Indian Institute of Technology Madras
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NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 4
Stable and unstable manifolds

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So we go back to recapitulate what we were doing the last time.

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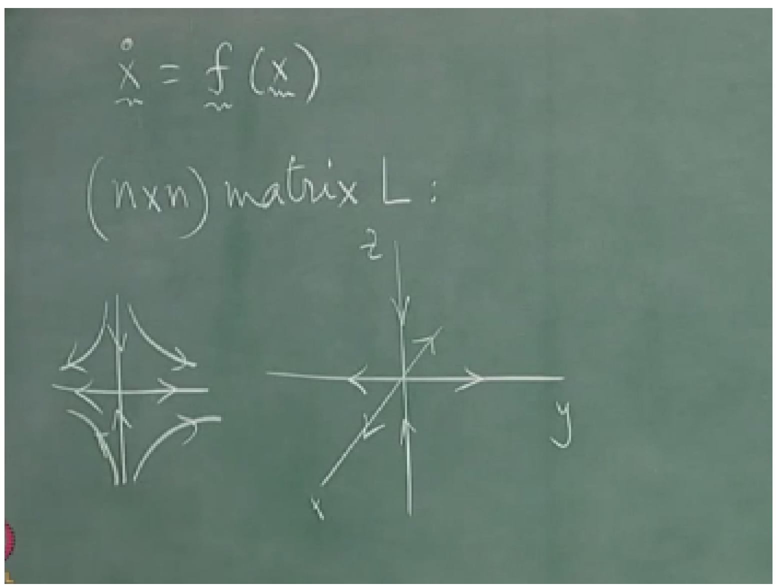

We discovered that the general dynamical system X dot is f of X has a set of critical points given by the roots of the vanishing of the vector field f of X and we saw specifically in the case when the dimensionality of the dynamical system was two we saw specifically that the matrix L the linearized matrix L in the vicinity of the critical point its Eigen values determined the nature of the flow near the critical point.

And we classified these critical points into saddle points nodes spiral points and centers those are the four main classifications and we saw that the flow was largely governed by the Eigen values as well as the initial conditions in the vicinity of these critical points I mentioned very in passing that the case when the roots of when the Eigen values are equal is a special case of the situation when they are unequal and the shape of the flow locally changes a little bit but the fact that it is a node if you have to rely in values of the same sign or a saddle point.

If it is two real Eigen values of opposite signs or a spiral point if it's a complex Eigen value with nonzero real part and a center if the roots are pure imaginary those facts stand there invariant under similarity transformations we also saw that for a general n x n matrix L the Eigen values are given as functions once you specify the trace of the matrix the trace of the square of the matrix.

And so on till the health power of the matrix the Eigen values are determined uniquely in terms of this set of invariant quantities and that would decide the nature of the flow while we focused on two-dimensional systems you could generalize the classification that we came up with to some extent you can generalize this to the n /n case although the number of possibilities increases enormously just to give you an example we talked of a saddle point in two dimensions which would correspond,

Say to having if the two axes are orthogonal to each other for instance to having a situation of this kind a flow of this kind would mean a saddle point in the two dimensional case more complicated possibilities occur in 3 / 3 or 4 / 4 or higher dimensional cases for example in the 3 / 3 case you could have a situation in which along this direction you flowed in and along the perpendicular directions the two of them you either flowing in both flow out in both directions or you flow out in one direction.

And flow in another for instance you could certainly have a situation of this kind flows out and this would correspond to a saddle point in three dimensions a general saddle point in which if this is the xy plane the xy plane spans a space the x and y axis span a space namely the xy plane which I would call the unstable manifold of this critical point and this the z direction would span the so called stable manifold of the of the critical point now I must mention a little bit of terminology here specifically we need to understand.

What is meant by spanning a space by unit vectors and all of us are familiar with this concept let me start by mentioning just to recapitulate to you a little bit from linear algebra let me tell you what spanning a space means a linear vector space a concept which I assume that most of you are familiar with at least intuitively is one where you have a set of vectors called the basis vectors of the space and any vector in the space can be uniquely expanded in terms of the basis vectors as a linear combination of basis vectors with certain coefficients the simplest example.

For example instance would be the x axis itself just a one dimensional linear space in which any real number could be written as some constant multiple of the unit vector in the x direction the xy-plane are to would be another example of a linear vector space in which any two unit vectors along any two mutually perpendicular directions would span the space any vector in the space any point in the plane it is coordinates can be specified uniquely in terms of these two unit vectors and multiples of these unit vectors the vectors forming a basis do not necessarily have to be orthogonal to each other they could be oblique axis they simply have to be linearly independent of each other and they must span the space in other words no direction must be left unspecified by these basis vectors.

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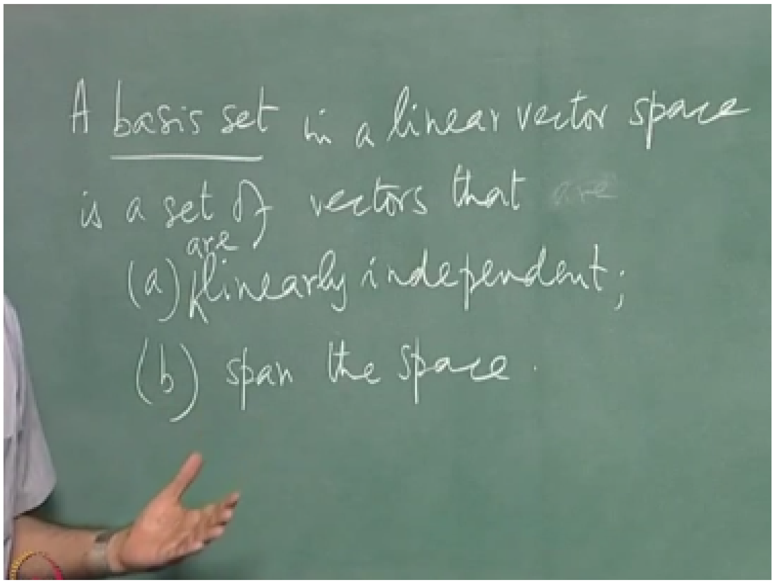

So let me write that down as a definition a basis set in a linear vector space or a linear space is a set of vectors that are a) linearly independent and b) that we should write here that are linearly independent and b) that span the space these two are different notions altogether just to set the frame let's give examples if you took unit vectors in three-dimensional Euclidean space and let me call them ex ey and ez in three dimensional Euclidean space.

