Indian Institute of Technology Madras
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TOPICS IN NONLINEAR DYNAMICS
Lecture 3
Two-dimensional flows

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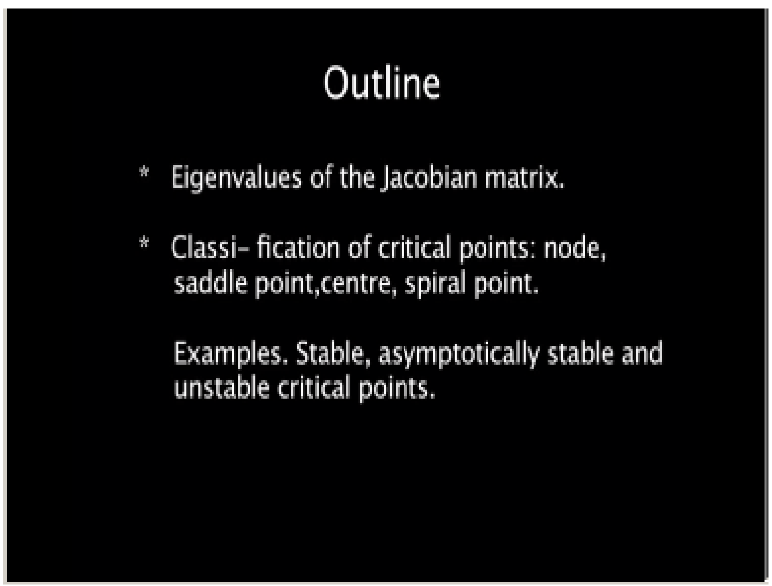

So we resumed our study of dynamics in the phase plane and you recall that we were going to analyze the behavior of a two dimensional dynamical system near a critical point which we have taken to be the origin.

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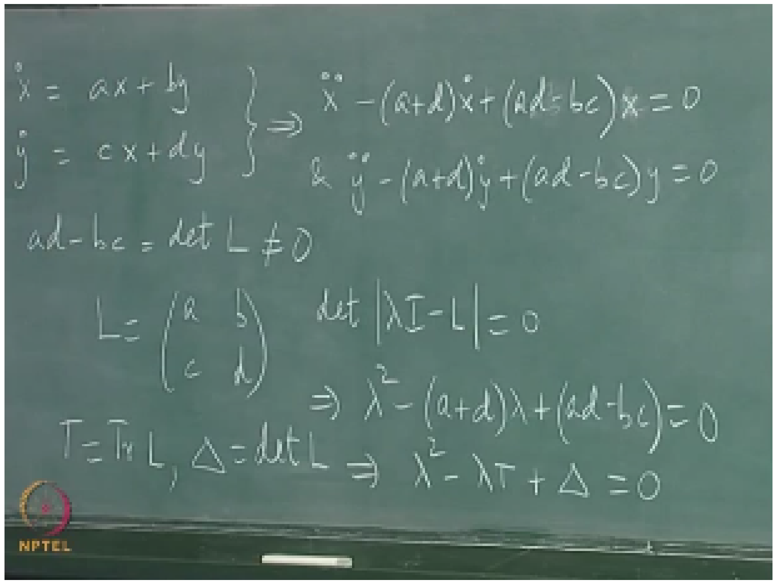

And the set of equations that we are interested in is a linear set of the form ax + b y and y . cx + dy where the constants ABCD are real constants and we also have assumed that ad minus BC is equal to the determinant of the linear matrix L is not equal to zero so this is the problem we would like to look at and understand the nature of the critical point at the origin based on the Eigen values of the matrix l.

And we pointed out that the general solution in this neighborhood could be written as linear combinations of the Exponentials of the two Eigen values of the linear matrix L L itself a rose in a nonlinear system as the Jacobean matrix at the origin of the flow itself of the vector field F itself but right now we are focusing on the linear system per se as it stands and we have seen that the general solution as I said is a sum of Exponentials.

Now if you eliminate either X or Y from the set of equations then this implies a second-order differential equation for X which is of the form X double dot minus a + d x . + ad -bc =2times X = 0 and similarly Y double .- a +dy . +ad -pc y = 0 X and y of course would differ in general in their initial conditions as would X dot and y dot and you would get different combinations of e to the λ 1 t and e to the λ2 T if λ 1 should happen to be equal to λ 2 then the solutions are of the form linear combinations of e to the λ T and T e to the λ T.

We have seen that too and now we are going to analyze the stability or otherwise of the set of equations depending on what the Eigen values of it look like notice immediately that if you have a matrix L whose Eigen values matrix L ABCD whose Eigen values are λ 1 and λ 2then we find the Eigen values by writing determinant λ times I -L = 0 which implies λ^{2}-a +T λ+ ad -bc = 0 do you notice anything interesting about these combinations this combination.

And that combination they are not arbitrary combinations of a B C and D they are very special combinations this is the trace of the matrix and that is the determinant of the matrix so this immediately tells you that if I set T equal to trace of L and Δ equal to the determinant of L this implies that the Eigen values are given by λ^{2} -λ T plus determinant equal to 0so the Eigen values of functions of the trace and the determinant of the matrix.

