Indian Institute of Technology Madras
Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 5
Hamiltonian dynamics (Part I)

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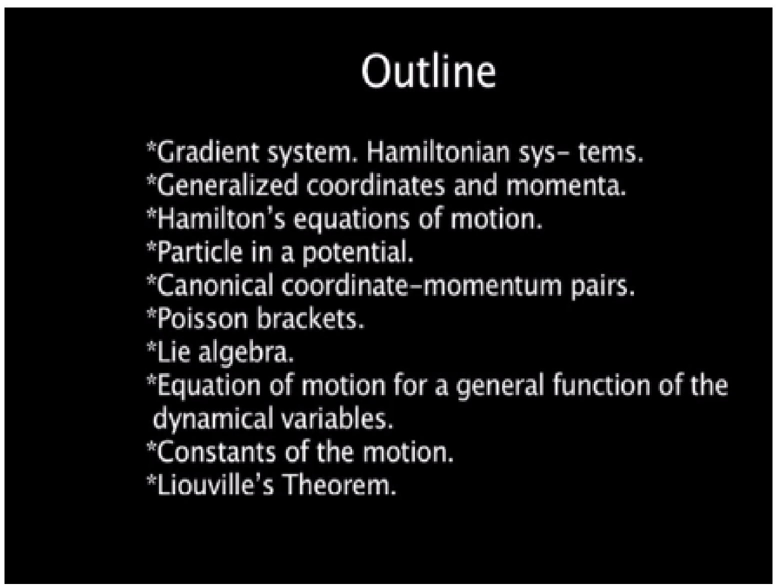

Let us spend a little time today digressing to a topic in dynamical systems more properly taught in courses on physics namely Hamiltonian systems and I would like to bring out to you how special Hamiltonian systems are and we studied Hamiltonian systems from the point of view not so much of mechanics as of our general study of dynamical systems and nonlinear dynamics before we talk about Hamiltonian systems there is another class of dynamical systems which I would like to dispose off or at least mention very briefly.

In passing and perhaps we might come back to such a system little later in the course and this is a class of dynamical systems called gradient systems.

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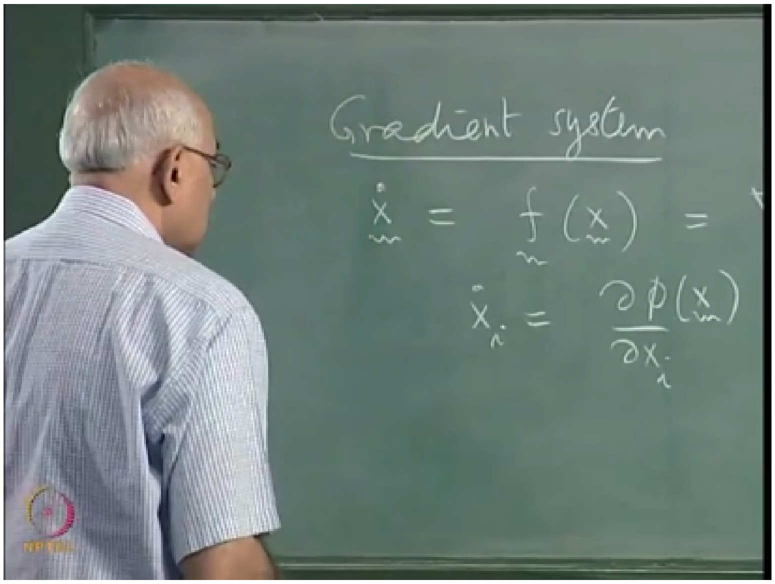

We call our particular kind of dynamical system we focused on namely those which satisfy autonomous couple differential equations of the form f of x where F is a prescribed vector field it is a function which has really got n components in n dimensional phase space F1 F2 F3 up to F sub N and these are functions of the coordinates of the variables x1 x2 up to x sub n therefore to specify the dynamical system you have to specify a vector field at every point in some region of phase space a gradient system is one where.

But this vector field has a very special form if this vector field is of the form the gradient of some scalar function of the variables x I call this a gradient system what's special about it of course is the fact that instead of specifying a vector field which means all its components n functions f1through to xn the F sub n we need to specify just a single scalar function ϕ and everything else all the components of F are determined in terms of this single scalar function ϕ, so clearly it is a very special kind of dynamical system.

Not all dynamical systems can be written in this form but those that can clearly enjoy a little superior status among dynamical systems because a single scalar function suffices to tell you the flow at all points, now of course the component of this if I take the i^{th} component this implies that xi. $\nabla\phi$ / ∇x,I and ϕ is a function of the variables x that is what is meant by a gradient system if you give me the function ϕ and I set ϕ of x equal to a constant it would in general be a surface in this n dimensional phase space, so if schematically this is what the surface looks like at a particular point x and this corresponds to the surface ϕ equal to a constant c say then what is the direction of the gradient of ϕ at this point the gradient is defined.

As the normal derivative the gradient of any scalar field is the normal derivative it is the direction in which this function ϕ has the maximum rate of change that direction along which it changes most rapidly therefore it is normal to the direction of the level surfaces of ϕ, so if I equal to constant is a surface the level surface ϕ equal to constant then this is the direction of the gradient of ϕ at that point which means the flow lines or the phase trajectories of a gradient system are always normal to the level surfaces ϕ equal to constant at each point.

That is a useful factor now e.$_0$ come across too many gradient systems they are somewhat rare in a certain sense but when they when they do occur then it is very useful to give a geometrical interpretation for the flow lines in this in this sense namely the lines ϕ equal to the flow lines are normal to the level surfaces of the scalar function phi single scalar function ϕ this is another very important class of dynamical systems called Hamiltonian systems for which once again a single scalar function determines the vector field F on the right hand side and now let me define a Hamiltonian system.