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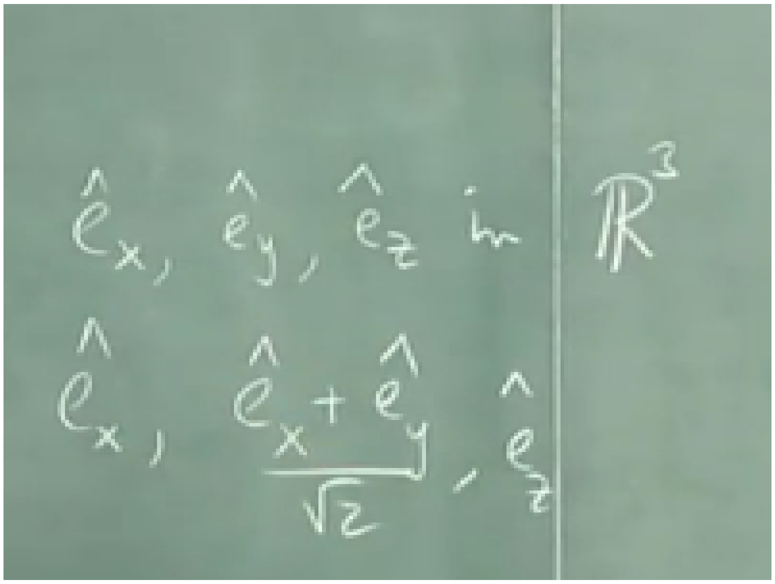

Let us call it R3 we know these form a set of orthogonal axes there are the normal in the sense that ex dot ex is 1 and so for ey in is it and ax dot ey is ey dot ez dot is and = ez got ex equal to 0 they are orthogonal to each other now these vectors are linearly independent of each other in the sense that none of them can be written as a linear combination of the others therefore they are linearly independent of each other on the other hand the combination ex + ey / $\sqrt{2}$ is not linearly independent of ex ey it can be written as a linear combination of these vectors.

These vectors also span the space because any point in three-dimensional space or any vector in three-dimensional space can be written uniquely as a linear combination of some scalar numbers times these unit vectors both these concepts are necessary so just as I said ex ey ez form a basis in r3 you could also have a basis which is for instance ex + ey/ $\sqrt{2}$ and ez this is as good a basis as the other what is the only difference between these two bases they are not orthogonal this vector is not orthogonal to this vector it is as if you chose the x-axis and the 45-degree line in the xy plane and the z axis as the unit directions as the special directions but nothing else is lost.

Is this set of vectors linearly independent this set of vectors is not linearly independent at all would this following set of vectors be a basis in the space yes indeed this would be a basis in the space do these vectors span the space they span only the xy plane they do not span three dimensional space these two vectors alone are linearly independent but they do not span r3 they do not span three-dimensional space on the other hand this vector this vector this vector and for instance ex + ey +ez √3 does this set span the space yes it does but they are not linearly independent.

So these two concepts are different from each other you may have a set of vectors which are linearly which is the vectors of which are linearly independent of each other but they do not span the space you may have a set of vectors which span the space but which are not linearly independent of each other such as that one yes I have not given the formal definition of a vector space I will buy and buy but since this is not primarily focused on linear algebra I have not done that but we will talk about this when the need arises.

So the definition is very simple there is a set of properties that elements of a set have to obey in order to become only a linear vector space and I am assuming that you we have at the back of our minds the intuitive notions we already know about two-dimensional and three-dimensional vectors in ordinary Euclidean space.

So I am merely illustrating the general properties have linear spaces in terms of these vectors in terms of these special cases so it is not as if every space can be written down if the basis set can be written down in a simpler manner but the point is the idea of a linear vector space is much more general than the geometrical idea of vectors in two or three or four dimensional Euclidean space it is far more general than that and as problem as we come across properties that we need we will use will define these things more formally.

So as I was saying the idea of linear independence and spanning the space are different ideas altogether and when a set of vectors does both then we say it is a basis set and of course basis sets not unique, to a given space three-dimensional Euclidean space you could choose a basis set of unit vectors in many ways in an infinite number of ways but the point is given a basis set you are guaranteed that every element of the vector space can be expanded uniquely in terms of this basis set of vectors.

This a set of vectors which belongs to a basis set but which is not the complete set for instance just a e_x and e_y what in general spanned a subspace x alone spans the entire x axis ax + $_e$y together span the XY plane and ex ey and together span all of 3-dimensional Euclidean space. So this is the idea we need at the moment. Going back to the three-dimensional case this is a saddle point you could have other instances as well for instance you could have a situation.

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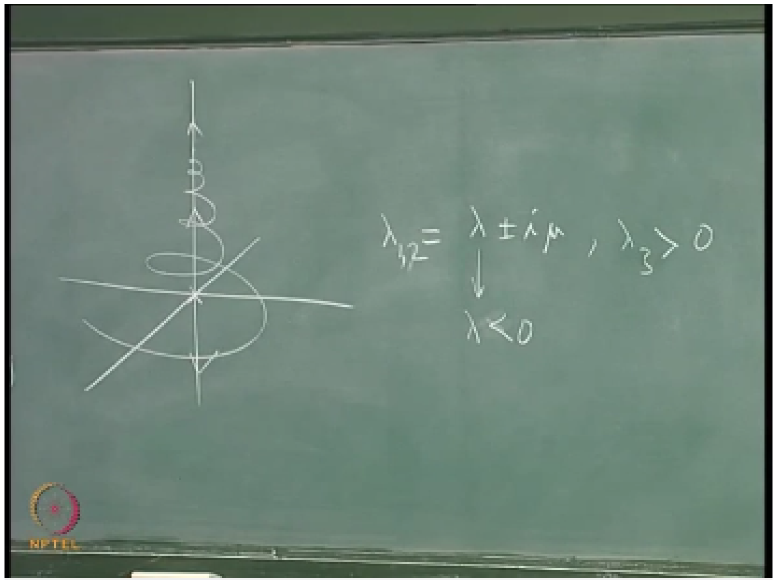

Where you flow away along the z axis but in the XY plane you flow inwards in what would normally be an asymptotically stable focus but as it comes inwards you may go away along the z direction may flow out and this is actually an unstable point although as far as the XY plane is concerned, if you ignore the said axis it looks as if you are going to flow into the X the origin in the XY plane but is really flowing out in this direction.

What kind of Eigen values do you expect here in the XYZ variables? What would be the three Eigen values taking in this case it is clear that the x axis the y axis and the z axis are themselves the principal directions, so what kind of what kind of Eigen values do you expect? Yes I would expect λ 1 , 2 to be =some λ + or –μ where λ is < 0 because they are flowing in towards the origin and I would expect λ 3 to be some positive number.

