yes I would like to explain this a little more.

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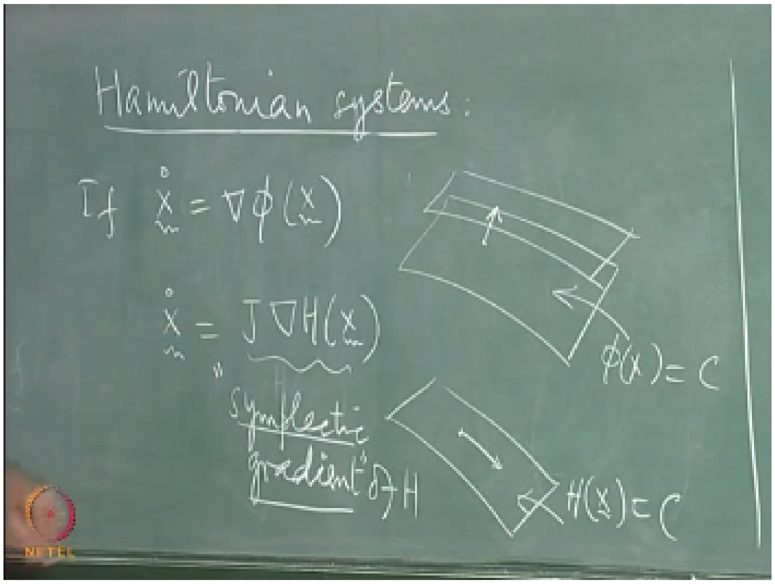

The statement is if we had for instance X dot equal to the gradient of some Π of X then this tells you the velocity in phase space namely the direction of X dot tells you in which direction X is going to change as a function of time at each instant of time if π of X = constant on this surface π of X equal to some constant and we are at this point in phase space then velocity vector is directed normal to this surface in this fashion.

So in the next instant the system moves to some other surface π of X equal to a different constant for instance and the motion is normal to the level surfaces π equal to constant at each point on the other hand for a Hamiltonian system X dot is J times the gradient of H of X and H of X is constant on the solution set which implies that if this is locally the surface H of X equal to C at any point then this velocity vector is guaranteed to lie on the surface rather than normal to the surface itself.

So it continues as time goes on for any given initial set of conditions this motion is restricted to the two and - one dimensional hyper surface specified by H of X equal to that value of the constant which it has owing to the initial conditions, this J ensures that the derivative of the first component of X is the partial derivative not with rest of H not with respect to the first component itself but the N + 1 first component and the second with respect to the n plus toothed component and so on it also ensures that the derivative of the nth component of X which is P 1 is - the derivative of H with respect to the first component.

So this - sign that this J introduces pair wise ensures that H itself is a constant of the motion identically, because if I substitute δ H over δ P I here and - δ H over δ Q i here this bracket vanishes identically now that's formalized by saying that the introduction of this gradient here this J here which makes this not the gradient of H but the simplistic gradient of H ensures that H itself is a constant of the motion this quantity here is called the simplistic we also saw that Hamiltonian flows have another very special property that is any volume element remains unchanged in time in other words the system is corresponds to a conservative dynamical system.

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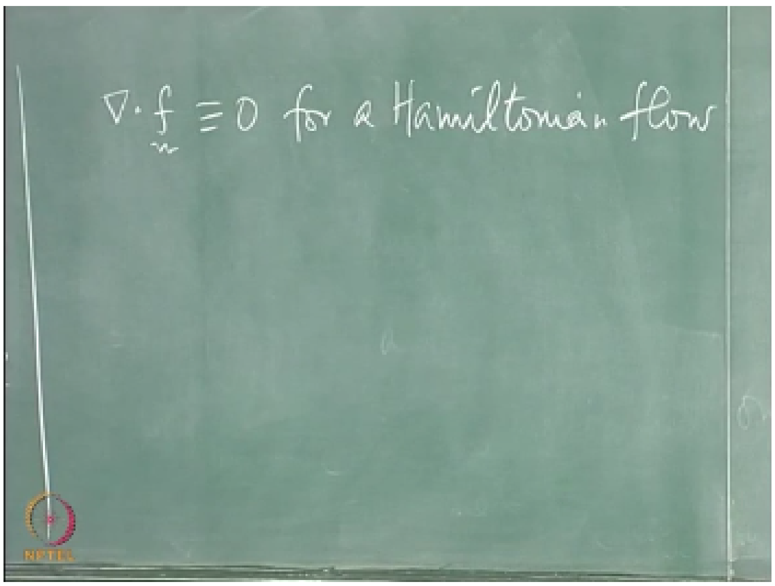

So we also saw that ∇. F is identically 0 for a Hamiltonian flow this implies - that volume elements are preserved as time goes along they may get distorted they may get substantially distorted but the magnitude of the volume of any set of initial conditions in any volume element remains unchanged as time goes on and therefore Hamiltonian flow is like that of an incompressible fluid in phase space, a fact which will have further implications what else can we say about Hamiltonian flows well here is where a very important structure called the pass some bracket gets introduced and let me define this quantity.