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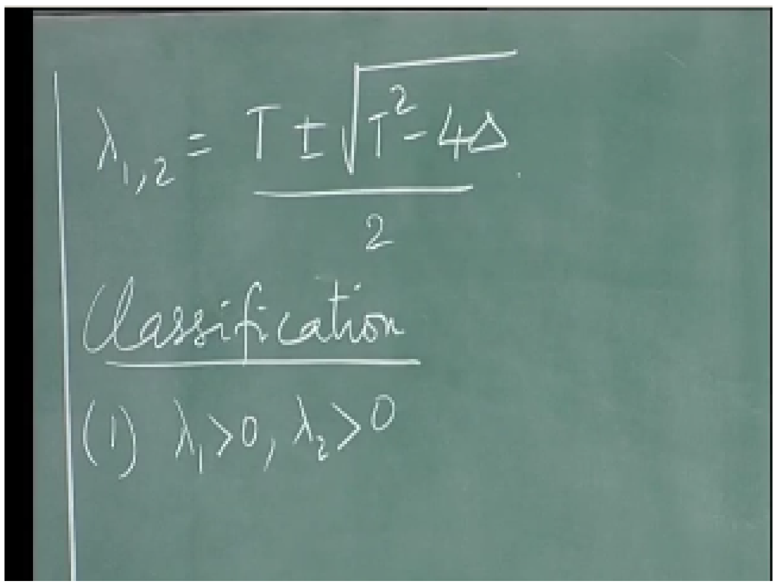

What does that imply λ 1, 2=T + -$\sqrt{}$ T^{2} -4 Δ/2 and recall that the solutions are Exponentials in λ 1 and λ 2 so what does this tell you what does it tell you about these what does it tell you about these Eigen values the fact that it is dependent only on the trace and the determinant and not on the actual all for not on all four coefficients but just these two combinations that is going to play a role in what we are going to say next what, what are these combinations what is so special about these combinations what happens.

If I do a similarity transformation on the matrix the trace does not change in the determinant does not change which means that if I transform these variables from X and y to certain linear combinations of x and y by a similarity transformation by coordinate transformation the Eigen values would not change and that has a physical implication as we will see shortly so right now yeah one is the sum of the Eigen values and the other is a determinant.

So it involves yes exactly the other is the product of the Eigen values because this quantity here the coefficient of the linear term is minus the sum of the Eigen values and the last coefficient is a product of the Eigen values these two quantities are independent of what sort of linear transformations you make on the matrix as we will see in a short while so the behavior of the Eigen values is going to be determined entirely by the behavior of T and Δ.

Therefore and we will take up this lesson more deep in more detail little later the entire behavior of the dynamical system near the critical point is going to be governed by just the combinations T and Δ therefore we could regard these two as parameters and ask what kind of behavior you have in the T Δ plane which we will do in a short while but first what are the various possibilities that you have for these Eigen values in terms of what kind of quantities can they possibly be and let us classify them.

So we now classify we start a classification first the first possibility of course is that λ_1 and λ_2 are both positive that is certainly a possibility so let us write $\lambda_1 > 0$ $\lambda_2 > 0$ the solutions as we have seen are Exponentials in λ_1 and λ_2 e to the λ_1 T and e to the λ_2T so as time goes on what happens to these Exponentials if both are positive they increase their growing Exponentials and therefore the flow would be away from the critical point in all directions and you would say this is an unstable critical point.

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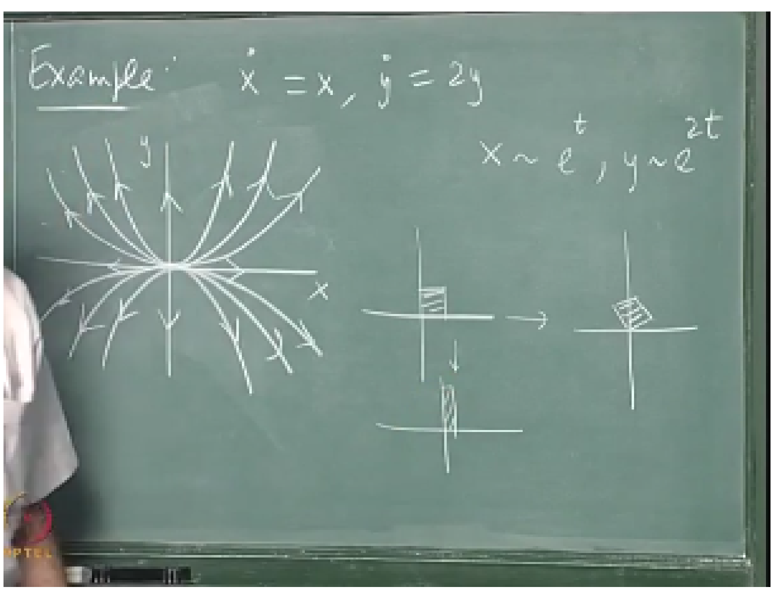

Let us look at a simple example and then we comeback and formalize this in terms of the actual classification here so the example that we are going to look at and each time on this side of the board I am going to give an example X = X Y = Y what is the linear matrix L now in this case just the unit matrix and the two Eigen values of + 1and +1 each repeated Eigen value what would you say is the solution to this.

Let us draw the phase diagram now here is X and here is y here is the critical point clearly unstable in this case because everything is going to flow outwards if I started with y equal to 0at any point on the x axis I flow outwards if I start anywhere here I flow outwards anywhere here I flow outwards and anywhere here flows outwards if I start at any point here it is clear you are going to flow away in this fashion.

So the phase portrait in this case is exceedingly simple and it just the set of lines flowing out in all directions what would you say is the stability of this critical point unstable and this kind of critical point is called an unstable no de what would have happened if the two Eigen values had been unequal in this case suppose this had been y dot equal to twice y what would the flow look like in this case the two Eigen values are 1 & 2they are both positive and the flow is it again outwards but what sort of shape would the flow lines have it is evident from this.

That X goes like e to the power T while Y goes like e^{2T} so therefore Y goes like x^2 what would the flow lines look like in this case but if y you start off with y =0 you are still going to be like that if you start off at x =0 you are still going to do this but if you start it off with finite values positive value say of both x and y the y is going to increase much more rapidly than the X like the square of X actually.