So it flows away along the Z direction such a critical point is called a saddle focus because it flows off and this kind of dynamical behavior plays a big role in real three-dimensional systems which would exhibit what will later on study as chaos, because what can happen in such systems if the system is nonlinear is that the system would appear to spiral in towards the origin in the XY plane but then moves away along the z axis eventually, gets really inject it somewhere in the XY plane and starts this whole thing all over again and goes off and the motion is eventually chaotic in some sense.

So this is one of the mechanisms by which chaos appears in three dimensional flows but the genesis of it is already buried in this kind of behavior of a critical point, my statement which I have not made precise at the moment is that in real three-dimensional systems which exhibit chaos one of the common mechanisms by which chaos appears is that you have a saddle focus of this kind.

Where things flow in a two dimensional subspace but then in the third direction it flows off eventually leaves this access moves off and gets re injected into this plane flows towards the origin and moves off and so on and does so in a chaotic manner in an irregular manner, in the real three-dimensional system and will come across an example of this later on. The next question is we have pointed out that near a critical point the story is you linearize the system.

Find the eigenvalues of the linearize matrix l the Y acoubian matrix and study it is the nature of its eigenvalues find out how many of them have positive real parts, how many have negative real parts and how many of them have zero real parts this can be formalized now and the statement is as follows.

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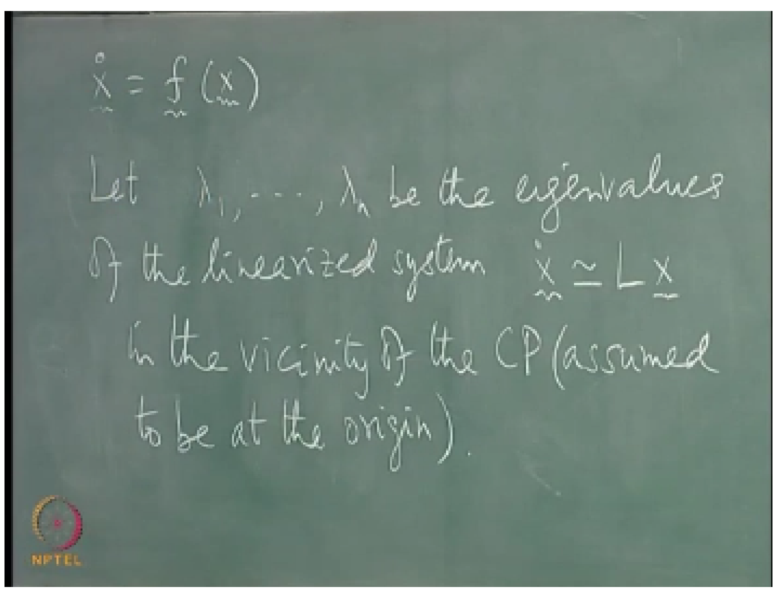

So let us let λ 1 up to λ n be the eigenvalues of the linearize system X dot is L x so we assumed there is a critical point at the origin I have shifted coordinate, so that I am near the origin and then I look at the coubian matrix which is non-singular remember by assumption and if it is perform X dot is LX the flow near the origin then the properties of the system near the origin are determined by the eigenvalues of L.

And the statement is the eigenvalues of L can be broken up into three classes, those with real parts positive, those with real parts negative and those with real parts =0.

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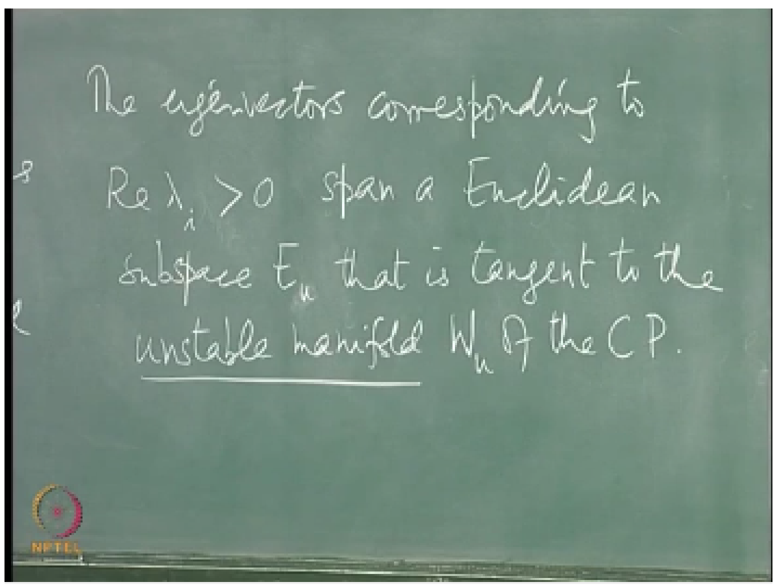

The set of eigenvalues the eigenvectors corresponding to those eigenvalues whose real parts are positive these Eigen vectors span a Euclidean subspace let me call it E u and the then the reason for the subscript u will become clear in a second that is tangent to what is called the unstable manifold of the critical point This statement is already intuitively clear to us all I am saying is if you go back to the original case that I talked about of a saddle point.

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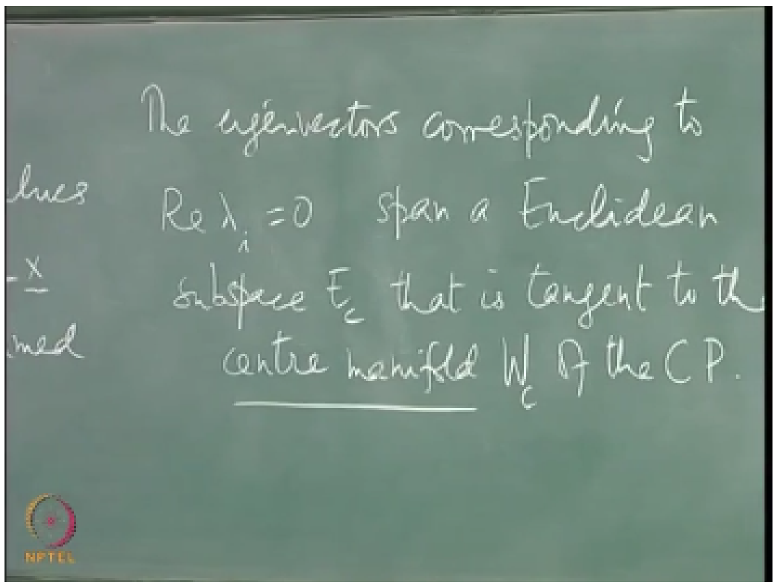

A saddle point in the behavior of the saddle point you have a situation of this kind if it is in two dimensions for instance there is a thing that is coming in this direction something that is going out in this direction in the close vicinity of the saddle point we have linearize this matrix the actual flow lines could be quite different from this, the actual flow lines could perhaps be like this and all that is happening is that this line is tangent to the actual flow line.