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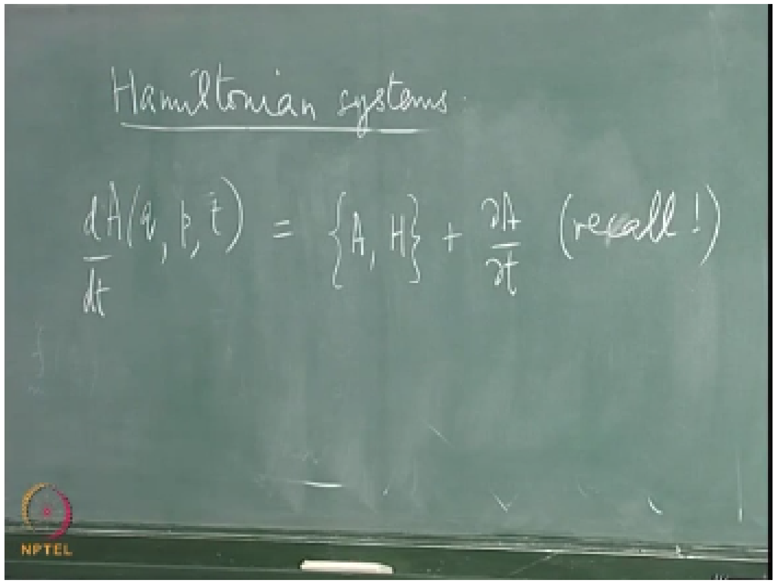

If I took any function a of all the Q's the P's and perhaps of time itself this is just a function of all the dynamical variables as well as maybe it has explicit time dependence and asked what is the rate of change of this quantity as a function of time this becomes equal to we've done this becomes equal to the pass some bracket on the right hand side, and therefore the importance of the Poisson bracket becomes very clear.

Now this is equal to a with the Hamiltonian plus δ a over δ e recall this and this quantity a is a constant of the motion if this right-hand side vanishes and if a does not have explicit time dependence then it said it b is a constant of the motion provided its possum bracket with the Hamiltonian vanishes or it was long commutes with the Hamiltonian. This is the reason why the possum bracket among other is why the possum bracket plays a fundamental role in Hamiltonian systems. Now it could be one of the very dynamical variables the original dynamical variables themselves what happens then of course we have the following relations.

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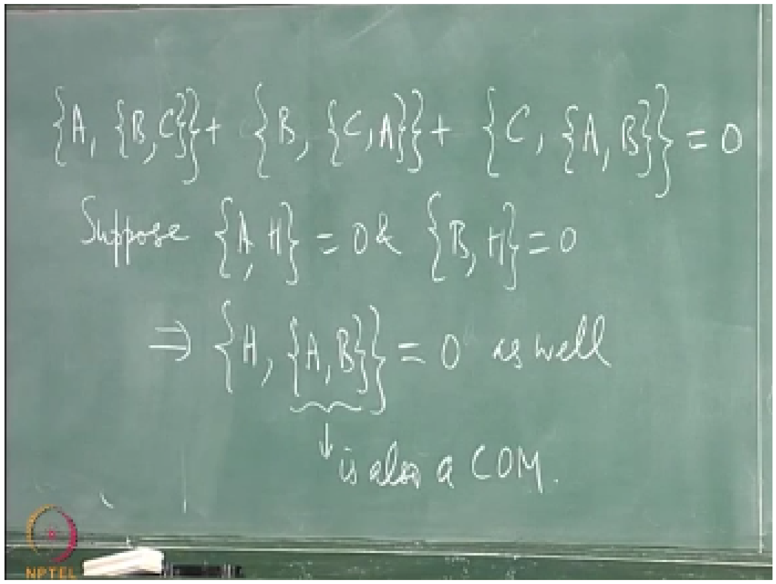

We could compute the possum bracket of any Qi QJ this is easy to calculate and this vanishes as we have seen and that's also kiai PJ from our basic definition of the possum bracket we also have Qi with PJ equal to the Kronecker δ δ IJ and that was the case in which we said a coordinate generalized coordinate Q and a generalized momentum P a canonically conjugate with each other provided that q with that corresponding P has a possum bracket which is equal to unity on the right hand side these were the standard basic canonical possum bracket relations defining a Hamiltonian system.

Now where does this get us how do we go about solving for the Hamiltonian systems how do we try to find the flow itself, one technique would be to use those constants of the motion which we can find by symmetries and I will say a lot more about symmetries and constants of the motion a little later one technique would be to find constants of the motion using the fact that you already have certain constants of the motion and this is based on the epopee identity recall that this identity said that if you had three constants three quantities a B and C then the possum bracket of a VC plus cyclic permutations is equal to zero two other cyclic permutations in which you have B with the possum bracket of C with a plus the third term equal to zero.

Now suppose you discover that a and B are constants of the motion so suppose a with H is 0 and B with H is also equal to 0 what does that imply what would that imply using this fact of C replace it with H what happens to their Kobe identity, let us write this out explicitly plus B let us see a plus C with a be equal to zero suppose instead of C I replace it with H and I tell you that a and B are constants of the motion what would you conclude this would 9/10 takes so this possum bracket vanishes identically, so does that and what is the third thing tell you exactly so it exactly precisely.

So this implies that H with the possum bracket of a with B is also 0, therefore this quantity here is also a constant of the motion this quantity could turn out to be trivial it could turn out to be a constant a numerical constant it could turn out to be the identity function it could be turn out to be something utterly trivial but there are cases where it could turn out to be a non-trivial function of the coordinates and the momenta in which case having started with two constants of the motion we found one more and in principle you could now find further commentator is further possum process on brackets and see whether you can find other constants of the motion.