So it is actually going to flow off in this fashion or in this fashion here or here, here therefore the phase portrait looks like this except for a flow along the y-axis itself the rest of the flows have a common tangent which is the x-axis case this too is an unstable node it is just as different set of Eigen values of course if I change this to 3y or any multiple of the exponent in X the coefficient in X you are going to have slightly different shapes of this parabola had the coefficient of x been larger than the coefficient of Y the whole situation would have been reversed.

And you would have had parabolas flowing out such that the y-axis is the common tangent but it's still an unstable node now what would you say would be the situation if you did not have decoupled equations of this kind remember that notice that in this case the coefficient of Y is missing here and the coefficient of x is missing there so it is already a diagonal matrix but in general I do not have a diagonal matrix el has got both x and y on both sides.

What would you say would be the flow in that case can we deduce what the flow would be in geometrical shape based on the fact that in the decoupled case it looks like this in general after all you could regard ax plus B Y and cx + dy as linear transformations of the x and y coordinates by the matrix ABCD which is non singular and what does it do if I take two axes x and y and I go to linear combinations ax +b y and cx +dy what sort of coordinate axes do.

I get in general it could be rotated but rotation still leaves the angle between the x and y axis the two, two axes 90 degrees but in general that does not have to be the case I could go to oblique axis more over the scale of X and the scale of Y are also changed because these are factors which actually change they magnify or D magnified in the most general linear transformation that you can induce which leaves the origin unchanged is precisely by a set of linear equations of this kind.

If I put u = ax + b y and V is cx +dy then in general the U and V axis an order right angles to each other the scale on one direction could have been d magnified or magnified and similarly for the other direction in pictures this means that I could start with a set of axes and if I have a little square of this kind and ask what happens to this after the transformation if it were a pure rotation it will just look like this still a square look like this if on the other hand it is not a rotation but a magnification in one direction.

And a contraction in the other direction it would perhaps look like this so I have contracted the X direction and expanded the Y direction on top of these two transformations there is one more possibility and that is to change the perpendicular orthogonal axis into public axis in other words I shear in some direction so that would look like this a shear would take it to some shape of this kind and a general 2x2transformation at general 2x2 matrix would take this little unit patch.

And in general push it in one direction squash it in some direction expand it in another direction rotate and shear and therefore the final product would actually be something like this perhaps so it is been rotated sheared and magnified or dilated in some direction or T magnified in the other direction how many parameters are there in this matrix in specifying algebra for real parameters here we must make sure that in this way of counting or decomposing a matrix a transformation a linear transformation into a rotation a dilation.

And a shear we stick to the same number of parameters and I put it to you that a general 2 / 2 linear transformation of this kind non singular matrix induces a transformation which is a combination of a rotation dilation or a magnification and a shear the number of parameters needed to specify a rotation is 1 just the rotation about from the reference axis x axis in the plane so there is one parameter here.

This has two parameters for the two axes that is one plus two is three and this gives the handle of shear that is one more that is four and that is exactly the same number there you can count this in several ways this is not a unique decomposition but this is one of the ways of decomposing a general 2/2matrix you can decompose it into a rotation a dilation and a shear if we accept that this is all it does then we can write down what the general flow would look like once you induce those transformations on this decoupled case and what would it look like.

It would simply look like some weak axis of this kind and perhaps etcetera so the moment I see a flow pattern of this kind I know it is an unstable node and it is just found from this simple case by a linear transformation of coordinates in this kind therefore we assert that both Eigen value is positive corresponds to an unstable node notice there are two different kinds of nodes in pictures one of them is this generic case where the two roots are unequal.

And typically you would have a common tangent to all the trajectories except for in a single exceptional direction on the other hand the earlier case we looked at had a star pattern at a radial pattern in which you had straight line emanating in all directions do not have to be straight lines but the fact is that all directions there are no exceptional directions at all, all directions the flow moves often there is no common tangent these are two subtle distinctions between further distinctions.

Between kinds of nodes that one has but is not really very relevant here and right now we are concerned with the fact that two positive Eigen value simples an unstable node of the we will come back to this we will explain what sort of flows would have this extra term and this is common tangent etcetera in specific instances in examples what would happen if both the Eigen values were negative less than 0 λ 2 > 0 all the arrows would be reversed.

Because you would have damped Exponentials and this means that as T tends to ∞wherever you start you are going to flow into the critical point at the origin as importantly therefore I would call this I would call this an asymptotically stable node the reason is I would like to be careful about the definition of stability so let us for the moment call it and as emphatically we will distinguish shortly between stability.

And asymptotic stability in there different concepts altogether this is asymptotic stability we are guaranteed that once a trajectory enters the neighborhood of this point the critical point then as T tends to ∞ it inevitably falls into this critical point that is asymptotic stability the third possibility is if one of the Eigen values is positive and the other Eigen value is negative once again all we have to do to study.

This is to look at a decoupled case a case where the matrix is already diagonal and then argue that a linear transformation of that picture would give us the true picture near a situation which corresponds to one positive Eigen value and one negative Eigen value so all we have to do is perhaps to do the following write x dot equal to minus x and y.=+ 2y in this instance X goes like e to the minus T flows into the fixed point where the critical point where as Y flows into 0 where as Y explodes outwards like e to the 2t.