Since we have linearize it you only capture this portion of it and similarly this part could be like so. I would then call this e you whereas this would be W and a statement is that these tangents at that point there I directions special directions which span a Euclidean subspace and since you can have more than one of them in higher dimensional cases I have called it a full subspace EU in this simple example two by two case, this is the subspace every point on this line moves off and that is it.

So the unit vector is along the unit vector along this direction, similarly if I replace the word if I replace this by something less than zero then this piece vector span the Eigen vector span a

Euclidean subspace piece of s here that is called the stable manifold and it is denoted by Ws, so here is W s on this is w face on the other hand this is e yes I am sorry it is the other way about this is Es this is WS yeah.

Now we have to define what is the meaning of the word the technical meaning of the word manifold, there are subtle differences between the words linear space a subspace of a linear space and a manifold for the moment, let me get over this by merely mentioning what you should think of as a manifold a manifold is a generalization of the idea of a curve or a surface, to an arbitrary number of dimensions.

In other words if it is a differentiable manifold it means certain smooth derivatives exist on this object whether it is a curve or a surface or a higher dimensional manifold that is all we need to know at the moment and the statement I made in geometrical terms was that if this critical point is a saddle point which you discovered by linearizing in the vicinity of this critical point, then there are two eigenvalues one of which is positive when the other is negative.

You could ask what are the eigenvectors corresponding to these eigenvalues, that eigenvector each eigenvector would define a certain direction and the eigenvector that corresponds to the Eigen value with negative which is negative would be this Es this direction a unit vector along this direction for instance. Similarly the eigenvector which corresponds to the positive Eigen value would be au along this direction.

But the actual flow lines of the vector field the true flow lines in the non linear system could be of this kind could be, for example could for example have the shape but you are guaranteed that when you linearize the eigenvectors you discover would actually be tangent to the true flow lines at that point, which implies that if you go arbitrarily close to this critical point then the linearize problem gives you an accurate indication of what the actual problem does what the actual system does in other words it is a good approximation to linearize.

And there is a fear which says that if this is all that happens if all the eigenvalues of L near a critical point or near any point for that matter have either positive or negative real parts then linearization gives you an accurate picture of the true flow arbitrarily close to this critical point. In other words it is a legitimate operation to linearize and understand what happens close to the critical point.

We have assumed that this is a differential vector differentiable vector field we have made this assumption here that this vector field has smooth partial derivatives, in each of the variables this is a primary assumption that is gone in our definition of the dynamical system, it is already been taken in. There are cases when this is not so at certain isolated points we will then handle them case by case but otherwise the moment I write this I am going to assume that each component of this vector field F has partial derivatives in each of the variables comprising X the vector X this is an assumption we make.

You can guess from what I am saying that the situation is going to be different if you had eigenvalues whose real parts are 0, they span a Euclidian subspace and the notation here is e sub C that is tangent to what is called the center manifold of the critical point, this too is true. However there is a catch and the catch is and this is a very important catch the catch is unlike the case of the unstable manifold Wu for which there is a unique tangent a sub u.

And the stable manifold WS for which there is a unique tangent subspace is found by linearization. In the case of a center manifold there is no guarantee at all that EC is unique what does that imply in practical terms because one of the main tools that I have is linearization near a critical point because I know, how to solve the linear equation X dot =LX but the moment I say that this is no longer unique what would you conclude.

It means linearization is no longer a reliable guide to the actual flow you have to look at the nonlinear problem in some generality and it is no longer possible to have reliable or accurate results from linearization, you have to go beyond the linear approximation. Notice that the case when the eigenvalues are actually 0 falls under this in some sense because we have said that no Eigen value should be 0, we already made that statement when we did the analysis in the two by two cases.

But it is possible that there is no imaginary part at all and these would also belong to the Center manifold and we have to be careful. So this sounds a note of caution it essentially says that linearization near a critical point is good it is a good approximation and you can rely on it as long as no I gain value of the linearize matrix has is not only not 0 but also has no real part that is 0. In the case of the two by two example the only case where this happened was when the eigenvalues were + or - I μ.

And the Eigen value itself we assumed was not 0 then they could be pure imaginary and it corresponded to a center and that is the origin of the word center manifold, but in a nonlinear system if I discover that there is a center manifold and I do this by linearizing and I discover that one of the eigenvalues one or more of the eigenvalues has zero as a real part, then I am on Mike I am cautious immediately I have to now worry about the two nonlinear the system to understand what the flow is.

And we will see by example what can happen how one could be misled by linearization but before we do that let us take the positive part of this statement namely if you do not have a center manifold on the problem near a critical point then linearization is good and this is essentially the content of a theorem called the Hartman Grobman theorem and let me just write the name of the authors down and we use this theorem very often implicitly.

Yes in the linear problem if you discovered that λ is + or - I μ and the problem was linear to start with there is no problem that is it, but if the problem is nonlinear and you have dropped the nonlinear terms and then you discover λ is + or - I μ it means you have to re-examine the problem because, it might not be a center after all if it is linear its unique quite right this question is if the problem is linear to start with and I discover that there is a center manifold that certain eigenvalues of pure imaginary is that reliable or not?

Yes it is it is because it is in some sense an exact solution to the linear problem, so it is certainly true.

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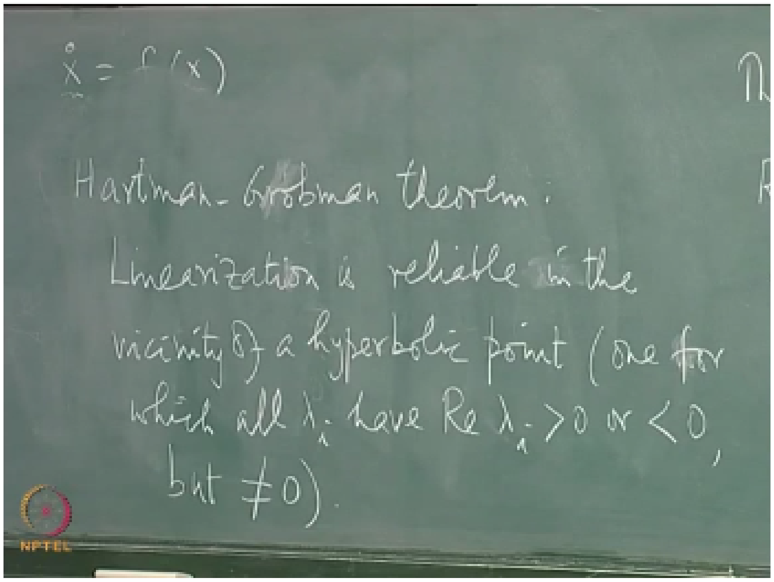

The theorem I am talking about is called the Hartman Grobman theorem which essentially says I am not going to write down the rigorous definition of it here but I refer you to the texts that I have given in the list, it essentially says linearization is reliable this is not the exact statement of this theorem but essentially it implies this, in the vicinity of a hyperbolic point by that I mean one for which all the eigenvalues have real parts that have that are not 0 that are nonzero.