So that is one possible use of these Poisson brackets to find other constants of the motion, but one should like to have a systematic way to integrate all the Hamilton equations at the same time this is not a trivial task and in general is not doable in general generically it turns out that for Hamiltonian systems with an arbitrary number say F or n degrees of freedom you have basically only one constant of the motion available namely the Ham ill's in itself which implies that the motion can be extremely complicated you do not really have much of a handle on it apart from knowing that for whatever initial conditions you start with the motion is restricted to a 2n - 1 dimensional energy surface very little else can be said.

But there are instances where the equations the entire set of equations is integral and here is one way in which it happens in fact it is the only way in which it happens for this I must introduce the idea of canonical transformations.

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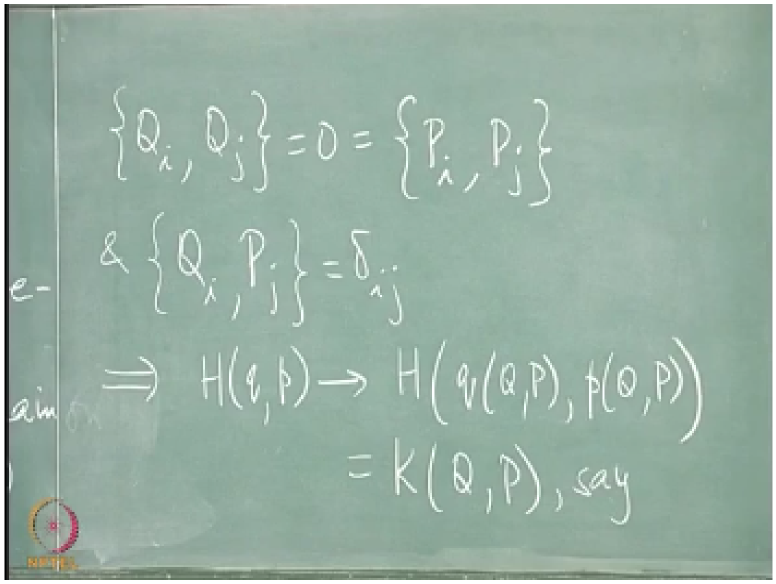

This is a mini subject in itself but we are going to just get a small flavor of it here. So what I am going to say is restricted to certain special cases, but then we will see how far this kind of approach takes us, the idea is the following suppose instead of starting with the coordinates and momenta Q and P I am able to find a change of variables a change of dependent variables from the set QP to a new set of variables and let us call those variables Q, P there are n of these and n of these so the total number of independent dynamical variables doesn't change it is still 2n but these new sets of variables these new variables are functions of the old variables.

And now let us suppose that this change of variables has a very special characteristic namely the acobian matrix determinant or other of the change of variables QP = + 1 suppose at all points where this change of variables is defined these functions of the old variables and these functions of the old variables are such that this 2n / 2n determinant has the value +1 identically what would you say happens to the volume element as written in the new variables it would be the same this would imply the flow is measure preserving volume elements remain an altar.

And suppose further that the change of variables is such that the following special property happens the possum bracket of Qi with QJ is zero as is the possum bracket with of P I with PG and further Qi with PJ = δ I J. In other words not only is the flow not only is a change of variables such that volume elements are unchanged but the possum bracket structure is also unchanged suppose this is so we are able to find such a change of variables notice that since this is plus 1 rather than - 1 not only our volume elements unchanged but orientations are preserved as well.

So elementary parallelograms if we define orientations for them they would also be unchanged under this change of variables, one is then guaranteed if this happens one is then guaranteed that the structure of Hamilton's equations is also unchanged under this change of variables this is not hard to show such transformations are called canonical transformations I will abbreviate them by C T at various points together these properties imply that the structure of Hamilton's equations does not change what does that mean it means that if H of Q, P goes over to a new function when expressed in the new coordinates which means this goes over to H of Q which is a function of the new variables.

So the Hamiltonian when expressed in the new variables is obviously some other function of these new variables.

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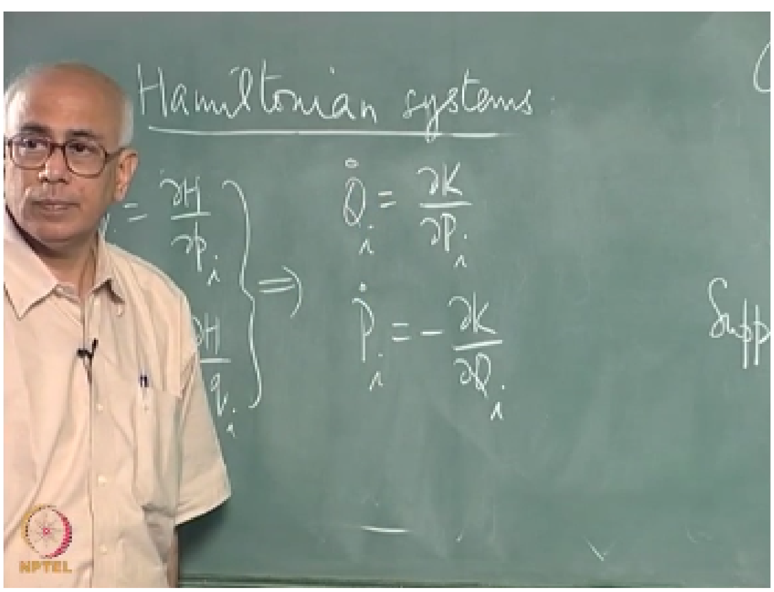

Let us call it k then we are guaranteed that under a canonical transformation if qi dot this δ H over δ P I and P I dot is - δ H over δ Qi this implies that Qi dot is δ K over δ P I and P I dot - δ K, so canonical transformations leave the structure of Hamilton's equations unchanged in other words it takes a hamiltonian flow to another Hamiltonian flow in the new variables. The necessary and sufficient condition for this is that the transformation be canonical in other words that the Jacobian determinant be equal to + 1 and the possum bracket canonical possum bracket structure be unchanged you then guaranteed that these equations do not change in form yet completely unchanged.