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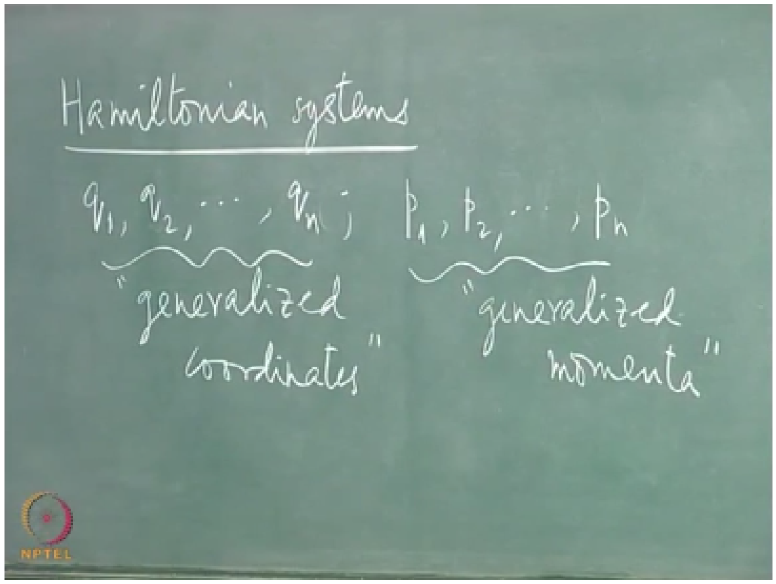

These are systems which are always even dimensional in a very specific manner they described by a set of dynamical variables which occur in pairs therefore the total number of variables is always an even number the first n variables are generally denoted by the Q's q1 q2 upto qn and these are generally called usually called generalized coordinates and along with them go a pair a set of variables p1 p2 up to pn which are called generalized momentum therefore the full set of dynamical variables x.

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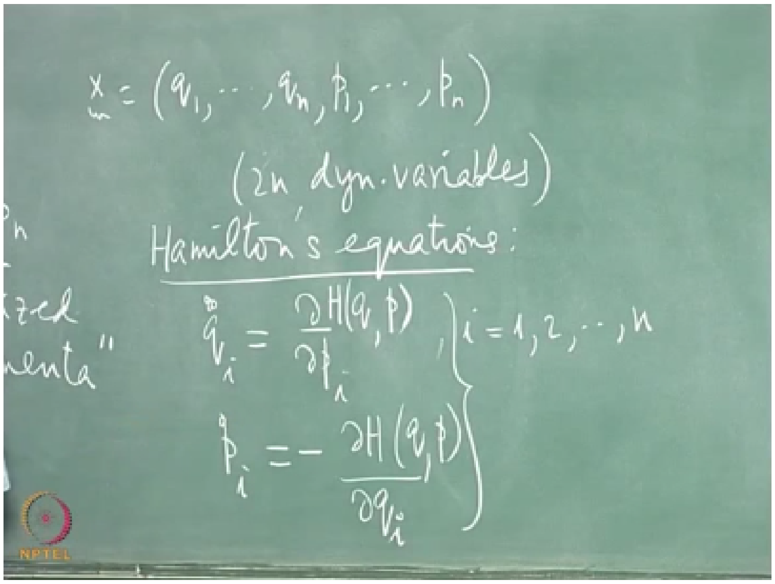

Is a set of two and variables q1 to qn p1 to pn to n dynamical variables the phase space of a Hamiltonian system is always even dimensional it is conventional to call Q's the generalized coordinates degrees of freedom, so this system has n degrees of freedom and it has to end dynamical variables something very special connects each Q to its corresponding P called its conjugate momentum P and that is a structure called the possum bracket and I will define this as we go along very shortly for the moment.

We will use this terminology which says that the Q's and PS are conjugate to each other q3 is conjugate to p3 q 4 is conjugate to p4 and, so on the more strict term is canonically conjugate in a standard manner and we will explain as we go along what we mean by this the equations of motion are of particular interest they call Hamilton's equations of motion and the most common occurrence of Hamiltonian systems is a set of particles moving for example under the influence of forces.

Which they exert on each other conservative forces which they exert on each other or a set of particles moving under the influence of some conservative external force or both mechanical systems very often turn out to be Hamiltonian systems in the absence of friction in the absence of dissipation, so Hamiltonian systems will in the sense we understand it will be conservative dynamical systems with the special cases of conservative dynamical systems and the equations of motion which correspond to our dynamical equations go by the name of Hamilton's equations.

And they read as follows qi . is ∇ /∇ Pi of a certain scalar function H of all the Q's and PS in general and let me abbreviate that by just writing it as q,p that stands for all two and variables in general and this is true for each eye from 1 to N qi. and the rate of change of Pi is again given by the partial derivative of the Hamiltonian but with respect to the conjugate generalized coordinate qi with a minus sign and this set of equations is valid for each eye it is not a gradient system because the gradient system would imply.

A gradient system would imply that each time derivative of each dynamical variable is the partial derivative of some scalar function with respect to the same variable, but that is not happening here the time derivatives of the Q's are partial derivatives with respect to the conjugate P's and vice-versa but with a minus sign and this minus sign is crucial it is very, very important this is the general structure of equations when you look at particles moving in a potential for instance yeah there is a question.

What exactly the question is where do these things come from this is the definition of a Hamiltonian system but as soon as I finish defining this I will go back to ordinary mechanics and show you that ordinary mechanical problems in the absence of frictional forces these are the defining equations these are Hamilton's equations they define a Hamiltonian system the original motivation was of course the observation that Newton's equations of motion in the case of conservative forces are essentially Hamiltonian equations and this is a generalization of the idea.

Which already one already has from Newton's equations of motion, so let me do this in a formal way and then we go back in a minute and see that ordinary mechanics is indeed a special case of Hamilton's equations of motion with a specific prescription for the Hamiltonian function but right now I am doing this in a very abstract way I am simply saying that if a dynamical system is even dimensional and has this kind of structure with certain additional properties which I am going to write down regarding the way the Q's and PS are related to each other then it is called a Hamiltonian system okay.

So let us step back and ask is this a gradient system no not as it stands but you still have this wonderful property that the entire vector field on the right hand side is actually defined in terms of determined by a single scalar function the Hamiltonian function that is going to cause a great deal of creator a lot of simplification it is going to lead to a lot of simplification and very interesting properties, now let us go back and ask what happens if I take a single particle which is non relativistic.

Ordinary Newtonian mechanics moving in some conservative potential or field of force what would you say are the equations.