If a system has at all points in its phase space the local eigenvalues of the linearize matrix el are of this kind I would say the system is purely hyperbolic, everywhere globally hyperbolic and such systems are what mathematicians generally analyze in rigorous terms and a number of rigorous theorems are known for that subclass of systems.

But I must say immediately that in practical considerations when we model real systems physical systems with by dynamical dynamics of this kind very often you discover the systems are not purely hyperbolic Center manifolds to occur and then you have to examine what the nonlinear problem is in its full glory and linearization may not always give you the right answer because you do not have a corresponding theorem of uniqueness for the center manifold.

Pardon me ah but it does not have to be periodic motion at all times that is the whole point, so you could have very often what happens is that a pair of Eigen values could cross the unit circle, could cross the vertical axis, could cross the case where real, so let me say this let me let me illustrate this.

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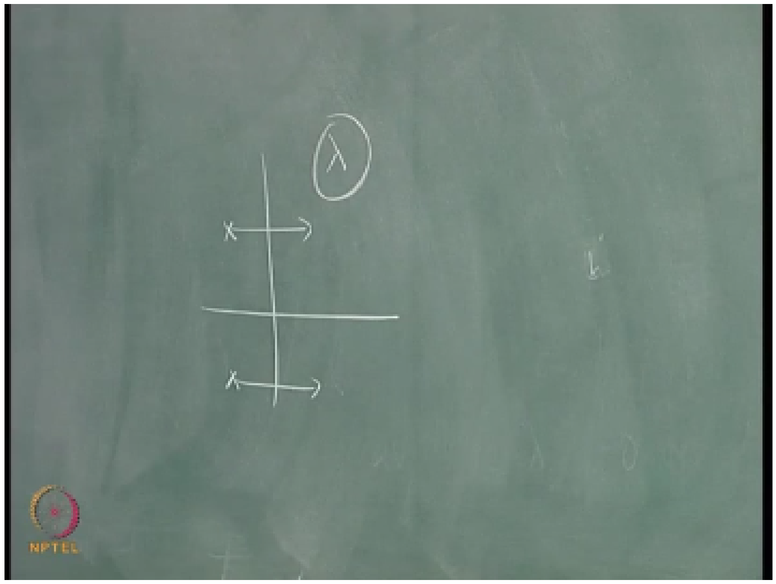

In the complex plane if you had a pair of eigenvalues of this kind in the complex Eigen value plane as you change a parameter these eigenvalues could wander around and move off in this fashion, so they go from something which belongs a pair which belongs to the stable manifold to something which belongs to the unstable manifold but in doing so they have to cross the center manifold and this is where the problem arises.

These modes would become soft in some sense because the real part disappears and then interesting physics happens, so this is the kind of situation which we will understand which will analyze at some length. As you can see I am heading more and more towards the geometrical description, we brought in concepts from linear vector spaces we have brought in the phase portrait we have talked about the fact that phase trajectories cannot intersect themselves the fact that when phase trajectories are closed curves you have periodic motion and so on.

So would like to formalize these statements a little bit more and a great deal of attention will be paid to the way the flow occurs in phase space, what happens to little volume elements as they move out. So let me take that up next and define a basic difference between two kinds of dynamical systems conservative dynamical systems versus dissipative dynamical systems, it is a very useful operational definition.

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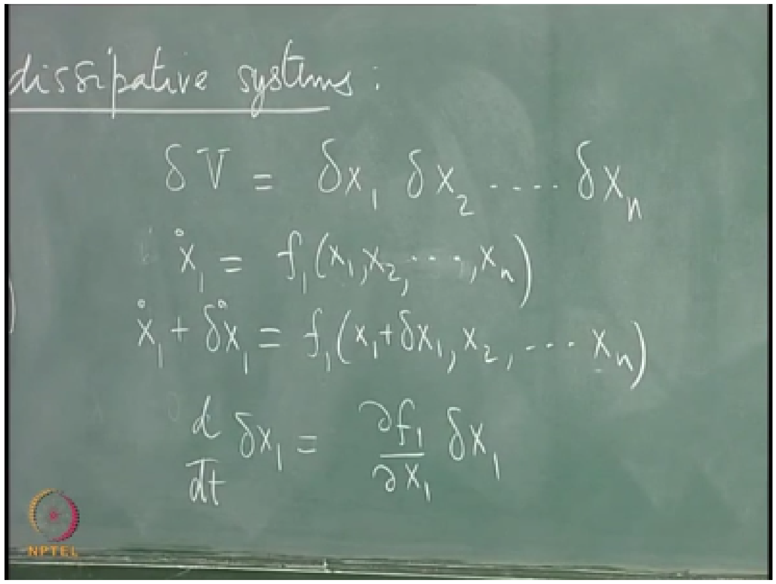

So let me do that the idea we already have from mechanics for instance is that a conservative mechanical system is one, where the total energy of the system is conserved and a dissipative one is one for which the total energy is essentially lost maybe due to friction or other dissipative forces, this is our knife intuitive idea of what we mean by conservative and dissipative systems.

We would like to generalize these ideas a little bit here first of all the systems, we are going to look at are not mechanical systems, there are many systems for which the whole the idea of a total energy does not exist at all we would like to consider cases where we put the whole thing in a much more general framework and one possibility a very important one is the following if you start your dynamical system from some point here.

And as time goes on it flows out in this fashion and you start with the neighboring initial condition in general this lies on a different trajectory and perhaps the trajectory here moves in this fashion and similarly for an initial condition out here it moves in that fashion, then the question you could ask is what happens if I start with a volume element in phase space corresponding to a whole set of initial conditions and I look at the fate of this element as a whole as time goes on.

So if I denote this as a little volume element here what happens to all the initial conditions that start at this point and move off in time perhaps at a later time, these volume elements are spread in an element of this kind or there is another possibility this element could shrink to a point or it could oscillate or it could remain completely unchanged all these possibilities exist. So one of the important things we are going to look at is twofold one what happens to volume elements as a whole as the flow occurs and two what happens to the distance between two neighboring initial conditions as a function of time.