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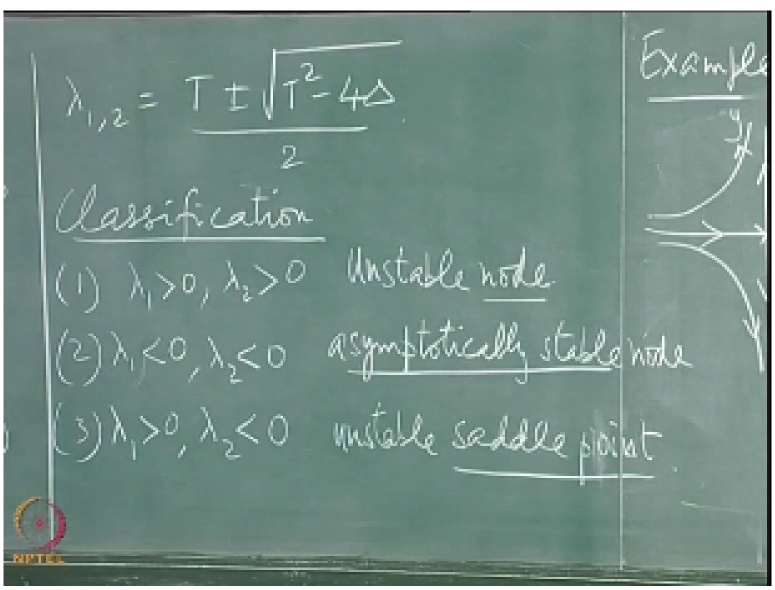

So we know the solutions what does the flow look like well on the x axis since it flows in you have this picture and on the y axis since it flows out you have that picture if you start with some point here some value of x which is not zero say some negative value then it is got to flow into ward zero but the Y has to flow outward and therefore the trajectories look like that the continuity this is what they would look like would you call this stable or an unstable critical point.

And call it unstable I have call it unstable because except for very special initial conditions in other words Y starts at zero and remains at zero if you are on the x axis to start with you flow in but wherever else you are on the plane you are actually going to go away to ∞ in some direction or the other therefore this is unstable what is the shape of these trajectories it looks like hyperbolas but actually they are not hyperbolas unless of course the coefficients are equal in magnitude then of course you would have something like XY equal to constant.

And they have be rectangular hyperbolas if you had X.- x and y .- y then of course XY is constant and then they would look like rectangular hyperbolas but right now that is not happening the Y variable is exploding much faster than the X variable is going to zero but they look hyperbolic in shape roughly in shape this is an unstable critical point and it is called a saddle point.

So that was this is a saddle point again if I had a situation where I have a general L with two real Eigen values one of which is positive and the other is negative the flow would look like a distortion of this picture by the same sort of coordinate transformation we looked at earlier and therefore in general perhaps this is what a saddle point flow would look like there be two directions in which the flow goes in another one in which it goes out.

And the rest of the trajectories would follow curves of this kind and the whole Plains in this case would be striated by these curves this fashion at that point itself is a trajectory by itself and it is unstable although I draw all these pictures with these lines going right through that point remember that that point is a trajectory by itself but you can come are bitterly close to it and that is the reason.

I draw these as continuous lines it is natural to call this the unstable direction and this the stable direction associated with this saddle point will say much more about this as we go along so that is what a general flow near a saddle point looks like in this linear case these lines are actually straight lines as you can see but in a general case with the non-linearity present the first thing that would happen is that these things although arbitrarily close very close to the critical point.

They would tend to be straight lines the actual flow lines would be curved in general so you would not have straight as into ads as you have here in a non linear system that too will play a role so we have the third possibility namely it is a saddle point here remember we ignored the case where one of the Eigen values can be 0 1 or both Eigen values can be 0 because I said we are going to look at all those cases where the matrix is non-singular.

Therefore it cannot have 0 as an Eigen value we have to look at it separately we have to ask what happens when the Eigen value one or more of the Eigen values is 0 what other possibilities exist have we exhausted everything complex Eigen values absolutely right we have complex Eigen values possible and the first thing that would happen is first thing we have to recognize is that since these coefficients are real the complex Eigen values would be a complex conjugate pair.

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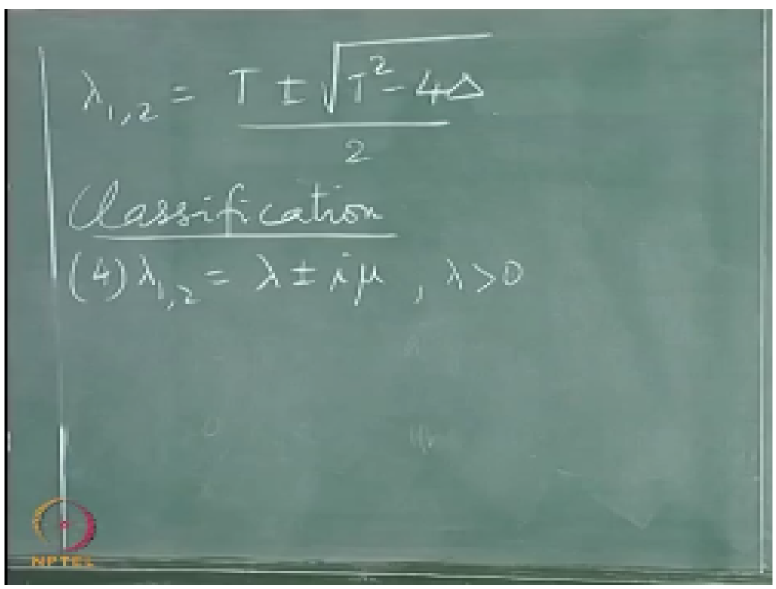

Therefore all we have to look at is a situation in which for λ 1 ,2 is some λ +- I mu where λ and mu are real numbers and λ > 0 the behavior of these trajectories is not really crucially dependent on whether mu is positive or negative it is irrelevant because e to the I mu T is going to be just cosines and sins of MU T on the other hand you have an e to the λ T and if λ is positive it is going to flow outwards if λ is negative it is going to flow inwards.