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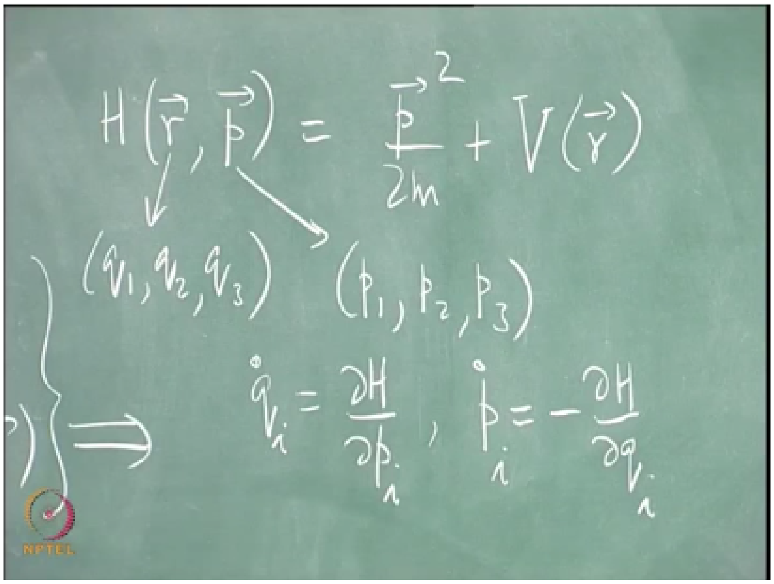

Of motion a single particle moving in some potential I could write down Newton's equations of motion and the way to write it down in our language is to say that the mass times the velocity which is dr/dt is the momentum p and the rate of change of momentum dp over dt is equal to the force on the particle if this force is time independent no explicit time dependence and depends only on the instantaneous coordinate of the particle then it's some F of R and if this force is conservative which means it is the gradient of some scalar potential which has the connotation of a potential energy.

Of the particle then this is equal to minus the gradient of a potential V which is a function of the coordinates of the particle this is what you would write down as normal equations of motion I have just written Newton's equations down but as a set of coupled equations the r/ dt is p/ m I could bring this to the other side and dp /dt is some function of R it certainly is a dynamical system in the sense we understand dynamical systems what is the dimensionality of the phase space here it is six dimensional because three coordinates and three momentum variables.

So six coordinates and it is exactly the structure that we had x. is some f of x some vector field on the right hand side, so we have written it as two coupled vector equations three dimensional vector equations in this fashion the question is it of that form is it of this form at all can we write it in the form of Hamilton's equations by hindsight we know we can because we know that in ordinary mechanics with conservative forces the Hamiltonian of the particle as the connotation of the total energy of the particle.

Which is the sum of it is kinetic energy and its but initial energy, so it right H as a function of r and p and these would be q1 q2 and q3 in this case and this the components of p would be p1 p2 and p3 these are Cartesian components then this is equal to p^2 over twice the mass of the particle plus the potential V of r and if I call these components q1 q2 andq3 and similarly this has components p1p2 and p3 it is immediately clear that these equations here really imply that qi. is indeed equal to ∇ H /∇ pi.

Because if I want to find out what d q2 /dt is we know from this set of equations here that d q2 / dt is p2 over m and that is precisely ∇ H /∇ p2 and it also tells us that pi. is - ∇ H / ∇ qi since the potential energy depends only on the coordinates when you differentiate with respect to the coordinates in Hamilton's equations you end up with the corresponding component of the force so this is an example of a Hamiltonian system and the way I have defined it there in general it generalizes the same idea to n degrees of freedom and the simplest way of thinking.

Of a Hamiltonian system is to look at the special case of a set of particles moving in a conservative field of force and that is an example the standard example of a Hamiltonian system there are many other Hamiltonian systems as well, but that is the simplest example so Newton's laws of motion for conservative systems are definitely Hamiltonian the prescription for writing the Hamiltonian down in conventional mechanics is fairly straightforward you start with the Lagrangian of the particle and then you make a certain transformation.

And go over to the Hamiltonian and generally in elementary problems it turns out to be the sum of t + V it is the sum of T and V but T is a kinetic energy and V is the potential in that is the way we have written it down here but it is much more general than that this CAF folding is not necessary a Hamiltonian system is defined by the set of equations Hamilton's equations that I wrote down the general equations together with a certain connection a certain structure called the Poisson bracket relationship between the Q's and PS and let me define the Poisson bracket.

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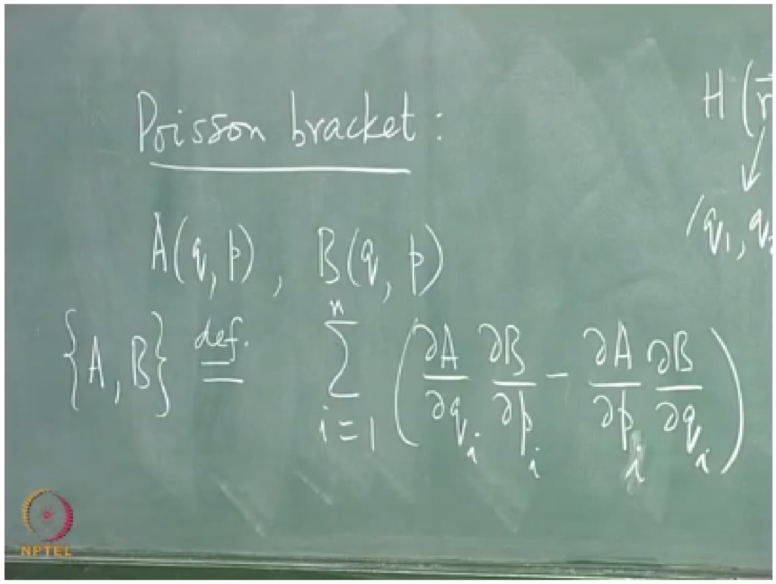

Given the dynamical variables q 1through qn and p1 through pn if I look at any function of these dynamical variables let us call them a of q and p and any other function of all the dynamical variables B of Q and P then the Poisson bracket of a with B is denoted by curly brackets A, B and by definition it is equal to a summation over all the degrees of freedom 1 to N δ a / δ qi δ B/ δ pi minus δA / δPi this is the definition of the Poisson bracket of a with B we need to understand a little more about the structure involved here.

But it is these partial derivatives in this particular combination if the Q's by assumption the Q's are independent of each other and the peas are independent of each other then.