Do they come closer together do they go further apart if, so what is the rate at which they move further apart the second question is going to be of crucial importance in all that we studied. So let us start with a set of initial conditions here and ask what the fate of this set of initial conditions is as a volume and so let me start with the volume element δ V which is =a product of infinitesimal differences in all the variables of the dynamical system δ X 1 δ X 2 through 2 δ X N and I write my set of equations down and track the time dependence of this δ V.

And the set of equations is X 1. is F 1 of X 1 up to xn and so on up to xn but if I start with an initial condition if I look at the point 0 X 1 but X 1 + δ X 1 then X 1 dot + δ X 1 dot this stands for a neighboring point near the X 1 this is =F 1 of X 1 + δ X 1 and all the other variables are unchanged. So let me complete that by writing X 2 subtracting one from the other it is immediately clear that D over DT of δ x1 is the difference between this quantity and this quantity here.

What would that be to first order in the infinitesimal δ x1? I want the difference between this quantity in that quantity to first order in δ x1 it is just the per you are completely right it is the partial derivative of F 1 with respect to X 1 multiplied by δ X 1.

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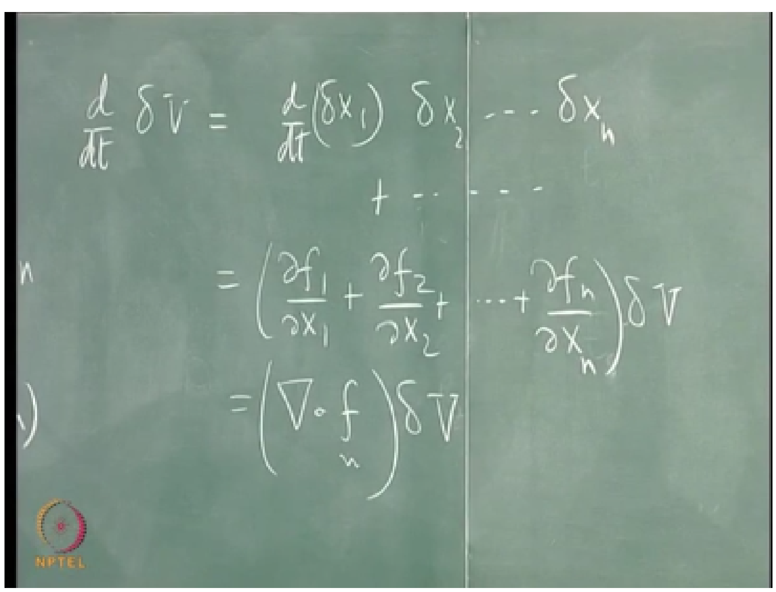

Therefore the rate of change of the volume element δ V this can be written down in an extremely simple form because this is =d over dt δ x1 multiplied by the rest of it + n - 1 other such terms in which each δ x 1 is differentiated in turn but that becomes = when I plug this in and similar terms it becomes =δ F 1 over δ X 1 + times δ V itself because what comes out as a common factor is just the product δ X 1 δ X 2 up to δ X N and that is just δ V what would you call this we have a vector field F as a function of X.

So what would this be is it the gradient the gradient is a vector yeah it is each component differentiated, with respect to the corresponding label or coordinate and therefore it is the divergence. So this is =∂.F δ and that is two point by point at every point this happens to be true, if I now define my conservative system as one for which the volume element in phase space whatever volume element you focus on does not change as a function of time then this is true if and only if the divergence of this vector field vanishes identically. And I define my conservative system in this fashion, so let me define.

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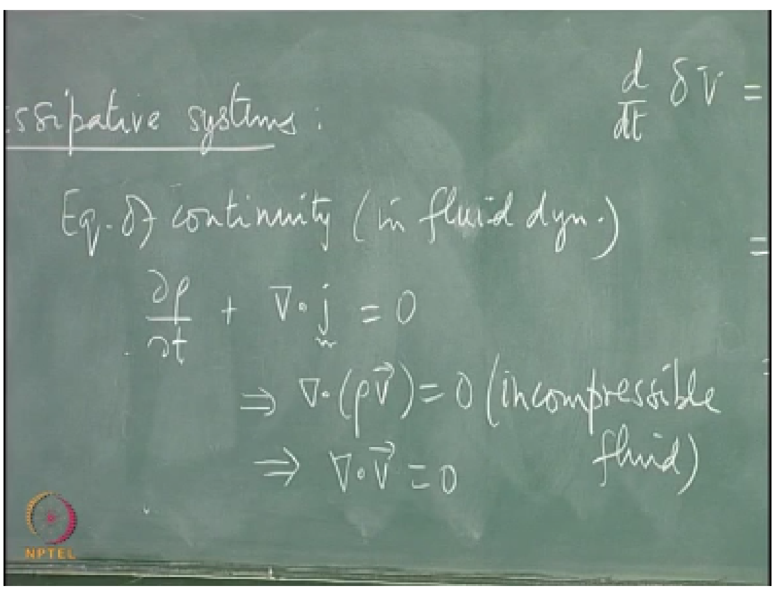

So the system is conservative if ∇dot F is identically zero what is this flow then remind you of the flow in phase space, volume elements do not change they could get distorted very badly nobody says a cube must remain a cube a small spherical element remains a spherical element doesn't have to happen at all, it will get very badly distorted but what would you call a fluid flow for which you have this kind of behavior.

Imagine a fluid for which the current is such that ∇dot J is 0 it is an incompressible flow because if you recall the equation of continuity for fluid flow it is of this form equation of continuity it is δ ρ over δ T where ρ is the density of the fluid + the divergence of the current =0 in the absence of flow in the absence of sources or sinks, so ∇ dot J = 0 and if the fluid is incompressible then δ ρ over δ T vanishes identically does not change with time.

And then ∇. J is 0 in fact you go further in the case of fluid flow just to recall to you in fluid dynamics, what is J in the case of fluids it is the density multiplied by the local velocity at the point so this implies that del dot ρ V =0 this is incompressible since it is incompressible ρ does not change with time either nor with space if it is homogenous and then therefore ∇ dot V itself is zero if ρ is constant in space and time then of course ∇ dot V itself is 0.

So the lesson that we learn from that is that the flow in phase space of a conservative dynamical system is like that of an incompressible fluid and exactly as in the case of an incompressible fluid these fluid elements could stretch could distort could be very badly wounded on each other but the total volume does not change. And I define a dissipative system as one for which this is not true, this generalizes a little bit the mechanical idea that the energy is the only quantity that has to remain constant.

