Indian Institute of Technology Madras
Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 28
Stochastic dynamics (Part V)

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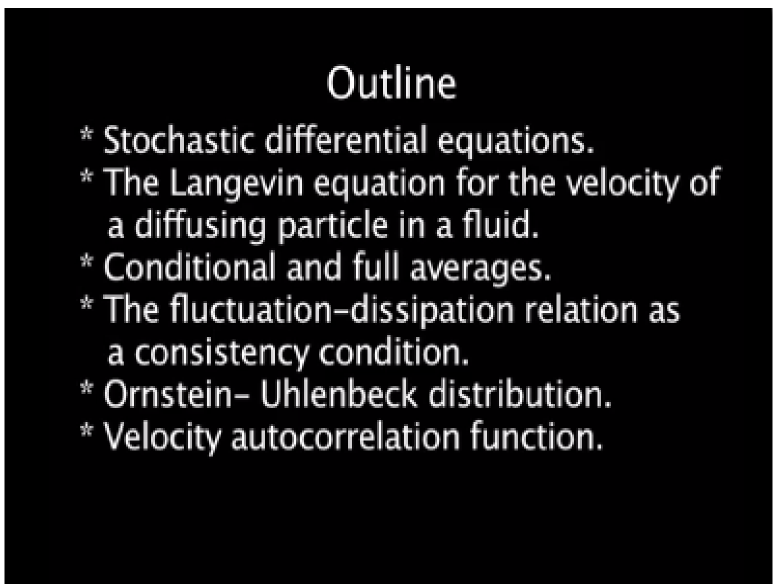

Here we have so far concentrated on dynamical systems which are generally non-linear with special reference to systems which are deterministic the sense that there were specific rules of evolution and then we looked at the nature of the solution the dependence on the initial conditions and sensitive dependence in the case of chaotic systems. We looked at both differential dynamics where systems were described by sets of differential equations of the first order coupled first order differential equations and discrete time systems versus the dynamical system was described by a map of some kind of recursion relation which gave that evolution in time.

But you see in real life most systems that we deal with would also have influences from outside or even internal sources of noise which cannot be given diled descriptions you can only give statistical descriptions for such influences, and then the resulting equations that you have no longer are deterministic equations but they would involve components which are random some kind of noise which is imposed on the system.

So the general dynamical system would have portions which are deterministic and portions which are pure noise with some specified statistical properties, and this would lead us to the area of stochastic differential equations stochastic difference equations and soon we haven't considered that very much in this course but I just like to introduce you to the very elements of stochastic.

So that you can make some contact with what happens in real systems some of the typical annoys is that one deals with, now of course we also took a little excursion into probability and a little bit about Markov processes and so on. So we will build upon that knowledge and try to go on from there to write down the most elementary stochastic differential equations. Here is one instance of where it can happen in a practical situation you know that if I have a set of chemicals reacting with each other in general you would have the concentrations of these chemical species the different molecular species as your dynamical variables and you perhaps have a set of differentially equations of the form if the concentrations.

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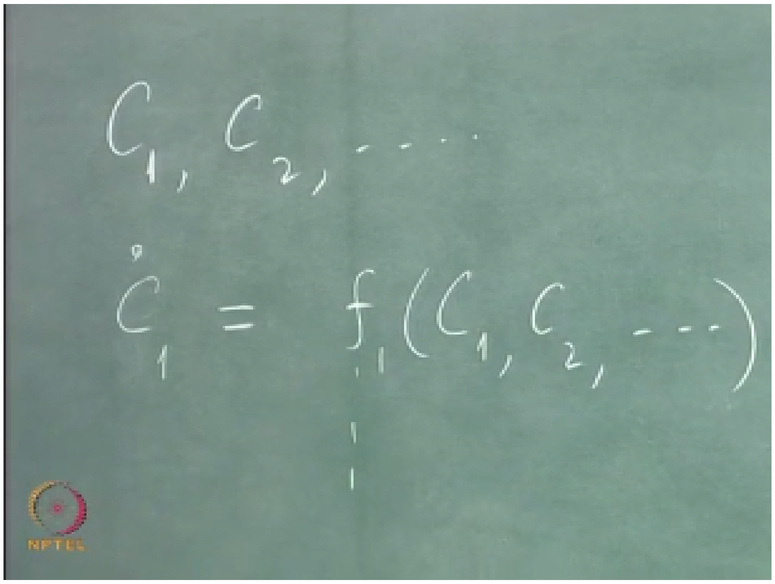