Now you might ask what is the use of this but the answer is it might turn out that you find the transformation of variables where some of the coordinates do not appear in this new Hamiltonian if for example capital Q 3 does not appear in K at all it is not a function of capital Q 3 then it immediately implies that P 3 dot is 0 and therefore P 3 is a constant of the motion.

You have therefore found a constant of the motion namely the momentum conjugate to the new coordinate Q 3 if any new coordinate does not appear in K it is called a cyclic coordinate or an ignorable coordinate and the corresponding conjugate momentum is a constant of the motion so let me write that down.

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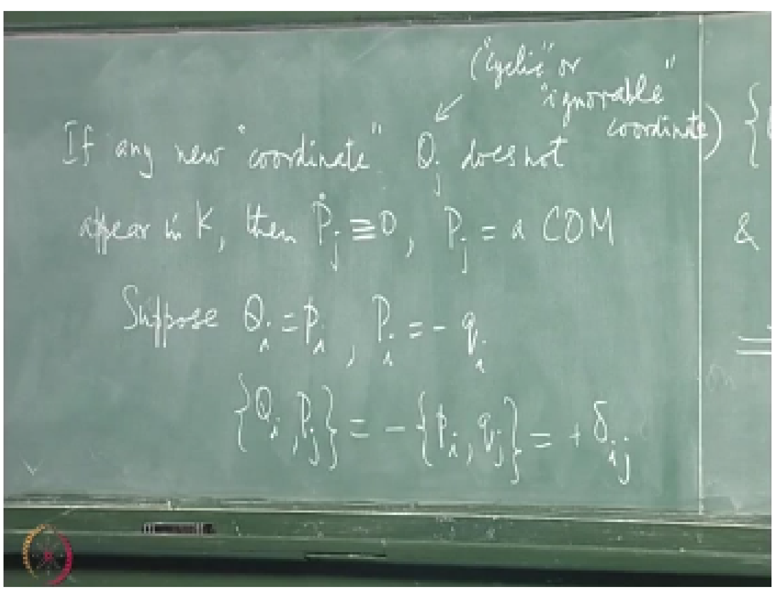

New generalized coordinates QJ does not appear in k then PJ dot is identically 0 PJ equal to a constant of the motion and if it does not appear in K then this coordinate is called a cyclic or ignorable ordinate an extremely happy situation would be one if you found more and more cyclic coordinates. So that this set of equations can be integrated the moment some particular new variable does not appear in K a certain time derivative vanishes and you have discovered a constant of the motion.

So one very happy situation would be if two and - one of these was zero and then of course the matter becomes completely trivial the vector field is rectified at once only one derivative is nonzero and the rest are all zero on the other hand that is a very rare situation hardly ever happens more common is a situation where by a clever choice of variables n of these equations have 0 on the right hand side and only n of them have nonzero quantities.

On the right hand side this tells you how such systems of equations are integrated completely in certain situations and this is called a transformation to action angle variables and we will talk a little bit about this to show you how Hamiltonian equations are integrated we need a criterion for it which I will state without proving here and then we will apply that criterion to discover how to solve this set of problems completely how to integrate these Hamiltonians.

Again I point out that this is not the generic situation most Hamiltonians are not integral in the sense that we do not have enough constants of the motion available to us other than they Hamiltonian itself perhaps a few more right in the beginning I pointed out that in the Newtonian problem of a set of n particles interacting with each other by forces by pair wise forces directed along the line joining any two particles that system has at least in no relativistic mechanics it has basically ten constants of the motion Galilean constants of the motion whereas the phase space itself is 6n dimensional this is the generic situation the typical situation but there are very important special cases where the set of equations can be integrated completely then of course the problem is solved in some sense.

So we will take it from this point and try to understand how this magic and what the necessary and sufficient conditions for it are in order to integrate the entire set of equations just before I do that here is a simple example of a canonical transformation which always works suppose Q I is just P I and P I is - Q I, I simply call these by new labels what happens to this what is that equal to it is equal to - P I with Q J but this is equal to plus δ I J because recall that the Poisson bracket of B with a is - the possum bracket of a with B because of anti symmetry and there is an extra – sign.

So this set of variables is canonical they have the canonical possum bracket relations and what happens to the determinant of the transformation what is the determinant of this transformation in this case, so δ Q comma P over δ this Y be determinant what does that become, how do you find that what is this a matrix equal to this is the determinant of a certain matrix and what is that matrix it clearly has zeros in the first block what's in the next block. So this matrix here this thing here is the determinant of a certain matrix and what's that matrix it clearly has zeros everywhere here and when you start writing those elements down what do you get from this just the unit matrix.

So just the unit matrix here you get - the unit matrix here and zero here and of course you recognize this matrix it is J itself, so this is equal to determinant J go-to plus 1 since it is an even dimensional matrix its determinant it is the only modular the determinant is plus 1 therefore this is a canonical transformation of course you do not gain anything much by it but it helps you understand that this division of the variables in two coordinates and momenta while we started with mechanical examples where you can clearly distinguish between a coordinate and its conjugate momentum that distinction from a mathematical point of view in Hamiltonian systems is quite artificial there are two sets of variables no question about it.