Because we are going to discuss systems for which there is no need to introduce the idea of it they may not be a concept such as the total energy, in a population problem there is rarely such a concept available right. This definition in some sense is a little too rigid because it says that this system has to have an identical F must be a solenoid elector filled identically at all points and then it is conservative but you could come across systems many physical systems for which you come close to a conservative system but it is not quite a conservative system.

Because it does not vanish identically del dot F does not vanish identically at all points but for instance suppose the motion is periodic for any initial condition and you take a neighboring set of initial conditions and all these motions are periodic and it turns out further that on the average over a full period for this volume element del dot F should happen to be 0, I would still say it is pretty close to a conservative system even though it does not formally look like one.

And let me give an example from population dynamics a very common one a very popular one and one which we will study in some detail or we should work out in some detail at least on the problem sets.

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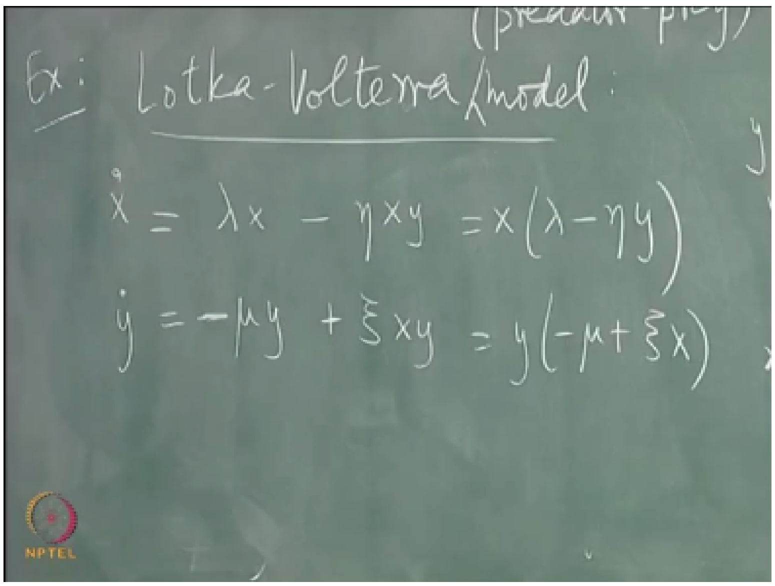

And this is the famous lotka –volterra in its simplest form this model has a large number of ramifications, but we look at the simplest form of this simple model here this model here and it goes as follows suppose you have a large population of some animal in the popular example is that of rabbits and it is large enough that you can write down the evolution equation of this population as a differential equation rather than a difference equation so we assume that the population X of rabbits at any instant of time is large enough.

That you could write a differential equation for it then left alone with plenty of grass and no predators X would increase and it will increase proportional to the existing population of rabbits multiplied by some constant which is the birth rate of rabbits, so x. is say some λ where γ would be a positive number the flow would be very simple in this case you do not worry about negative x at all it is not physical, if you start with any X that is positive it is going to exponentially blow up in time.

I am going to move along the x axis and blow off exponentially like e to the λt okay so it is an unstable fixed point at the origin and, now suppose you have some prey predators which eat these rabbits say foxes that is the original example so this model goes by the name of the predator and let us say y denotes the population of foxes, and if you had no rabbits around since forecast foxes do not eat grass they would actually become extinct they would die down, so this thing here will be some minus mu times y when u is a positive number.

And this is a decaying population so they actually die down and what this looks like is in the xy plane there is a flow line of this kind this is a critical point and as a flow line which goes down like this and you would say this is exactly what you expect of a saddle point because, if you start with the population of foxes here and a population of rabbits of this kind the rabbits would increase in population and the foxes would decay and this is what the flow lines would do, and now let us introduce interaction between the two species.

When you do this then since the foxes prey on the rabbits the population of rabbits is controlled by a loss term, now also has a loss term in the rate equation which would be proportional to the existing population of rabbits of course multiplied by a rate which is proportional to the existing population of foxes and it would appear with a – sign, so minus some new constant and let us call it maybe a toe or something like that times xy the foxes of course would prosper since they interact with the rabbits and they would increase at a rate.

Which is proportional to the existing population of rabbits with a plus sign and therefore this would have a+ some other constant says I x xy where all these four constants are positive constants there is also natural death which is taken into account here, but it is not taken into account here so certainly you have to add a term which depends which says that a certain population is depleted there is not only natural death but they could also be competition for the same food in which case to 2λX.

This involves interaction with another species but if there is for instance competition for the same food what kind of modification would you have for this rate equation, well you would have a loss term which would be proportional to maybe x2 because the same population is competing for the same amount of the same resources, so you could add competition you could add interaction you could have all sorts of complications to this but in the simplest form this is it as it stands we have not allowed for natural death.

Except to say that there is natural breeding and in any case in spite of natural death as we know if a population is left unhindered and has infinite resources then it actually increases geometrically in some sense, so this is a good pretty good model for the population of two interacting species a predator prey model where are the critical points well 0 is 1 and it is a saddle as we can see by linearization near 0 this is positive and that is negative it is clear x is the unstable direction and y is the stable direction comes in.

But the moment you put in these two other constant be there is a new critical point yeah it is quite clear there is another critical point somewhere here.

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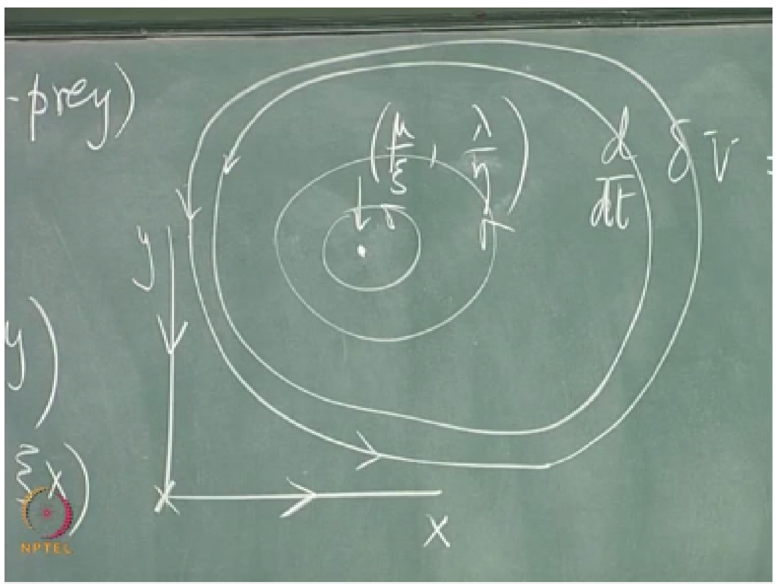

Which is of the form μ / ζ and λ / y this is another critical point at that point where you have this an extra critical point in the first quadrant then of course the question is what happens to these flow lines as we move along in this fashion and I leave you to show and this is not hard to do you have to linearize in the vicinity of this point.