Are of the form C 1 C 2etcetera at some given instant of time then you would perhaps have a description for the way the reaction proceeds in terms of differential equations which say C 1 dot equal to some function of C 1 C 2 etcetera, and soon for the remaining concentrations. So this would give you a very gross description of what happens in the system and one of the ways you would write down these functions is to write the individual reactions down and then use perhaps the law of mass action along with the rate constants to tell you the rate at which the concentration of each species changes now that is a dynamical system in the sense we have understood it set of coupled first order differential equations which is possibly some non-linear terms on the right hand side.

And then the question would be is there an equilibrium is there an attractor etcetera what kind of steady state or equilibrium solutions you can have for such systems that would be one brand of questions. But of course you could also look at this at a more local level and you could say well.

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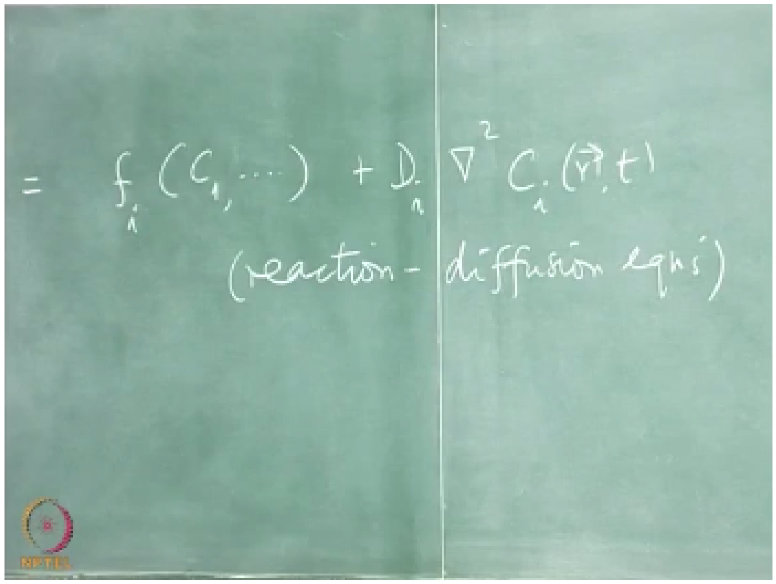

Perhaps the concentration C I of the, I species changes from place to place if you coarse-grain the system and it is a spatially extended system in which case this becomes a function of R and T. Then you could ask how does this change as a function of time and typically you would get equations of the form δ / δ T of this would be equal to again but I have some function of all the concentration C_1 etcetera some deterministic portion which would be governed by maybe the law of mass action or whatever the set of reactions concerned gives you plus there could be diffusion of the species, and that would be described perhaps by writing a diffusion constant I for the species I times ∇^2 C I the function of r and t.

Notice that we have not put in any noise here at all this is still a deterministic system but it is specially extended so these are partial differential equations and now you have a much more complicated problem than before because you have a degree of freedom corresponding to each of the species at every point in space. And so this certainly is much more complicated than the earlier kind of system we talked about these are called reaction diffusion equations, their solution poses several challenges you now really have to deal with much more complicated system of partial differential equations of this kind.

But it is still not a stochastic differential equation it is still completely deterministic you could say okay I do not have something as regular as this but I have noises I have perturbations external forces and so on, in which case you would have genuine stochastic differential equations let us look at the simplest of these. So let us now change horses and go and look at the simplest kind of stochastic differential equations that one could possibly have.

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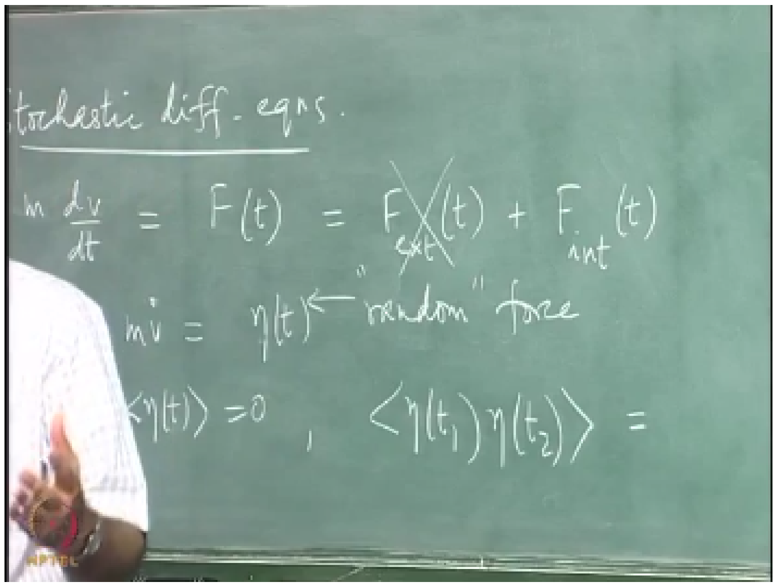