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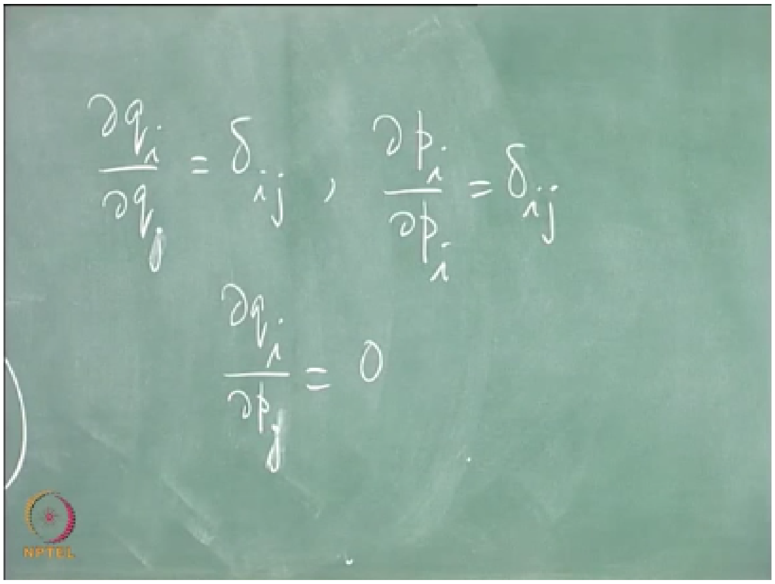

It follows if these are independent dynamical variables it follows that $\delta q_i / \delta q_j$ is 0 unless I equal to J in which case it is equal to1 therefore this is equal to the δ_{ij} similarly $\delta p_i / \delta p_j$ is δ_{ij} because the ps are independent of each other unless I equal to J in which case it is identically equal to 1 and the Q's and ps are independent dynamical variables, so $\delta q_i / \delta p_j$ is always equal to 0, so you have two sets of dynamical variables half of them are regarded as generalized coordinates.

And the other half as generalized momentum they are independent dynamical variables you need all of them to describe the dynamical system, so this is not surprising and the fact that this relationship or that holds code simply says that different Q's are independent variables you need that many independent variables to describe the system once you have that you could ask what is the Poisson bracket of particular Q with a particular p, let us figure that out.

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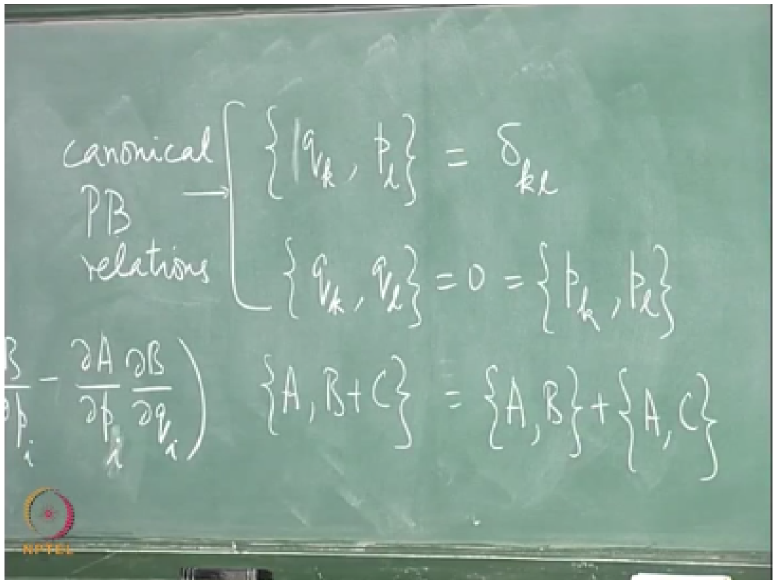

So what is the Poisson bracket of qk with pl for instance what is this equal to all you have to do is to plug it in here and do a summation over I and compute what the answer is using the statements, we had earlier about the P's and Q's so what do you get well this is going to become a delta ki and this is going to become a δLiI so it is zero unless K is equal to I and this term becomes zero unless L is equal to I, so the answer is δ kl because this term would anyway be zero if you differentiate the Q.

With respect to a P or a P with respect to Q the answer is zero and therefore it is equal to the Canonical δ what is the Poisson bracket of QK with QL both these quantities are, now Q's in the numerator and at least one of these will vanish therefore this is zero and the same is true for pk this set of relationships between the generalized coordinates and the generalized moment a of a dynamical system of a Hamiltonian system are called canonical Poisson brackets, so these are the canonical Standard Poisson bracket relations you might ask why am i defining a Poisson bracket.

Why is this what is the use of this and we will see very shortly what the practical use of this is but meanwhile this relationship here this way of defining Poisson brackets has got mathematical structure to it very deep mathematical structure notice it is a bilinear operation in some sense you give me two functions A and B of the dynamical variables of a Hamiltonian system and I do something which involves both of them it is a bilinear structure in general the right hand side is some other function of all the dynamical variables.

And it is also got certain interesting properties of the following kind A with B + C if you took two functions the sum of these two functions as the second member of the Poisson bracket it is quite clear this is equal to A with B plus a with C it is immediately obvious that this is, so if I multiply B by some constant which is independent of all the dynamical variables then of course the constant comes out and a with KB is equal to K times a with P but K is some constant and this quantity is also anti-symmetric.

So you have a natural anti symmetric bilinear structure here yes absolutely there is nothing great about it at this stage the fact is that these things are enormous implications once you identify a Hamiltonian system then you guaranteed certain relationships automatically, so what we have done is the other way about I started with mechanical examples and I told you Cartesian coordinates are independent of each other and I wrote down certain relationships between them but the fact is that structure generalizes much more general context than just the usual ones you are used to.

So simply that it is a very useful and a very general frame work for understanding a huge class of dynamical systems of conservative dynamical systems right now we are talking about some mathematical properties of Poisson brackets and we are writing, now relationships which are fairly trivial they follow fairly straight forwardly from the independence of the coordinates and the moment a between each other plus the canonical Poisson bracket relationships, but these have deeper implications as we will see.