But which ones you call the coordinates and which ones you call the momenta is largely irrelevant because you have a canonical transformation which is guaranteed to let a Hamiltonian flow go into another Hamiltonian flow in which it turns out that the original coordinates and moment I have simply exchange roles but they appropriate - sign there it is as valid a system as the earlier one it is as valid in other words if you solve the original thing you have solved this one too so as it is it does not mean anything it does not tell you anything but it gives you a certain symmetry of Hamiltonian systems which is going to be crucial in solving such Hamiltonian systems.

So it is trying to tell you that this flow in this 2n dimensional phase space is a very special kind of flow we saw that volume elements were preserved many other things are preserved as well in this flow and the natural language in which one writes this down is that of differential geometry which we are not going to assume here. But let me say it in words, so that you begin to see how men strained the Hamiltonian flow is what is constant is the following.

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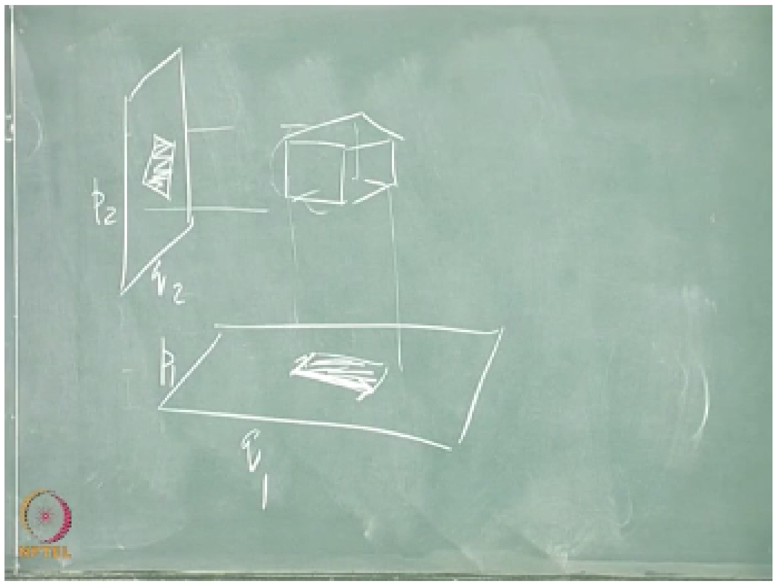

If this is a volume element in phase space and let me draw it simply to fix our ideas as some kind of three-dimensional object in two n dimensional phase space I have this volume element of initial conditions and as time goes on this element flows in phase space like an incompressible flow it might get distorted squeezed out in certain directions compressed in other directions but the magnitude of this element does not change at all the volume does not change and I projected this element I project a shadow of this on to a particular Q P space.

So let us say this is Q 1 P 1 this plane I project this volume element at any instant of time and I get a little shadow patch of this kind I project it on to another plane which is the Q 2 P 2 plane and let us say I get a shadow patch of this kind I add up all these shadows the areas of all these shadows and that remains constant as a function of time you are guaranteed that under a Hamiltonian flow the sum of projections remains constant but what is this sum of projections if this is the direction along Q and this is the direction along P and this is δ q SE q 1 and this is δ p 1 then clearly this parallelogram has an area which is the magnitude of δ Q 1 cross δ P 1 regarded as vectors this cross product and you sum all these areas and the answer is guaranteed to remain constant.

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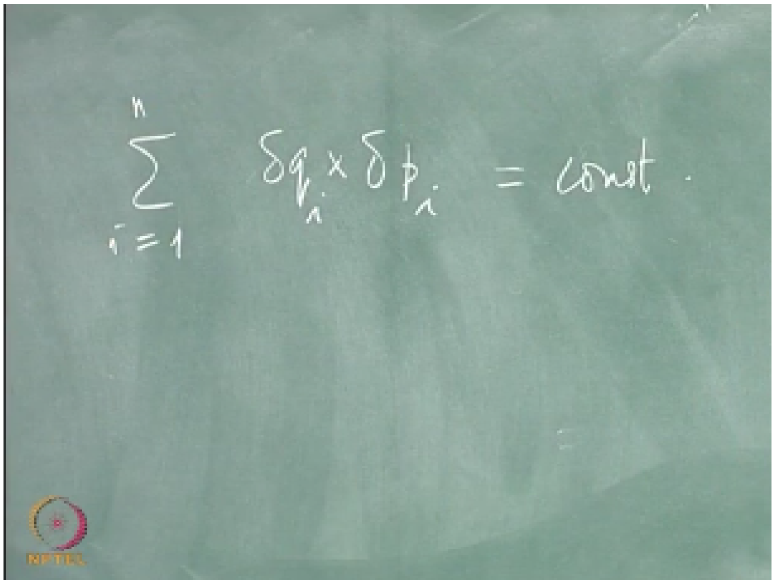

So schematically it says that under the hamiltonian flow a summation from I = 1 to N δ Q I δ P I and this is a cross product in the sense of a parallelogram the area of a parallelogram being written as a cross product of the two vectors defining the sides of it that quantity is equal to a constant this immediately implies that a certain quantity called a for form but it is actually for differentials of this kind for infinitesimals also remains constant and then something which involves six of these differentials etcetera till finally the entire product of all to n δ Q's and δ PS also remains constant which is Lewis theorem.

So that theorem actually follows from the invariants under the flow of this area this quantity here one can show this rigorously using simple differential geometry but I want you to appreciate the fact that the flow in phase space is what you have to think about and it is a very special kind of flow it has there all counts they are not constants of the motion in the sense of quantities which are dynamic of functions of dynamical variables being constants of the motion these are infinitesimals.