So I start with extremely simple examples of such equations and then we go on from there to look at more complicated things and let me in fact use an example which is familiar to all of you the example of a fluid say an ideal gas like the gas in this room and try to write really a microscopic equation for a single molecule we assume this to be completely classical and I try to write a differential equation or its equation of motion including the effects of collisions due to the other molecules and try to see where this gets us.

Now what would you say obviously if I took a simple system of this kind like an ideal gas or a simple fluid and I write Newton's equation down for a given molecule we will assume these to be ideal point particles point masses, then let us make life even simpler by focusing on one particular Cartesian component say the X component of the velocity and then the what you write is to say the mass times DV / DT this is the acceleration is equal to the force on this particle and the simplest assumption you could make is to say that I focus on one particular particle of a system like the gas in this room in equilibrium which is in thermal equilibrium.

So I know there is a Maxwell in distribution of velocities and then I tried to write down a microscopic equation of motion for a single particle given tagged particle molecule this is equal to the force that acts on it so let us write this as the force and this force of course would depend on the time it will change from time to time this is some time dependent force here in this fashion and what would you say this force is equal to well it would be the vector sum we talk about a single component.

So it will be the resultant of the external force on the system if any maybe you have got in a gravitational potential or an electric field or anything like that some external force if any that could of course change with time plus the internal forces due to the system itself on this on the particle and this would come due to the interaction of this Molecule with all the other molecules in the gas or in the fluid and we are going to make a very simple assumption and simply say for the moment you can always handle something with an external force let us do something in the absence of the external force and ask what kind of internal force you have here.

So now this internal force since its due to all the other molecules and I am going to now simply assume there is just elastic collisions that happens at completely random instants of time, and so a simple assumption would be to say MV dot is equal to without this external force an internal force there and it is a random force of some kind let me call that random force eight of T. So this is a random but it is not enough to say that is random moment I say something is a random variable I have to specify statistical properties of this random variable otherwise I cannot do anything more.

So we have to now start saying what kind of force is this what kind of random forces is make the simplest possible assumptions this force is changing extremely rapidly because on the average the molecules in this room each molecule undergoes collisions such that maybe the time between collisions the mean time between collisions is of the order of picoseconds. So certainly within any measurable time of even a microsecond it is undergone a large number of collisions completely and its motion is totally randomized.

So one possibility is to say that this force here a very reasonable thing to say is that the average value of this force is zero at all times because the gases whole is not going anywhere therefore the average force is 0 it is a reasonable assumption to make independent of time simply zero you could also make the assumption that this force has no memory at all the different collisions is very unlikely the same two particles keep repeatedly colliding with each other re-collisions are very improbable.

So a very reasonable assumption is to say under these circumstances, that this force is completely uncorrelated what happens from one instant to another have nothing to do with each other. So a reasonable assumption would be to say that η (T1) η (T 2) two different kinds this thing here it factors into a product of η (t1) average of η (t1) times average of at it or t2 because they are uncorrelated with each other when two random variables are uncorrelated then the average of the product is the product of averages since they have nothing to do with each other.

So this thing here is 0 for T 1 ≠ t2 and of course when T 1 = t2 this thing here is a nonzero number it could even be unbounded, because we have in mind something of this kind if I plot this correlation function as a function of the time difference between t2 and t1I would expect that if t2 is bigger thant1 and expect that this thing comes down goes off like that physically this is what I would expect such that the whole area under the curve is finite because that would specify a correlation time.

So a crude assumption 0^{th} order assumption is to say if I look at this force on some time scales much larger than the characteristic time scale of the DK here this thing looks like a *δ* function to me and therefore a good assumption is to say this is equal to some γ of *δ* of t1 - t2 this is the white noise approximation this simply says that the force ε is *δ* correlated there is no memory zero memory time and γ is some constant I have yet to determine this but it is some kind of constant here at the moment.