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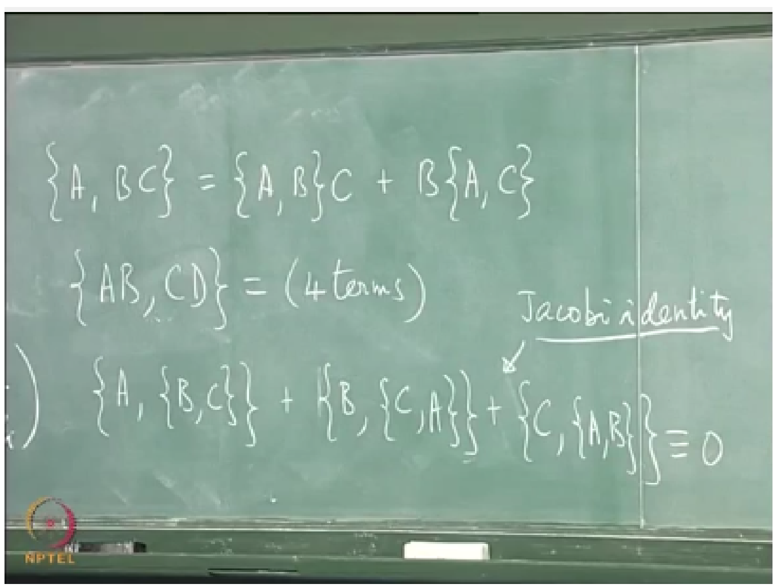

What is the Poisson bracket of a with a product of two functions so if I took A width BC and asked what this is equal to you would have to use BC here and BC here and use the chain rule of differentiation repeatedly and then the simple exercise to show that it is A with b times c plus b times let us see this is a chain rule for differentiating functions that is all that's been used and then it leads to an identity of this kind what would this be what could A B with C D be equal to what will you get on the right hand side.

You get four terms so first treat a B as a unit and use this formula for decomposing it into Poisson brackets of AB with C and AB with D and then use the formula again for decomposing this into two more terms, so you get a set of four relationships so this is four terms on the right hand side there is one more property of Poisson brackets which is crucial importance and that is the following if you took three functions a b c of the dynamical variables and computed a with the Poisson bracket of B with C and then took the Poisson bracket of a with this plus the Poisson bracket of B.

With the Poisson bracket of C with A plus the process on bracket of see with the Poisson bracket of a with B, what would you guess is the answer it is zero it turns out to be identically zero on using this property, so there is this anti symmetry when you take three of these guys the anti symmetry of the Poisson bracket leads to this property it is easy to establish do these properties you have got a set of objects a b c etc. Which are functions of the phase space variables in a Hamiltonian system and among those functions i have defined a certain bilinear structure called the Poisson bracket.

I take two the two of them at a time and i create something a new function that function is an T has this anti symmetry property and it also has this property which is called the echo be identity and it has the usual properties such as a with the Poisson bracket of A with a scalar times B constant times B is the constant times that of a with P the Poisson bracket of a with B it has the anti symmetry property and it has the property a with B + C is a with B +A with C does this remind you of something does this structure remind you of something these four properties -they remind you of anything else any other example where you have exactly the same sort of properties yes.

If you took ordinary vectors in three-dimensional space in Euclidean space and it took the cross product of two vectors what happens then do they have identical properties.

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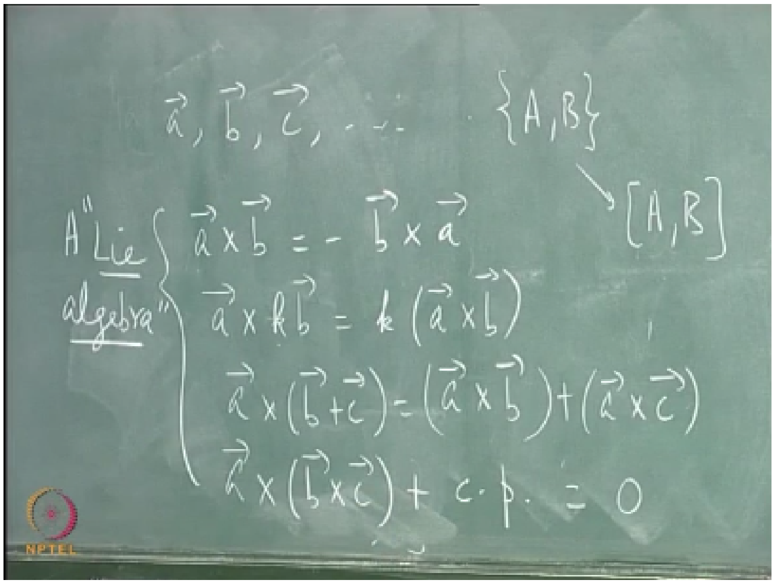

If I took vectors A B C etc. And I computed the cross product of two vectors is this not equal to one minus B cross a and is it not true that a cross some scalar times B is K times a cross Band certainly this is true this is true - and is this true, if I replace the Poisson bracket operation by the vector product or the cross product of two vectors is it not true that A x B x C + cyclic permutations turns out to be 0 identically on the right hand side, so exactly the same set of properties mathematically a bilinear operation which is anti-symmetric satisfies these properties.

And satisfies the ecobee identity anything else you think of any other structure which does exactly the same any other class of objects which have similar properties suppose ABC etc. square matrices n by n matrices of some given order and instead of the Poisson bracket instead of A, B as the Poisson bracket I replaced it by the commentator of the two matrices this stands for a B - B a then the commentator of two matrices is anti-symmetric the commentator of A with B is minus the commentator of B.

With A and this property is valid for commentators and, so is this any set of elements for which such a bilinear product or bilinear operation exists satisfying the anti symmetry property these properties as well as the ecobee identity is called alley algebra, so it says functions of the dynamical variables of a dynamic Hamiltonian system formally algebra under the Poisson bracket operation a very useful concept we are not going to deal, so much with the algebraic properties here in this course.

But it is a useful thing to know that there is something deeper a deeper mathematical structure to Hamiltonian systems under the Poisson bracket operation, so that is responsible a lot of the properties actually are mathematical properties which have physical implications which follow from this li algebraic structure of the functions of the dynamical variables under the Poisson bracket operation there is one immediate use of the Poisson bracket and that is the follower in conservative mechanical systems.

Such as those determined as where the Hamiltonian is simply determined as a kinetic energy plus the potential energy we know that the energy is a constant of the motion I now show you that the Hamiltonian itself for autonomous systems is a constant of the motion identically that is easy to see because.