That as a matter of fact if you are sufficiently close to this point it turns out to be a center that is not hard to establish and the flow lines look like this, arbitrarily close very close to the center the linearized problem would lead to ellipses in the xy plane, but of course as you move further away there is no reason why these flow lines should be ellipses this curve should be ellipses this would correspond to periodic motion in x and y, but the interesting point is for arbitrarily large values of x and y even far away from the initiate from this critical point.

These flow lines are actually parts of closed curves and they would come back and close on themselves wherever you are these are the only two critical points and this is a center that is a saddle point because we are not interested in what happens in the unphysical regions, and a center of course this kind of thing implies that the populations of x and y oscillate and this is the way this model was first proposed yes.

This is a nonlinear system a genuine nonlinear system and it is yes it means the real part is zero exactly so that is a good question his question is since we know this is a center by linearization a two-by-two system and it is truly a nonlinear system it is got other terms as well other than linear terms how are we sure that this is the way the flow lines look like how are we sure that this remains a center even.

If you include the nonlinear terms that is the question and in this instance it can be completely settled this problem can be solved completely and it turns out that this is indeed a center nothing happens to it and indeed you get a picture of what happens in it is vicinity by linearization around it this is one of those instances where nothing strange happens it does work it's interesting to show that all these curves are closed curves and as, I was saying this is the way the problem first arose when fish population.

In the Adriatic were compared it turned out that the populations of two species of fish one of which preyed on the other would fluctuate and oscillate at a certain interval with a regular period so when the prey fish was plentiful the predato fish was not and vice-versa and the fisherman in the Adriatic actually had a problem on their hands as to why they should be so with such regularity and it was mathematically analyzed vetoes Voters was a very famous Italian mathematician they analyzed.

This model and discovered that indeed there's a simple explanation which has to do with the way they compete with each other and you can see what happens in very physical terms if you start here at a large value of Y the population of prey is very small and therefore the predator population starts dropping because the predator population drops the prey starts growing and once the prey grows up to a certain point because there is predation upon it then the predator population starts increasing when that becomes too much the prey starts dropping because too many of them are being depleted and so on and it continues forever in other words.

This is a problem where coexistence is possible so at this point a certain equilibrium population of predator and prey fish of species can actually coexist with each other in stable equilibrium this is a stable point so this can actually go on forever unless you introduce other terms in the Hammond the problem the point .I was going to make and. I will stop with that was that if you changed variables and said certain variables equal to each other the simplest instance you could perhaps take a thing like this set all these constants =1 then the center is actually at 1 , 1 in this nonlinear system it's actually a nonlinear system.

You could ask is it conservative or is it dissipative I see no mechanism for dissipation in the conventional sense I do not see a node I see only saddle points and centers I do n0ot see a spiral point I do not see any nodes what happens to ∇ . F well if you work out Δ if you have to differentiate this with respect to X and this with respect to Y and add the two what do you get when X – y.

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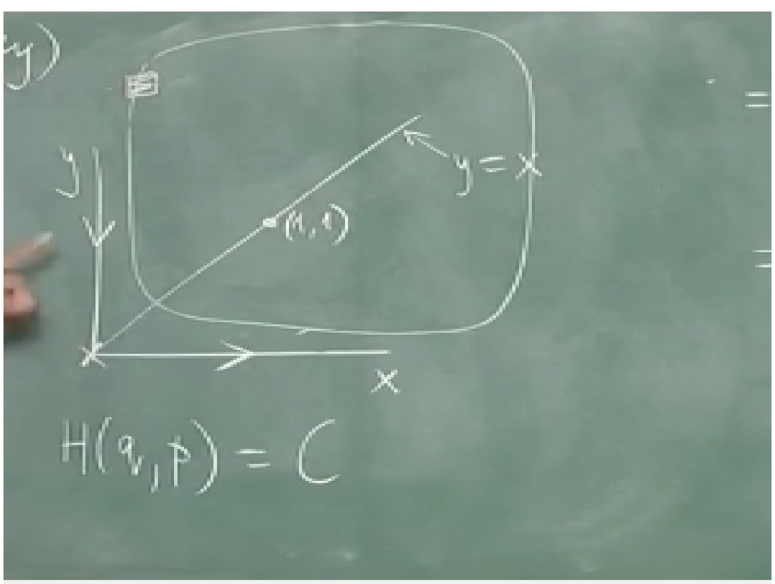

So this is X - y that of course is not identically = 0 it is = 0 on this 45 degree line and here is the center at 1 ,1 it is zero out there but then the phase trajectories which we can draw in this instance would perhaps be something like this and they'd be quite symmetric about this 45 degree line and in this region of the line X = Y it is clear X is bigger than Y so it is positive and in this region it is negative if therefore you started with a set of initial conditions of this kind there would be stretching in one direction and contraction in the other direction.

But on the average this distorted square as it goes along it gets distorted but if you took the average value of what happens to this quantity as you did a full cycle you would end up with zero so whatever ∇ . F does and it is positive here is exactly compensated for by whatever it does on this side where it is negative by the same amount and therefore the average goes to 0 you there for suspect that this system is really a hidden conservative system it is really not dissipative it goes on forever there is no mechanism for dissipation here although formally this is not identically =0.

It really is like a dissipated conservative system I urge you to do the following and we check this out set let me think log y = Q log X = P so we change variables in this fashion and rewrite the equations of motion in terms of Q and P and you will discover what happens to this critical point where would this go to go off to - ∞ because log zeroes tends to - ∞ where would this go it goes to the origin the origin is therefore a critical point in the QP plane in a finite part of the QP plane and you discover.

This turns out to be a Hamiltonian equation system with a certain Hamiltonian which I will leave you to find out and once you do that then of course you know that the Hamiltonian for autonomous systems is itself a constant of the motion and therefore you found the constant of motion and for a two-dimensional system on a phase plane how many constants of the motion do you need in order to find the curves themselves the phase trajectories themselves just one absolutely right you need just one because one equation between two variables gives you a curve and those are the phase trajectories for different values of the constant.

So you can find an H of QP and this = constant gives you in the QP plane the equations to these curves the phase took the phase trajectories and you could transform back now to the x and y variables to write down what these curves look like the alternative is to try and solve this set of equations and that' is not so easy because it is actually nonlinear far easier to find this curve this quantity far easier to find the constant of motion.

So that is the exercise find the equation describing these closed curves and once you have got the equation of course you can check explicitly whether they are closed curves or not so this would be a simple way of testing it out so we take it up from this point next time.

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