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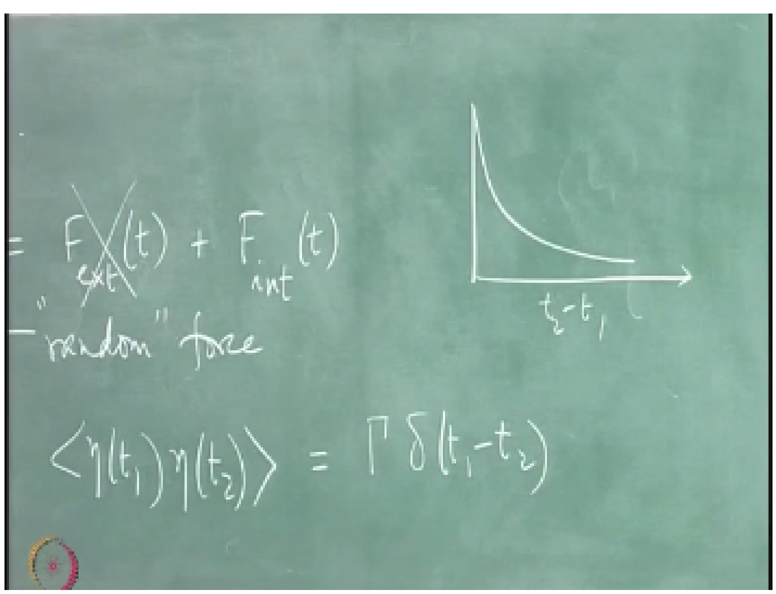

So it does satisfy the requirement that is0 when T 1 ≠ D 2 moreover we have seen earlier but in the case of a stationary random variable in other words a random variable whose statistical properties do not change with time the average value is independent of time and that satisfied here it is certainly 0 and the correlation function is a function not of t1 and t2 separately but a function of the time difference t1 - t2 or t2 - t1 in fact it is not hard to show that it is a symmetric function of the time difference that is because you can subtract from each of these time arguments the same amount of time if I subtract t2 then this becomes ε of T t1 - t2 ε of 0 here if I subtractt1 then it becomes ε t2 - t1 ε of 0 here.

But these are classical variables you can commute them neither order therefore it is easy to see that as a function of t1 - t2 it is symmetric in other words it is a function of modulus t1 - t2 and that is exactly what the δ function is this thing here is symmetric under the chain of sign of its argument. So this is a good model as it stands we still have to see where it gets us but there is a good model, now if I make these assumptions look at what is going to happen what consequence we have immediately it is an extremely simple model but we must try to see what it does for us.

So I start again with the equation of motion with these statistical properties and now I ask what is this imply for me what kind of average behavior will this imply for the velocity but remember that, now I have to specify initial conditions to solve a differential equation I have to specify initial conditions that is easily done the solution of this equation the formal solution of this for a given initial condition.

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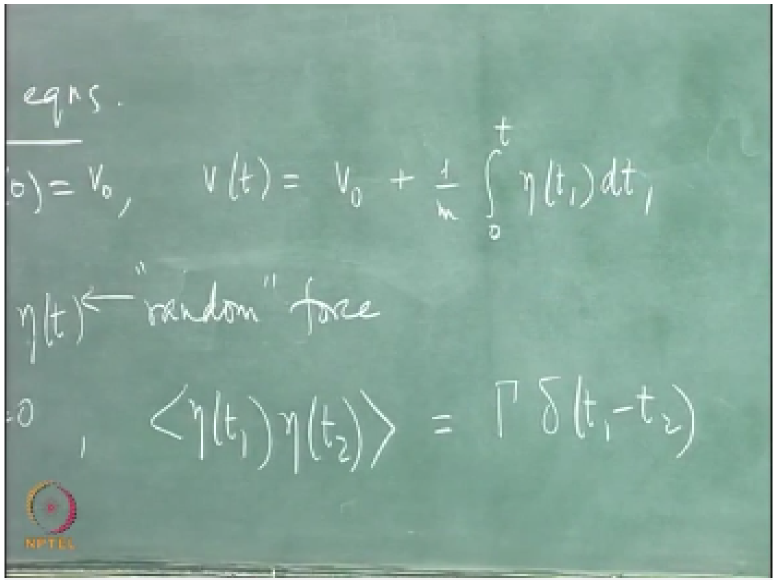

So given the initial condition v of 0 equal to $=V_0$ so I focus on some particular particle whose velocity at T = 0 happens to be V_0 that is the specified initial condition and I asked what is the fate of this particle of this molecule for this initial condition this equation implies that V of T = V_0 we integrate this equation out trivially it is V_0 + 1 / m integral 0 to t 8 of T 1 I simply integrate this equation that is the solution remember the system is in thermal equilibrium. So I know that there is an equilibrium Maxwell in distribution of velocities.