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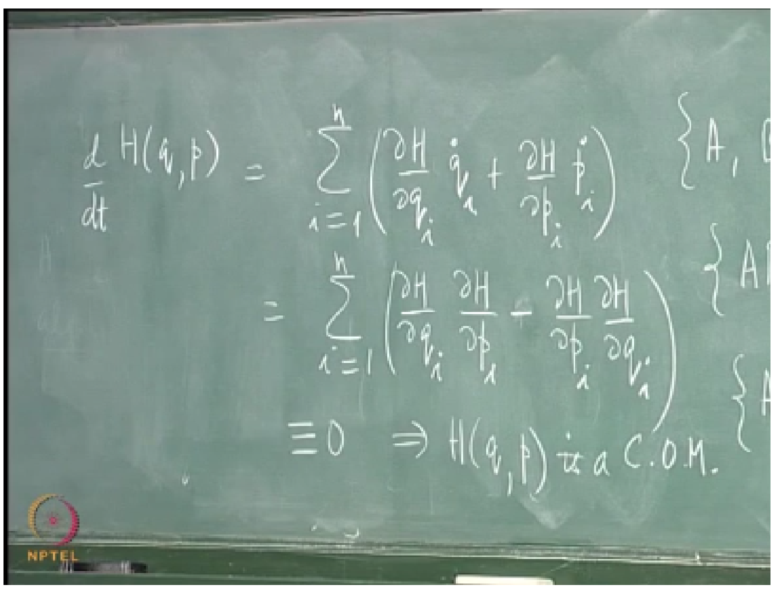

If I start with H as a function of all the Q's and PS and I compute D over DT of this quantity this is equal to by definition since it is a function of the Q's and PS this is equal to a summation over the degrees of freedom δ H over δ Q I multiplied by Qi dot plus δ H over δ P I multiplied by P I dot since it is a function of the independent variables Q 1 to Q 1 and P 1to P n its time derivative total derivative is just this set of partial derivatives multiplied by these total derivatives, but now we use Hamilton's equations of motion.

Therefore on a solution set on a solution of the equations of motion this becomes equal to the $\sum$ δ H over δ Qi and what was the equation of motion for Qi . δ H / δ pi + δ H / δ Pi and what was the equation of motion for Pi. – the all important -and of course this is identically equal to zero which implies that the Hamiltonian of a Hamiltonian system an autonomous Hamiltonian system is a C is a constant of the motion you will appreciate the importance of that relative minus sign in producing this result what is the equation of motion of any function of the dynamical variables that too can be written in a very simple way and that will bring out the use of the Poisson bracket so let us suppose a is a function of the Q's and PS and I ask what is the rate of change of this function of the Q's and PS.

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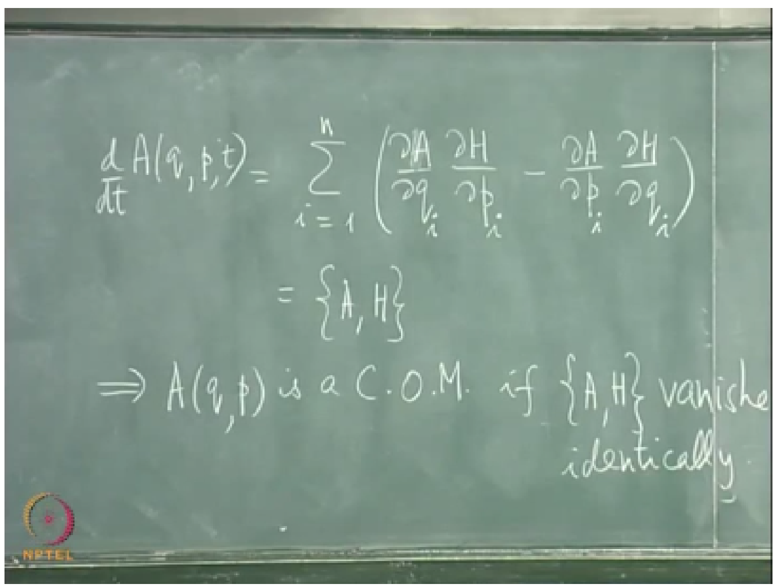

And I use exactly the same technique as I used earlier what would this become this would become a summation from I equal to 1 to N δ a over δ qi times qi dot but qi .is δ H over δ Pi put that in I + δ A over δ Pi. dot which is- δ H / δ QI, so they can get minus but what is this equal to the Poisson bracket of a with H that is a remarkable fact it says the rate of change of any function of the dynamical variables is just the Poisson bracket of that function with the Hamiltonian of the system that tells you the fundamental role played by the Hamiltonian of all the functions of the dynamical variables.

Of such a system the Hamiltonian is predominant plays a fundamental role because it controls the rate of change with time of every other function when is this a constant of the motion yes when a Poisson commutes with H when the Poisson bracket of a function with the Hamiltonian is zero it vanishes identically then the function is a constant of the motion and the converse is true as well as long as we are dealing with autonomous Hamiltonian systems of course you could generalize Hamiltonian systems to non autonomous systems and then this is no longer true because.

If you have a non autonomous and let me take a digression for a second and talk about non autonomous systems although would imply that there is also explicit time dependence on this side what would happen then what would happen then what would the right-hand side be modified to plus the partial derivative of H with respect to time the explicit time dependence gets differentiated and of course this portion vanishes identically and you are left therefore with is equal to δ H / δ T

And if H is non autonomous that right-hand side is not identically 0 and therefore the Hamiltonian is not a constant of the motion not surprising because it is actually explicitly a function of time and it is not a constant of the motion in this sense what happens if I consider a function a of the dynamical variables which also has an explicit time dependence independent of whether the system is autonomous or not, so independent of this I consider a function which has got explicit time dependence.

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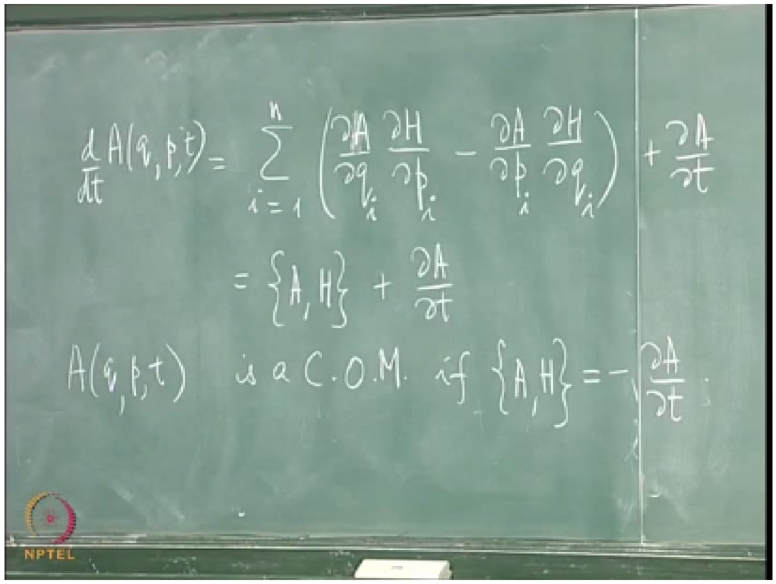

What should I add on the right-hand side I should add plus δ a over δ T therefore it is this plus δ a over δ T we need to modify this statement then we already saw that you could have constants of the motion which had explicit time dependence because the time dependence from the explicit T dependence could be cancelled by that of dynamical variables the other dynamical variables in this combination, so we now come to the conclusion that a of Q P T is a constant of the motion provided A, H.