So it says that this fluid element flows in such and such a manner it flows such that all these magical properties happen for each fluid element they are not constants of the motion in the sense of functions of all the Q's P is being invariant under the flow those are actual integrals of the motion these are little infinitesimal differentials and various combinations of them remain invariant the last of which there are n of these and the last of this is in fact the volume element in phase space.

So you could ask the converse question given a flow in which these various differentials are constant what kind of flow must it be and the answer is it must be a Hamiltonian flow, in other words the two n dimensional vector field F is given by the symplectic gradient of some scalar function called the Hamiltonian of the flow. So there is a very rich and intricate mathematical structure associated with Hamiltonian flows we do not have time to touch upon this and it is outside the scope of this course.

But it is important to remember that such a thing happens that this exists let me come back to canonical transformations and point out what happens in a canonical transformation and why it is so important.

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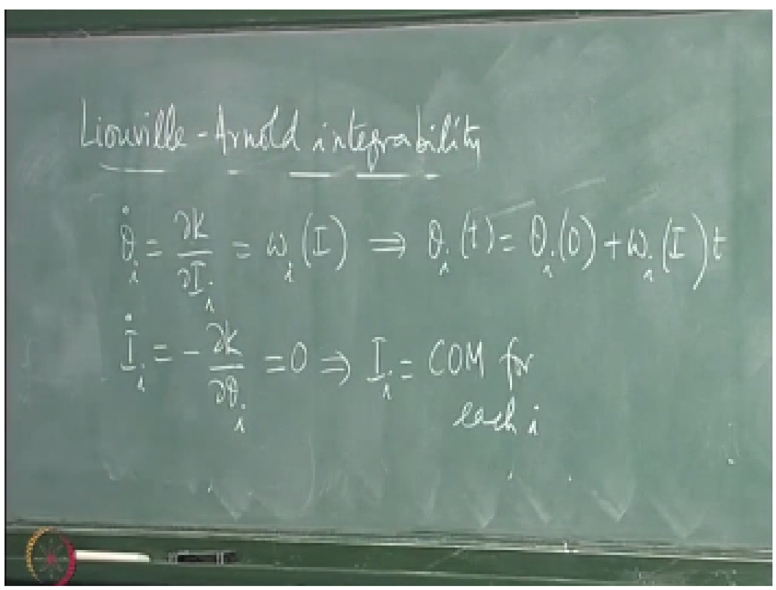

And this goes under the name of integrability Liouville-Arnold integrability of hamiltonian systems, and the statement is the following it says if there exist n constants of the motion n Co n F 1 up to F n that are functionally independent of each other and that are in involution the phrase in involution implies that the possum bracket of any of two these quantities is 0 identically in other words this implies if I possum bracket FJ is identically 0 then the set of equations can be integrated this is a loose way of stating the Liouville-Arnold criterion but I will explain what this means term by term.

So what is it really telling you recall that the flow is in two n dimensional phase space in order to integrate this set of equations and specify what the future values are given initial values you need to have 2 n - 1 independent constants of the motion in a 2 n dimensional space the mutual intersection of these 2 n - 1 surfaces would in principle give you the phase trajectory for any given initial condition but for Hamiltonian flows the following magic happens the Liouville-Arnold criterion says you do not need to have 2 n - 1 constants of the motion if you manage to find n of them that suffices.

But the catch is these n have to be compatible with each other in a certain sense and that sense is that they must have vanishing Poisson brackets with each other they must be in involution with each other it is, now we have to say what's meant by can be integrated the answer is that this set of Hamilton's equations explicit solutions to them can be written down in principle for all time in other words the full set of equations can be explicitly integrated not numerically solved. But explicitly integrated and solutions can be written down in explain at forum this is what I mean by explicitly can be integrated this of course is true in principle in practice there are many slips between the cup and the lip and we will see what the implication is what is meant by these constants of the motion and how one goes about it.

Now the question is this a matter of is this a theoretical limitation is this a constructionist kind of constructive statement in other words is there some constructive way of finding these F's and doing the integration what kind of statement is this. But right now when I make it this statement says if there exists these constants of the motion you are guaranteed that a solution exists for the problem that is the level at which this statement is being made it does not quite tell you how to find these constants of the motion it does not also tell you having found them what do you do next how do you go about integrating.

The second question will answer right away the first question is much harder what is the guarantee that a given system has or does not have these n constants of the motion that has to be examined on a case-by-case basis there is no general way of stating when a given Hamiltonian system is integrable that is the first part but now let me let me complete that having said that I should also point out that the set of Hamiltonian systems for which this can be done is quite restricted.

But it is a very important set and the importance lies in the fact that for cases when it is not integrable you can do perturbation theory you can develop perturbation methods for answering questions of what happens to Hamiltonian systems where such a set of constants of the motion does not exist, yeah you, yeah yes good question. So can we say that the problem is actually unsolvable in the sense that you cannot the answer is yes this is a necessary and sufficient condition for the global integrablilty of Hamiltonian systems.

So if you cannot find these or you can prove that such a system set of constants of the motion does not exist in a particular case you are assured that it cannot be integrated for all the initial conditions. So it works the other way too this is a necessary and sufficient condition for integrability in the sense which we will elaborate upon yeah yes we will come back we'll come back to this and I will give you examples where it is integrable we will see specific examples where the system is integrable and examples.