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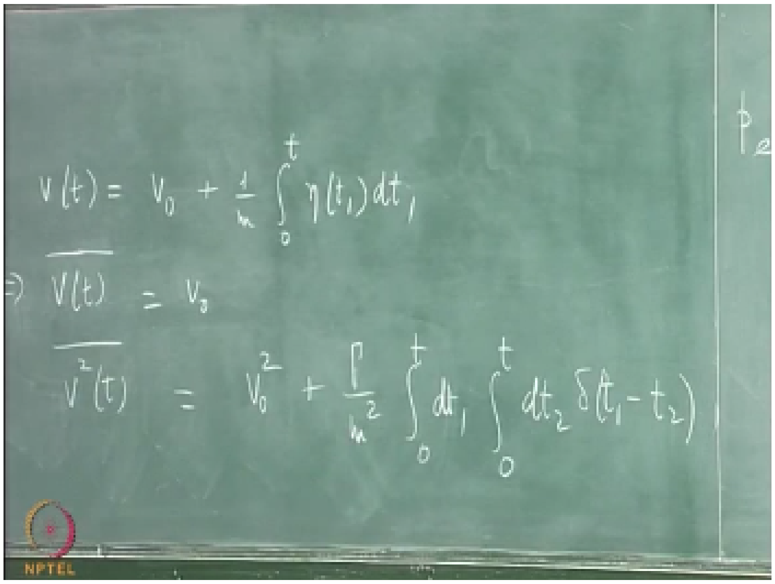

So I already know that there exists an equilibrium distribution P equilibrium of V which is e to the - MV ^{2} / twice $k_B{}^T$ normalized appropriately the normalization constant in this case is that of the Gaussian so it is 2 π K T to the power half, so that is the probability density function of the velocity in equilibrium that is the Maxwell in distribution it should really be called the Maxwell in probability density function but of course in physical applications we very often use the word distribution to mean the density function instead of the cumulative distribution function we just call it the distribution.

So I know that but remember this is a conditional quantity here if the velocity is found to be $V_0 = 0$ then the velocity at time T is given by this, now I averaged over the entire system overall those particles whose initial velocity is V_0. So essentially averaged over all realizations of this random noise and let me call that average with denoted by an overhead bar because it is an average conditioned upon the fact that the initial velocity of whatever system particular I am looking at is V_0 some given V_0 later I will relax that and averaged over all possible V 0 as well with the Maxwell in distribution.

So what happens here if I do this implies that the average value of V of T for which I do not use an overhead bar is equal to V_0 because this is a given number there is nothing to average over it has nothing to do with the noise plus the average of this quantity here this is an integration it is like a summation and averaging is also arithmetic summation procedure. So they commute with each other and the answer is this is 1 over m times the average value of heat of t1d t1, but we saw that the average values 0 we assume that it was 0 and therefore this is it as it stands.

But this is already extremely unphysical because it says if I start with a particle with velocity V_0 then these particles that subset of particles which start with velocity V_0 remain with the velocity V_0 on the average for all time it is clearly not true it is clearly this is unphysical, so something is wrong somewhere and we will see what is wrong but you see we still have to discover where we went wrong.

So it looks very reasonable that all we said was this thing here is due to all the other particles and they are completely randomized very fast but then if I take this conditional average it leaves you with just V 0 here. So we already begin to see there is something wrong here, but let us see what further it says we also find that if I Square this, yeah why should this velocity be the same I would actually expect it why, why should it be so why should it remember its initial velocity I start with the set of particles whose initial velocity is V_0 and now I discover that their average velocity remains me not at all times.