Is equal to - δ a so the right hand side vanishes then the total derivative of this function with respect to time vanishes and you have a constant of the motion so this is the test this then is the test of when something is a constant of the motion we are going to look at some other remarkable properties of Hamiltonian flows or Hamiltonian dynamical systems and the most important one of them for our purposes is the fact that it is a conservative system in the sense of what we meant by a conservative system namely.

That volume elements do not change in phase space under the flow let us verify that recall that when the dynamical system was given by.

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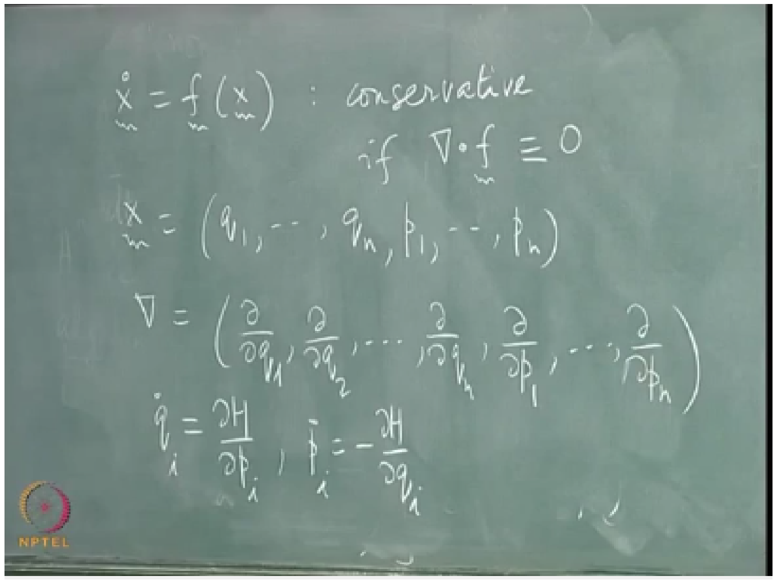

x is f of x then this was conservative, if del . F vanished identically see but this was identically equal to zero then we called it a conservative dynamical system well we now have a Hamiltonian system for which our x is really Q 1 to Q n P 1 to P n it is a ton dimensional dynamical system what is the gradient operator in this in this terminology in terms of the Q's and Ps the gradient operator is one which has components which are the partial derivatives with respect to all the dynamical variables so it simply stands for δ / δ q1 δ / δ q2.

Now another qn p1 and recall our equations of motion the equations of motion were Qi .was δ H / δ Pi but Pi .was minus δ H / δ Q therefore if I simply compute this divergence here it gives us a very simple result therefore δ. F in this case was equal to what it is equal to δ / δ q1 F 1 but F 1 is δ H / δ P1 and soon so this is equal to summation I equal to 1 to N δ /r δ Q 1 qi δ H / δ Pi that takes care of the first n contributions to del dot F and the remaining n are δ / δ P1 times minus Delta H over δ q 1 and so on.

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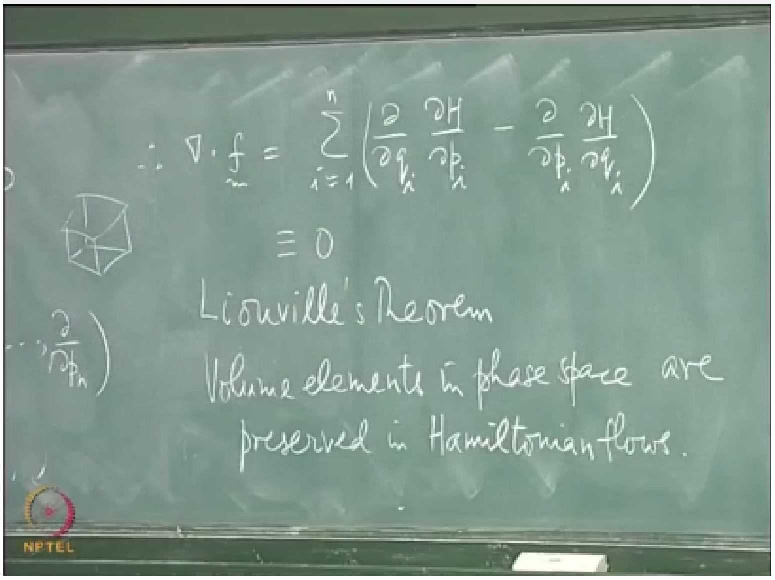

So you also have minus δ e over δ Pi δ H or δ Q ion using Hamilton's equations of motion but what is this equal to it is identically 0 since you can take partial derivatives here in either order this function is integral then you can take from partial derivatives in either order therefore this vanishes identically which says that Hamiltonian systems have flows such that the volume elements in these systems are preserved in time the flow is like that of an incompressible fluid or the volume is preserved in Hamiltonian flows in phase space and this goes by the name of here or Preserve is a better word.

Again a fact of great significance it is not just volume elements that are preserved many other things are made retained as constants also under Hamiltonian flows so it is very restrictive these conditions the conditions which are imposed by this set of equations plus the Poisson bracket structure between Q's and PS are actually very restrictive and they have very deep implications there is a very rich mathematical structure to Hamiltonian flows some of which we will uncover subsequently okay I start by saying.

It is not it is not a result of this structure no not at all what is not a result of this structure it is not I am not saying it is an additional constraint I am saying that you define a Hamiltonian system as a set of two and dynamical variables in which the variables are paired off pair wise that each q with its conjugate P such that the Poisson bracket of each Q with the p with its conjugate p is one and he Poisson bracket of each Q with all the other variables is zero and similarly for the piece together with that you specify a certain function of all the dynamical variables called the Hamiltonian.