But there is going to be an oscillation in the sine of X and why because of e to the I mu T present there because that is going to read two cosines and sins which would change sine as T increases what then would the flow look like again.

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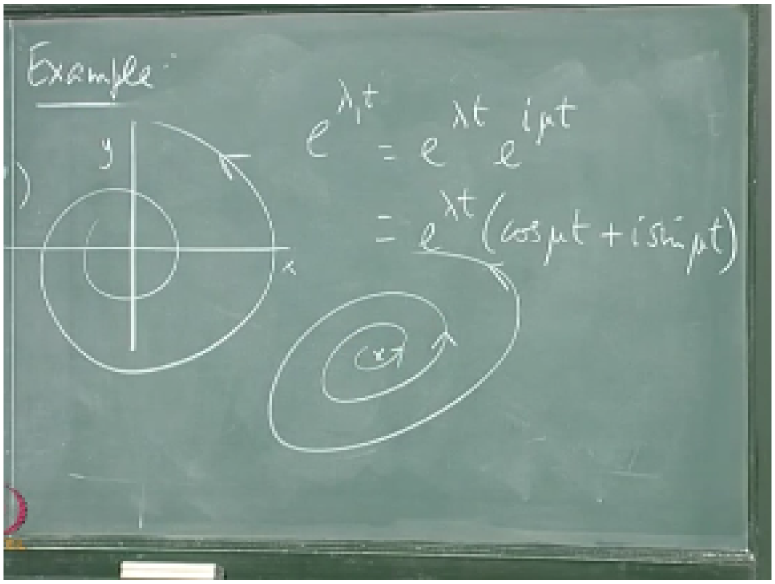

I go to the face plain here X and here is why if the flow has a positive value of μ it means as time goes on x and y would change sign but would flow away from the origin and they would oscillate with an increasing amplitude and go off so what sort of curve would you expect you would expect a spiral quite right.

But it would be an outward standing spiral it would go off in this fashion you know that because e to the power so let me write it here e to the λ 1t for example would be e to the λ T e to the I μ T which would be e to the λ T times the cosine of μ T + I sine and of course super positions of e to the λ 1 t and e to the λ 2 t would involve these cosines and sins and the coefficients would be adjusted.

So that the linear combination is real because we are looking at real variables but the fact is because of this cosines and sins although this increases as a function of time monotonically these functions would change sign they would oscillate between positive and negative values and all the while the amplitude would increase so in a picture like this if I start at some value here of X a little later the value is larger and magnitude.

But opposite in sign and then once it comes back here it is again larger in magnitude but opposite in sign and the same thing is true for y so it is intuitively clear that the picture is actually that of an outward spiral outward moving spiral in this case therefore is called an unstable spiral point there are other names for the spiral point one of which is the word focus occasionally called a vertex as well but focus is much more common.

But I would like to stick to the word spiral point the phrase spiral point because this is, is evocative it tells you exactly what the, the trajectories look like once again if I have an linear transformation on this spiral what would what sort of distortion would this spiral undergo what would it look like that is not hard to see it could get squashed in one direction turned around and extended in the other direction maybe.

Therefore it would perhaps look this fashion this would be the general flow around the spiral point which is unstable and it is unstable because λ is positive and exactly as in the case of the nodes you could assert that if λ 1 2 were of the form λ +-μ with $\lambda < 0$ the arrows would be reversed and the system would flow into the critical point as t10 2 in tends to $\infty$and this would lead to an asymptotically stable spiral point the arrows would just be reversed whether the spiral is found in words clockwise or counterclockwise depends on the details of the problem.

It would actually depend on the direction in which the phase trajectories are traversed that would depend once again on the coefficients ABCD that is it but this is what happens when you have a complex conjugate pair of points there is one more possibility what is that there are subclasses here between in these cases when I have for instance two equal roots both of which are positive both of which are negative or one of them is positive and the other is negative.

And they equal in magnitude those are subclasses of what we looked at but what would happen there is one more class which is distinct λ is both the Eigen values of pure imaginary they form a pure imaginary complex conjugate pair so λ could be zero and yet determinant l is not 0 because the Eigen values are +- I μ then that is the final case just write that down 6 λ 1, 2 = +- I μ where mu is a real number.

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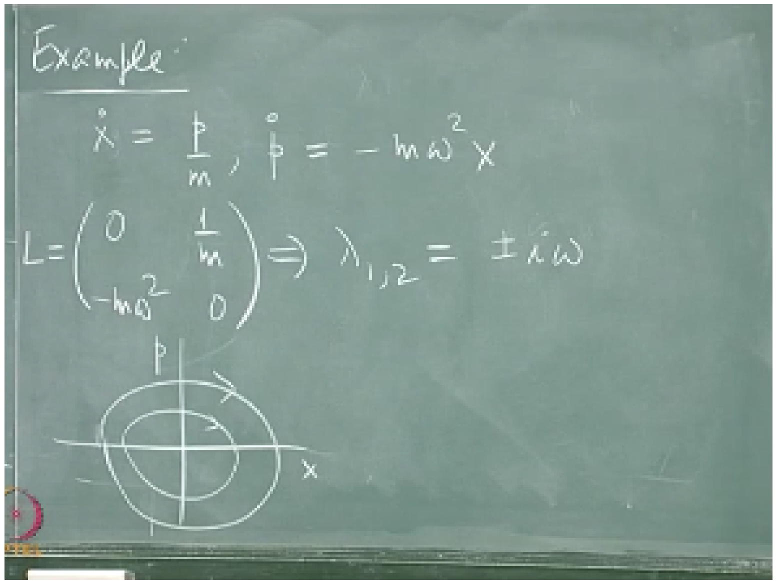

The simplest example of this is in fact the simple harmonic oscillator we have studied that over and over again but let us write that down and that if you recall corresponds to a problem in which X . P over m where P is the momentum of the oscillator and P dot which is the rate of change of momentum is equal to the force on the oscillator which is minus M ω^2 X or -K X where K is the spring constant what is the matrix look like.