On the other hand if V_0 is very far from zero I expect that the average velocity should actually be zero according to this yes, so therefore I would expect it to go to zero that is precisely the point I would expect things to go to zero, no if this V_0 happens to be extremely large positive I would not expect it to remain that I would expect it to transfer momentum to other particles and slow down I would not expect that it remains as it is because in an elastic collision take it in one dimension in an elastic collision between two equal - particles their velocities actually get exchanged.

That is the problem that is exactly the problem, so it is going to tell us that this is it is going to get worse and let me let me do this and show you what is going to happen here let us find the square of T V^2 T what is this going to give us? Well first there is a V_0^2 and then there are two cross terms where this multiplies that and vice versa but this is a constant and the average of β is zero, so those things go away plus 1 over m squared integral 0 to t dt 1 0 to t dt 2 η of t1 η of t2 averaged over all realizations of this noise subject to some initial the tag particle having a velocity V_0.
But that has nothing to do with the noise here this is a large heat part in which you have your tag particle sitting and therefore this bar here is exactly the same as far as ε is concerned as the full

average over the system and that we saw was γ of δ t 1 - t 2, so let us put that in and this γ comes out and you have a δ of t 1 – t2, but this integral is trivial to do justice replays t1 / t2 and that is the end of it, so this integral goes away and you are left with this which gives you a t.

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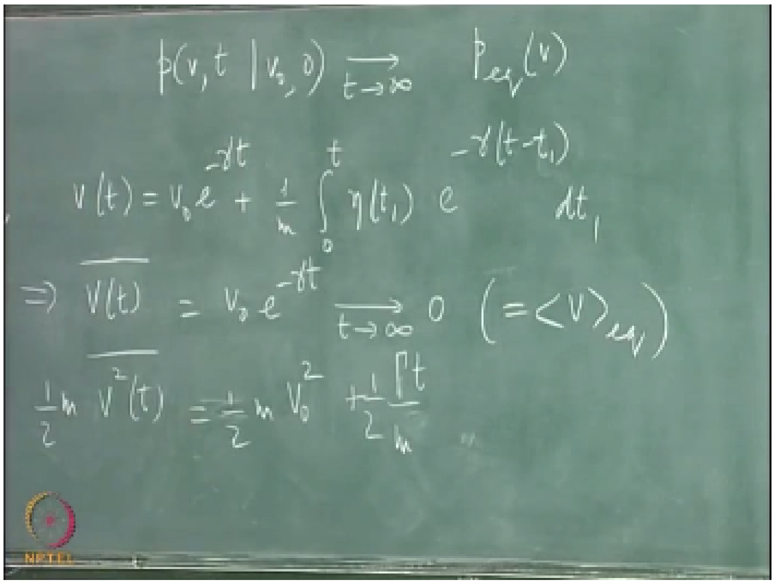

Now if you go back here from this distribution we know the following we know that be in equilibrium is zero because this is a component of the velocity it is an even function of V and therefore we know the average velocity is zero every component of it is zero we also know from this Gaussian it follows trivially that this is kb t / m which is just a verification of the fact that one-half MV squared average is half K T which is in accord with the equal partition theorem every degree of freedom translational degree of freedom as an energy half Katie on the average in no relativistic free particle situations.

But what is this telling us this is crazy because if you find out a fine one-half M this to be the average energy x 1 half times m on top sorry this is this cause of E and this gives me the effective temperature of the system then it simply says you leave a beaker of water alone in contact with the heat bar and its temperature spontaneously increases to infinite values given enough time which is clearly absurd.

So not only was this bad this is disastrous where are we wrong wherever on of course you could immediately questioned this the assumption made on ε and say well ε of T cannot be δ correlated because that is unphysical it says that the correlation time is zero and there is no physical force whose correlation time is zero could be very short, but it cannot be zero strictly this is true but on the other hand we have managed to separate time scales I tell you as a matter of hindsight that the actual correlation time of this random noise would be like the correlation between the time between collisions that is of the order of picoseconds or lesson the other hand if I look at the system on the time scale of microseconds or more then this is certainly a good approximation does not matter at all.

But the force is it is random but there is something missing in this random force and a moment's reflection will tell you that if a molecule has instantaneously a very high positive velocity it is clear it suffers more collisions per unit time raring it than other than pushing it forward. And therefore the right thing to do is to go back to the model and say that MV dot $\neq\varepsilon$ of t alone but also has a systematic part which returns it which is proportional to its velocity instantaneous velocity but is in the opposite direction and that is best model by saying it is - on this side something proportional to the instantaneous velocity with a constant γ C I would like to make this constant γ have the dimensions of time in verse, so let us put an M here.