And the equations of motion have this structure that is my definition of a Hamiltonian system and from that everything else follows including the fact that the Hamiltonian is a constant of the motion for autonomous systems the fact that any dynamical variable or function of any dynamical variable its rate of change with time is given by the Poisson bracket of this variable with the Hamiltonian for explicitly time independent variable functions and you will theorem which says volume elements are preserved various other things are preserved as well we would not go into that for instance.

If this is a volume elements schematically you have some volume element of this kind in phase space in two n dimensional phase space you could project this volume element onto each kewpie plane, so I take each cue and it is conjugate P and that forms a two-dimensional plane and I project the shadow of this on to the two onto this QP plane and I take the sum of all these shadows of these areas so I project on to q1 p 1 q 2 p 2 q 3 p 3 up to q and p N and I take the sum of all the areas and it turns out as this volume element flows not only does it maintain it is volume the size.

The magnitude of this volume is not changed it might get distorted but it maintains this volume, but in addition the sum of all those shadows those areas is also preserved it is also constant that is a very remarkable fact so it is not very intuitively very clear why this should be, so but it follows all these properties follow from the structure of these equations it is not a gradient system as you can see but I could write, it as something which looks like a gradient system and there is a mathematical structure here called the simplistic structure which I am not getting into right now.

But I was a little later on mention a few properties of this extra structure as we go along and as a prelude to that let me do the following let me define a matrix Ja.

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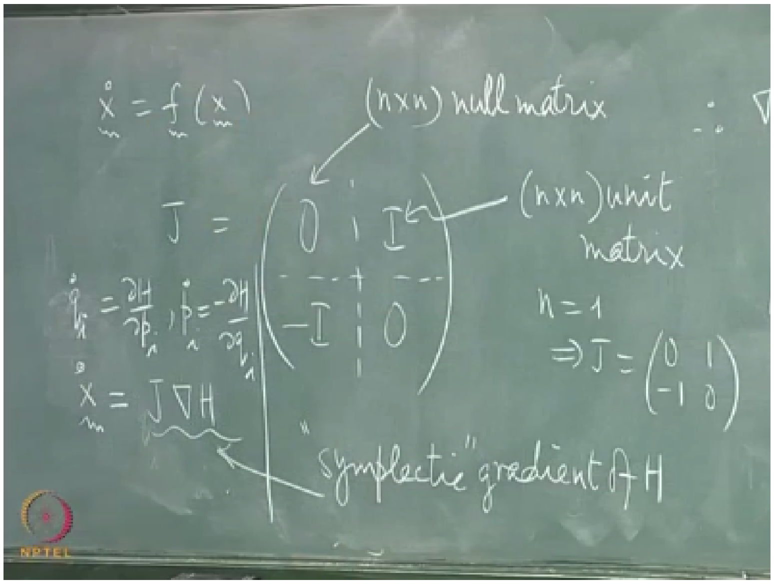

2n / 2n matrix with the following kinds of elements so let me define this J as a 2nby 2n matrix with n by n blocks here and I have the null matrix here this stands for n by n null matrix I have the unit matrix here and this stands for the N by n unit matrix minus the unit matrix here and the null matrix once again here, so it is partitioned into four blocks in by n blocks and it looks like this and I call this matrix J it is a numerical matrix it has just zeros and ones or minus ones as the elements this is a diagonal block here one here.

And this is a diagnosed with minus one zero these are null matrices then it is not hard to see that the equations of motion in this case in the Hamiltonian case which are x i qi dot is δ H / δ Pi and P I got is - δ H / δ q I this set of equations can be written almost in gradient form in the form x dot is equal to J times the gradient of H where by the gradient I mean A2 and component differential operator with components δ / δ q1 up to δ / δ q 1 and then δ / δ P1 up to PN and once I put this matrix J here then it is nothing but the gradient.

This twists in such a fashion that the Q's the rate of change of the Q's depends on the derivative with respect to the PS and vice versa with a minus sign that is the reason for this minus sign here, so if I wrote this x as a column vector to n-dimensional column vector q1 up to PN and similarly for the gradient of Hand then multiplied it by J you would end up with just this so this J does the required trick of converting the part that the rate of change of the Q's will depend on derivatives with respect to the peas and the peas with respect.

To the Q's but with a minus sign and this structure has a name it is called the simplistic this prop this matrix J has remarkable properties what is the square of J well it is immediately clear from this that the transpose of je is minus J because these guys would go there and those fellows would come here and what do you think is the square of J I leave you to figure this out incidentally when n is 1 then a large number of the properties become quite transparent because then you have a 2 by 2 matrix which is essentially for n equal to 1implies J = 0 1 - 1 0 and of course you could square this very trivially I leave you to figure out what J squared is then we look at some.

Interesting properties of Hamiltonian systems in this language the advantage is that it looks very much like a gradient system so you can write these things down much more compactly in this form one final comment we started with mechanical systems and said you have the coordinates and you have the moment a and they are quite physically distinct from each other, but now in a in an abstract setting when we talk about Hamiltonian systems and systems defined by just a set of 2nvariables of this kind pair it off two at a time then the distinction between what you call moment.

A and what you call coordinates can be actually lost completely you could make changes of variables of these dynamical variables in such a way that what looks like moment a initially could turn out the coordinates belong to the set of coordinates and vice-versa provided of course you put in factors for the right dimensionalities of these variables I bring this up because the conventional notion of momentum as a linear momentum is very restrictive because after all you could also have an angular momentum you could have generalized coordinates which are angles and the conjugate.

Momentum would be angular momentum so the momentum variables could have dimensions of angular momentum not necessarily linear momentum and the generalized coordinates could be angles which are dimensionless they do not have to be lengths so the advantage of looking at it this way is that you can go beyond the normal mechanical examples and look at the actual structure of the dynamical per se which is very intricate very interesting and has very crucial properties.

So we have seen today to summarize that Hamiltonian systems forma special class of conservative dynamical systems autonomous Hamiltonian systems Louisville's theorem asserts that the volume elements are preserved in phase space which conforms to a definition of what we said was a conservative system the rate of change of any dynamical function of the dynamical variables is given by the Poisson bracket of these variables with the Hamiltonian and the Hamiltonian itself is a constant of the motion and the Poisson bracket idea enables us to identify other constants of the motion we will take it from this point.

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