This is a linear system in which L = 0 here a 1 over M here - M ω^2 here and a 0 here this implies λ 1 ,2 equal to what are the two Eigen values in this case + - I ω and all as all of us know the solutions are linear combinations of e to the I ω T and e to the - I ω T what is the phase trajectories look like in this case instead of y I have P here they do not necessarily have to be circles.

The scales on the two axis may be different and their generalizations of circle so they really ellipses of some kind and so the entire plane is striated by these laminated by these ellipses concentric ellipses would you call this critical point here would you call this stable or unstable or asymptotically stable what would you call it I centrally would not call it unstable because the trajectories do not disappear from its neighborhood.

They do not blow away from it on the other hand they do not fall into it either it just goes round and round so we call this a stable critical point and this particular case is called a center so it is a stable center notice that I have set stable and not as imported so the time has come now to distinguish between stability and a asymptotic stability and these are two different concepts altogether one does not imply the other as we will see in a second.

I define a critical point to be stable if the following is true and this is do able rigorously but let me give an operational definition which is easy to understand stable here is the critical point and here is some neighborhood of this critical point.

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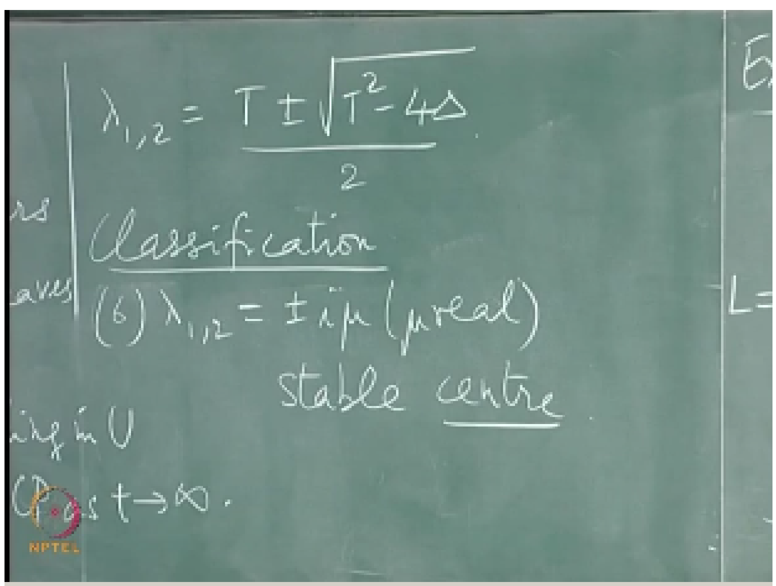

Then a stable critical point is one where a trajectory if it has once entered this neighborhood never leaves this neighborhood so something which has entered it may do all kinds of things inside here it may fall into it, it may also keep going around or it may do this but it never leaves this neighborhood I call that a stable critical point if this is neighborhood is you once a phase trajectory enters the neighborhood you it never leaves it, it does not have to be asymptotically stable.

This one you would say this is the situation I would say that this critical point these trajectories tend to this point and therefore we are going to have asymptotic stability but stability does not require that it suffices that a trajectory which one centers once it enters this neighborhood remains in this neighborhood forever and that is exactly what happens here if this is a neighborhood there exists a trajectory there exist trajectories which remain in this neighborhood forever.

The concept of stability depends on the neighborhood so like everything else you going to have to perturb the system a little bit away from the critical point and ask what the trajectory is to always that is how stability is defined in any case it is not a concept applicable to a single point but to a region yes you need to know if there are other critical points in there so the question you are asking really is how big can this neighborhood be in this case there is just one critical point.

So the entire plane there is a single critical point and no matter what neighborhood you give me there exist trajectories which having once entered the neighborhood never leave this neighborhood altogether if I have more than one critical point then the question that you raise comes up and we will see what happens when you have multiple critical points but right now just think in very elementary terms.

If I look at mechanical examples I would say that a stable equilibrium of a in a potential problem is if the potential has a minimum but the idea of a minimum is not a concept restricted to one point it says something about not only the function at that point and its slope but also about its curvature at that point so it is really a non-local concept in that sense talks about a neighborhood of this point that is always going to be the case.

So I say that a critical point is a stable one if there exists a neighborhood of this critical point such that once the trajectory enters this point neighborhood it never leaves it as t 10 to ∞what then is asymptotic stability well asymptotically stable again a critical point this point is asymptotically stable if a trajectory it starts in this neighborhood tends to this point as T tends to ∞ you guaranteed that every trajectory that starts in this neighborhood tends to that point as T tends to ∞.

So every point every trajectory starting in U and to the CP as T tends to, so something starts here it is guaranteed to fall into this asymptotically as T tends to ∞ does asymptotic stability implies stability it is clear that stability does not imply asymptotic stability because it could keep going around does not have to fall in but does asymptotic stability implies stability not necessarily because you could start here you could in fact do a thing like this.

So it could be a spiral it goes out very far away but it is guaranteed to come back so it starts off as this goes around and eventually falls into this point here so it certainly does not have to be stable there for stability and asymptotic stability they are not concepts they are independent concepts all together you could have a trajectory that stable as well as asymptotically stable you could have a critical point which has the property.