But you cannot find these constants of motion they don't exist and therefore the problem is not by this theorem cannot be found as a global solution cannot be formed we will come back to this but we take this step by step this is the first step first step this is a necessary and sufficient condition. Now I have to explain what I mean by functionally independent of each other we know that the Hamiltonian is always a constant of the motion, so the square of the Hamiltonian or e to the power of the Hamiltonian or the Hamiltonian cube plus the Hamiltonian squared they are all constants of the motion.

But they are not functionally independent of H this must really be independent of each other in other words none of them should be writable as some function of the rest in which case are completely functionally independent. The second point is there are certain technicalities here what kind of constants of the motion are permitted what kind of functions of the dynamical variables are permitted as it stands here these are assumed to be isolating integrals they are supposed to be single valued functions analytic in some sense if they have singularities then of course you again have problems and then these theorems don't hold good in the form in which I have written them now.

So subject to those caveats which we will take up as we go along one by one the first statement is and this is what one has to pause and marvel at that it is sufficient to know n constants of the motion and the entire set of equations is integrable in the following sense, what the theorem actually says is this can be integrated this means there exists a canonical transformation exists a canonical transformation to a new set of variables for which the conventional notation is theta 1 up to θ n I 1 up to I n we call with said new variables capital Q's and capital PS.

But these very special sets of variables which you have when the system is integrable are denoted by θ’s and I’s these are called angle variables and these are called action variables and it is a canonical transformation. So we are guaranteed that possum bracket relations are valid we guaranteed that the structure of Hamilton's equations is unchanged such that the Hamiltonian h QP goes over to a new Hamiltonian which we denoted by K which would have been a function of all the θ’s and I’s.

But the magic is does not depend the new Hamiltonian does not depend on any of the first n variables the theta variables depends only on the so-called action variables which are collectively denoted by I here we are guaranteed this that there exists such a set of transformations, notice that since it is a canonical transformation we also have the relations theta I theta J zero is PI with IJ and theta I with IJ is δ IJ.

Because it is a canonical transformation and you are guaranteed that the structure of Hamilton's equations is unchanged and what does that imply if this magic happens then Hamilton's equations in the new variables become theta I dot is δ K over δ I sub I and I . = - δ K over δ θ π for each I from 1 to n, but this function does not depend by assertion we have already asserted that it does not depend on any of the angle variables therefore these quantities vanish identically all n of them which imply immediately which implies that I I to constant of the motion for each I.

So we have trivially integrated n of the two end differential equations of the Hamiltonian system but we can go further once this is, so this is a function of all the action variables and it is partial derivative with respect to any one of them is also some function ω let me call it ω sub I of all the action variables which I collectively write as I because this is a function of I alone just the action variables and it is partial derivative is also a function of the action variables alone.

But this is now a constant of the motion because the I is our constant for any given set of initial conditions and it says the rate of change of any θ is a constant which implies at once that θ I of T is the initial value this + ω I which is of course a function of all the action variables multiplied by

T and therefore you affected the actual explicit integration of all Hamilton's equations all the 2n equations have been integrated out completely.

Now let us take stock of this what does this mean well if that magic applies if the Louisville Arnold theorem applies the criterion applies and we are able to find these n constants of the motion in involution then the assertion is that there exists a canonical transformation does not tell you how to find this canonical transformation but you are assured that there exists such a transformation such that the new Hamiltonian has only dependence on the action variables and all the angle variables become cyclic coordinates they are ignorable coordinates which at once means that all the action variables are constants of the motion.

And the angle variables have a simple linear dependence on time what determines the value of all these constants of the motion, the numerical values what determines these numerical values the initial conditions you have to and pieces of data initial data n of them go in determining these end constants and the remaining n go in determining the initial values of these angle variables. So what I have it achieved in the 2n + 1 dimensional extended phase space you need to have 2n constants of the motion to specify the motion of which at least one must be time-dependent.
So that there is motion as time goes on n of them are these action variables they are true integrals of the motion the remaining are constants of integration these quantities this immediately implies that θ I - ω I T is a set of n time dependent constants of the motion which together with these first integrals these guys specify 2n some of which are time dependent to n constants of the motion in the 2n + 1 dimensional extended phase space and the motion is totally determined in terms of these constants of the motion.

So it is in this sense that the liberal Arnold criterion tells you the system is integrable completely integrable, how do you find the original p's and q's once again remember this was a canonical transformation with determinant equal to + 1 the determinant is not 0 and therefore in principle the transformation is invertible given the Thetas and the eyes I should be able to go back in principle to the P's and Q's and they may themselves have very complicated motion but dependence is on time but the angle variables are very simple motion very simple dependence is on you also begin to see that this kind of integrability is applicable in general to bounded motion in Hamiltonian systems.

Generally periodic motion or motion which is comprised of many characteristic frequencies which may or may not be commensurate with each other, so the motion may not be strictly periodic but may be quasi periodic just like if you took two simple harmonic oscillators at right angles all of you are aware that if the frequency ratio of these two oscillators is a rational number, then the system comes back to its initial value initial position and the motion is truly periodic on the other hand if the frequency ratio is irrational then the motion is not periodic in all the four dynamical variables.

But only so in two of them at a time Q 1 and P 1 and Q 2 and P 2 separately and you get these complicated figures when you project this motion on to the Q 1 Q 2 plane. So in the same sense this kind of integrability for bounded Hamiltonian motion implies that in general the motion is quasi periodic with a characteristic set of frequencies which are one might call the omegas and cells but the important thing to note is that these omegas depend on these constants of the motion.