But it is not necessary centers particularly are stable not as important and spiral points stable asymptotically stable spiral points could have behavior like this they need not be stable but they would definitely be asymptotically stable the statement is that they are given a neighborhood there exist this sort of behavior I can make this much more rigorous but I am just trying to do this in Euro stick terms right.

Now till we have a few examples under our belt and then of course we could look at more tests for stability and so on so one of the things we are going to do is to have a kind of nonlinear test for stability which does not depend on linearization as we have done here and that is Lyapunov direct method and we will talk about it a little later yeah did anyone have a question so as a Center in particular is stable.

But not as important this is an important point and we will see why in a short while this exhausts all possibilities of those cases where the determinant is not zero and now can we make some sense out of this I already mentioned that everything seems to depend on T versus ΔT and Δ just these two combinations these two combinations which are unchanged under similarity transformations on the matrix and therefore it is clear that.

The nature of the critical point is kind of independent of these transformations it does not depend on the particular choice of coordinates x and y linear combinations of this x and y would still leave the nature of the critical point unchanged that is very important to know because this is a very robust property that we are talking about okay now what can we say about this classification in general terms we can draw a little picture in parameter space.

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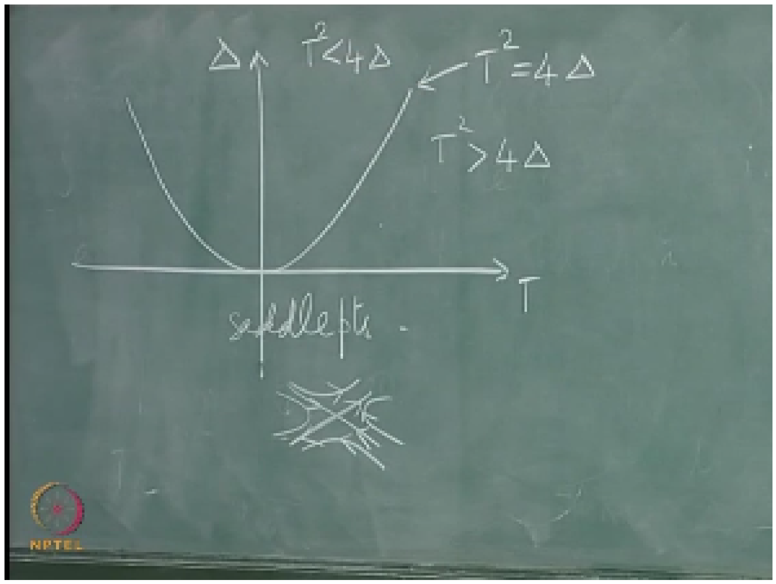

So I draw tea on this axis and the determinant Δ on this axis it is obvious from here that the case $\Delta = 0$ is on this line and that is the case I am going to ignore at the moment and the curve $T^2 = 4\Delta$ would separate regions where the Eigen values are complex from those where the Eigen values are real and what is the curve $T^2 = 4\Delta$ it is a parabola so on this curve T^2 4Δ and outside $T^2 > 4\Delta$ whereas inside $T < 4\Delta$.

And then the roots are complex what sort of behavior do you have here where Δ is negative if Δ is negative the square root of this number is certainly bigger than T in magnitude and therefore one Eigen value is positive and the other Eigen value is negative therefore below this axis you have entirely saddle points one positive eigen value another negative Eigen value so this whole place is saddles saddle points.

And the flow looks like this but they are on this side of the graph or on this side does not matter you just have saddle points and the whole thing is unstable everywhere in parameter space in this region you are guaranteed to have unstable saddle points what happens when you move up here if you are here say $T^2 > 4\Delta$ certainly but what happens now to these Eigen values both are positive it is an unstable this place is unstable.

But what sort of unstable point what sort of unstable point do you have two positive Eigen values unstable nodes so you have unstable nodes therefore the picture would be perhaps something like this what would you have here you would have stable nodes asymptotically stable nodes so as importantly stable nodes in this region and these are unstable nodes in that region what would you have here.

When $T^2 > 4\Delta$ T is positive remember you are on the right hand side so T is positive and this becomes pure imaginary so the square root gives you +- I times something or the other but then you have a positive real part therefore you have unstable spiral points so unstable spiral points flowing away and what would you have here you would have stable as emphatically stable spiral points so in this case.

The flow is in words what you have on the vertical axis t0 you have sentence so on this K in this case right here you have Center all along this point you have centers so essentially you have this sort of behavior we now come to a very important concept this picture is sufficient to tell us why this is so why this happens to be so if I start somewhere here and I perturb these parameters a little bit I change ABCD a little bit I may wander to this place.

But what was an unstable node remains an unstable node similarly if I move around a little bit here nothing much happens if I move across this line I go from an unstable node to an unstable saddle point if I move across this line of course is a big change I go from something that is asymptotically stable to something that is unstable but then we have crossed the line of degenerate or higher-order critical points where L becomes singular.

But with that exception everywhere else if I cross this line what was an unstable node becomes an unstable spiral point what was an asymptotically stable node becomes an asymptotically stable spiral point but if I am on this line and I perturb parameters a little bit if I move to this side it becomes unstable if I move to this side it becomes asymptotic lease table so you can see that this line of points the line of centers is structurally unstable in the sense that small change of parameters can completely change the qualitative behavior of the critical point.

So we have our first statement which is somewhat general it is centers are structurally unstable when mathematician says structurally unstable they mean that if you change parameters slightly then there is a qualitative change in the behavior of the system and this is exactly what happens to centers what sort of physical motion does that example of a center correspond to you have pure imaginary Eigen values so what is special about this motion periodic motion periodic motion.

So this implies that periodic motion is structurally unstable small changes can get you completely away from periodic motion I add a little bit of friction here I had a tiny bit of friction here and this becomes this thing has another term M γ X dot where γ is the friction constant assuming the friction to be proportional to the instantaneous velocity with the retarding coefficient γ and I use an M γ there.

So that γ has dimensions of time inverse then you immediately see that this thing here has - M γ here at this point or sorry in γ times X. so let us call this - γ P γ here and then of course the Eigen values not equal to +- I ω what happens to the Eigen values what are the Eigen values of this the -γ the γ- γ +-I ω right therefore the Eigen values do not look like this at all what sort of critical point is the origin.

Now asymptotically stable spiral provided γ is positive provided the friction acts as a retarding force the friction does not act as a retarding force but pushes you along in the same direction even further if there is positive feedback then of course it becomes unstable so this means that the phase trajectories in this case would reduce to something like this.

This is the case of ordinary friction γ >0 so introducing a small γ pushes you from here to this region to asymptotically stable spirals if γ had the wrong sign then of course you would have become unstable and it would move into the other direction altogether so that is why periodic motion is so fragile it is not robust at all conditions have to be just so in order to have periodic motion and they are really the exception rather than the rule this has further implications in dynamical system.

For example or I will say it in loose terms now we will look at it more rigorously later heartbeats if they are too regular that is not very good so if you look at the heartbeats of young infants with very robust hearts they are actually extremely irregular in some sense they are on something like a chaotic at tractor which is very robust structurally stable small perturbations do not push you off this attractor on the other hand if the heartbeat gets extremely regular.

And becomes periodic you have to worry because this implies that a small change could cause you to have an explosive growth and either direction which is not very good so again this is the lesson of some generality centers are structurally unstable this is important to remember here now once we have this classification under our belt we really can generalize this and go on in very many directions and the first question I want to ask which will dispose off right away is that the Eigen values depended on the trace.

And the determinant in this case if you go to higher dimensions instead of 2x2systemsXY systems if I look at higher dimensional dynamical systems n-dimensional systems then the liberalized matrix would be n / n it would then have n Eigen values and once again we expect that the Eigen values are independent of similarity transformations.

They stable against similarity they are invariant under similarity transformations which would imply that the Eigen values should be writable in terms of quantities which are invariant under similarity transformations of the matrix in the 2/ 2 case we know that the trace does not change and the determinant does not change what happens in the 3x3 case you need 3 such combinations where are you going to get them from once again the trace of this matrix does not change and the determinant does not change.

But you need one more where is this going to come from and in the end by in case we need to have n invariant combinations where are these going to come from I need n of these quantities the determinant is just one of them I am willing to put that down but that does not give you n of them for the 2 by 2case that was sufficient whether the other is going to come from well we do know we do know that the coefficients in the secular equation would be the product the sum of all the Eigen values the products two at a time three at a time.

And so on and so forth so what invariant quantities are they functions of you need to put everything in terms of quantities which do not change under similarity transformations no not quite them in, in terms of L what, what are the quantities that would be unchanged what happens if I square the matrix L and then take the trace what would that be for the two by two case what is the square.

What is the trace of l^2 well imagine you have diagonal zed well you do not cannot always diagonalizable matrix but imagine you have diagonal zed it what is l^2 look like in the diagonal form exactly $\lambda 1^{+} \lambda 2^2$ and what is the trace of that what is the trace of our diagonal matrix with elements $\lambda 1^2$ and $\lambda 2^2$ just the sum of these two so it is quite clear that if you give me $\lambda 1 + \lambda 2$and $\lambda 1^2 + \lambda 2^2$ I could certainly find $\lambda 1$ and $\lambda 2$.

Because I have straight away find $\lambda 1 \lambda 2$ by combining these two and then I could find $\lambda 1$and $\lambda 2$ independently so what would be the generalization of that absolutely it would be trace of l^2 trace of L^3 up to the trace of L to the power it is easy to check that ad -bc in the 2 / 2 case can really be written as a combination of the trace of l^2 and the trace of L the whole squared so these are the invariant combinations you guaranteed that these combinations do not change under similarity transformations.

And that is what the Eigen values are functions of after all the statement is that if you give me $\lambda 1 + \lambda 2$ up to λn $\lambda 1^2 + \lambda 2^2$ up to λn^2 and similarly $\lambda 1^n$ sum till λn^n then I can find all the λ s in terms of these quantities they are invariant so this generalizes what we know for the 2 / 2 case just aside remark when can you diagonalizable a matrix we can always find its Eigen values by writing determinant $\lambda I - L = 0$.

And that is an algebraic equation of the nth degree and it has n roots in the complex plane what would be the case and what would be the condition for diagonal zing a matrix by a similarity transformation when can I take a matrix M and find the similarity transformation s such that this is a diagonal matrix and of course once it is diagonal its elements are just the Eigen values of the matrix.

And what conditions can you diagonalizable a matrix by a similarity transformation this is not always possible you can always find the Eigen values that is a different problem from diagonal zing the matrix there are lots and lots of matrices which are not diagonalizable by similarity transformations a sufficient condition for you to diagnose a matrix.

A sufficient condition is that the matrix M commute with its transpose with its conjugate summation conjugate if this is true by this dagger I mean the hermitic an conjugate of this matrix which is the transpose complex conjugated that is a sufficient condition for m to be diagonalizable by a similarity transformation such a matrix is called a normal matrix so we stop at this stage and take it up from here you.

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