And therefore if you have different constants if you start with different initial conditions then in general the characteristic frequencies of the system also change this is a feature which is not present in the harmonic oscillator or linear problems of this kind, where the frequencies are given to you right in the beginning here the frequencies are energy dependent their initial condition dependent in general but the motion is solved it is completely solved for.

Now you could ask is it so that these eyes are the f1 f2 f ends themselves is that true the answer is after all they are in involution with each other as well the answer is in general these action variables are not the constants of the motion f1 through FN themselves but they be constructed from them you can construct from these constants f1 through FN you can construct appropriate action variables which would then be canonical variables in the sense that those possum bracket relations are valid and that is the crucial point this thing has to be valid here.

Now let us point out one final thing what is the meaning of having two functions in involution with each other what does this imply and this matrix J we talked about gives us a very simple way of understanding this geometrically, in fact most of Hamiltonian mechanics is something called simplistic geometry it is understandable in geometrical terms in phase space we already saw that the flow itself was always along the level surface h equal to constant.

Suppose I take two quantities a and b and take their possum bracket and ask can I write that in compact form the answer is yes when it is done in a very revealing form.

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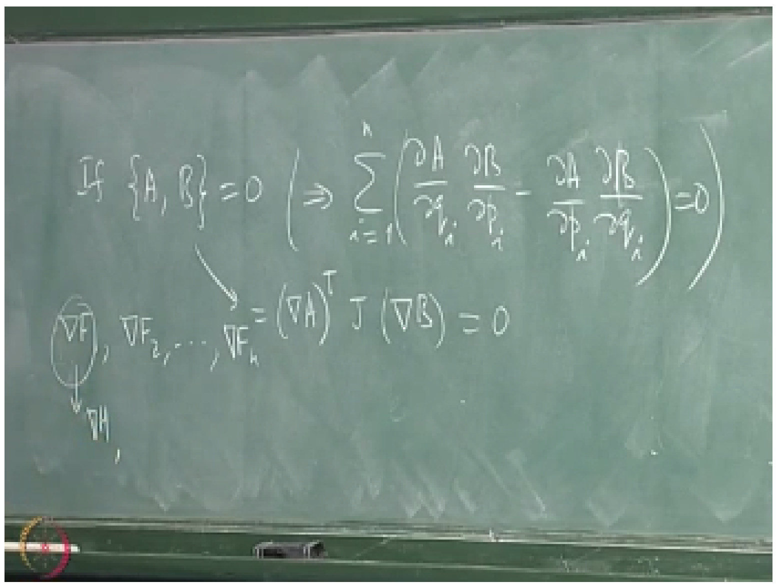

So if a possum bracket be equal to zero this of course means that summation I = 1 to N δ a over δ Qi δ B /r δ P I - δ a / δ P I δ Qi is 0 this is what it implies using this matrix J we can write this in an extremely simple form this quantity is also equal to clearly certain derivatives are acting on B, so this is also equal to the gradient of B in phase space with these two and derivatives δ / δ q1 up to δ / δ PN this thing is a certain vector and can be written as a column vector δ B / δ Q 1 all the way down to δ P over δ P n this vector with AJ here and the gradient of a here with a transpose to show that this is a row vector δ a / δ q1 horizontally all the way up to δ a / δ P and the transpose of this is a row vector here times the J matrix times δ B and that is what the Poisson bracket is.

So it says this is equal to 0 if the possum bracket vanishes, so it is as if the gradient of a and the gradient of B are not quite normal to each other which is what would happen if the dot product vanished but in a symplectic sense they are normal to each other because once you put the j in between here this is 0. So what does that tell us?

Whenever you have a constant of the motion it implies that the motion is on the level surface of this function because we saw that a constant of the motion implied that it does not change as the flow occurs and now in addition to the level surface of the Hamiltonian if you have other constants of the motion then the flow is on the level surfaces of both these flows and the fact that this dot product pseudo dot product vanishes implies that these vector fields gradient of a gradient of B and so on are in some sense independent of each other they are linearly independent of each other.

So we really have a mathematical structure for Hamiltonian flows if the Liouville Arnold criterion inside is which says that these n quantities gradient f1 gradient f2 gradient F n of which the first could be chosen to be the Hamiltonian itself. So this is δ H these n vector fields form a kind of basis on the surface on which the flow actually happens that restricts the kind of surface you could have.

So very special thing happens once this magic of integrability occurs and in some sense the flow is reduced from a to n - 1 dimensional energy surface to an N dimensional object in the action angle variables space that n dimensional object is the generalization of a torus it is an in torus, and we will see why this comes about and we will also see why these are called angle variables and what the significance of action variables is and this will also help us explain a mystery which you might have come across when you did elementary quantum mechanics.

When you are told that integral PD Q = n times Planck's constant and this was the Bohr quantization rule this will help us get a handle on why that happened why this thing happened and incidentally Planck's constant as the physical dimensions of action which is energy multiplied by time and that is exactly what the reason why these are called action variables and it is action that is quantized in Planck's constant in Bohr quantization in the original quantization.

So we also have a connection with semi-classical quantization at the same time and all this follows from the structure of integrable hamiltonian systems, so we will come back to this and I will get back here and explain what canonical transformations to what these transformations mean really what kind of system what kind of geometry is implied by these transformations and then we will look at special examples of various cases including some problems we know how to solve already like harmonic oscillators and collections of oscillators, because the lessons we learned there are generalizable to more difficult cases this is what we will take up next.

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