This is also random why because this is a random driving force that makes the velocity random therefore this is also random but it is a systematic part of the random force it is a specific function of the output variable and we made an assumption we have said that this raring force is linearly proportional to the velocity instantaneous velocity itself and that of course immediately tells you that V. + γ V = 1 / m in the absence of an external force you could of course always add an external force do this but this is the solution in the absence of an external force when the system is in thermal equilibrium.

And what is the solution to this equation and γ is a positive constant that is crucial what do you think is the solution to this equation, well all we have to do is to solve this first order linear differential equation with that initial condition instead of this thing here all we need to do is to put in the integrating factor which is V_0 e to the - γ T website and since the integrating factor is e to the - γ T in this case this gets multiplied by a to the - γ t- t1 that is the solution which obeys this boundary condition the initial condition right.

You could put in other models but there is a simplest thing what made me decide on it was simply experience based on what happens in a fluid I know that the friction for small velocities is proportional to the velocity itself that is the assumption I me and it is directed oppositely this is called the Langevin equation and why does it get us well for a start please notice what happens to the mean here it is this term here it is not that, that goes to 0 because the average of γ 0 so indeed you have this what happens to this at T goes to infinity goes to 0.

So you seem to be on the right track we seem to be getting back equilibrium but we have to still check this out, so this goes as T tends to infinity 0 which happens to be this we have further corroboration for that because how would I compute the full average of v full average of V of T would be this average averaged over all possible v zeros because this is a conditional average it is an average over that subset of particles whose initial velocity is V zero I take that subset and I use all realizations of the random force it of d.

But now I would like to get a full average velocity it does not matter what initial condition I choose so do you agree that this quantity here is equal to DV 0 V of T bar or all possible values of V 0 with what weight factor with what distribution with the Maxwell in because the system is in thermal equilibrium to start with at t equal to 0. So I put in P equilibrium of v-0 and what you get for this integral this is from -infinity to infinity well notice this is V_0 e to the - γ t e to the- γ T does not figure in integration this is an even function of V_0 you put a V_0 so you get 0 straight away, so we actually we seem to be on the right track as required.

So it is actually restoring the equilibrium but there is another lesson we will learn from this is a very important one to find this V equilibrium I can do in two steps one of them is to find the partial average over a subset whose initial condition I prescribe and then I either average over all the initial conditions or I simply wait for long enough I do not average overall initial conditions I say I do not care about the initial condition I wait long enough and the system forgets its initial condition and indeed it goes to 0.

My initial assumption is that a system is in thermal equilibrium with the Maxwell distribution of velocities that follows from equilibrium statistical mechanics independent of this time dependent stuff completely this thing here I would like to find the average of the velocity of the particles in this room at any instant of time and I start by saying at T equal to zero the system is in thermal equilibrium I do not do anything to it now I know the average value must remain unchanged with time.

Now I am trying to build a model for what happens if I ask time dependent questions namely if I start with the particle whose initial velocity happens to be v-0 and I watch it what can happen to this on the average well this model tells me it is average velocity will decay from V_0 to the value zero given enough time with some time constant γ inverse but I could be impatient I could say no to find this average value at any instant of time given the fact that the distribution is known at e equal to 0 all I have to do is to find the conditional average the partial average for a given initial condition and then averaged over all initial conditions with the equilibrium distribution.

So either I let t go to infinity or I let this happen and I must get the same answer and the reason is I expect I expect that if I if this process by which this whole thing is happening is a mixing process I expect that the conditional probability density that the velocity of a particle is V at time T given that it was V_0 at time 0 this thing here I expect that T tends to infinity this memory is lost and I would expect this to go to P equilibrium of V. But if that happens then the averages should also follow the same rule I would expect that if I wait long enough I should get equilibrium average in this fashion.

So this is what we have to corroborate whether this happens or not well it seems to work at the level of the mean but the real test is at the level of the mean square because that is what involves physical quantities like the energy average energy of the particle. So let us try to find this quantity now B^2 of T and where does that get us in the launch of our model well this is equal to I have to take this solution square it and then take the average value the first term will be very trivial it is V 0square e to the - 2 γ T is square this guy plus a quantity which is the product of this times that averaged over a η that is 0 because average of β 0.

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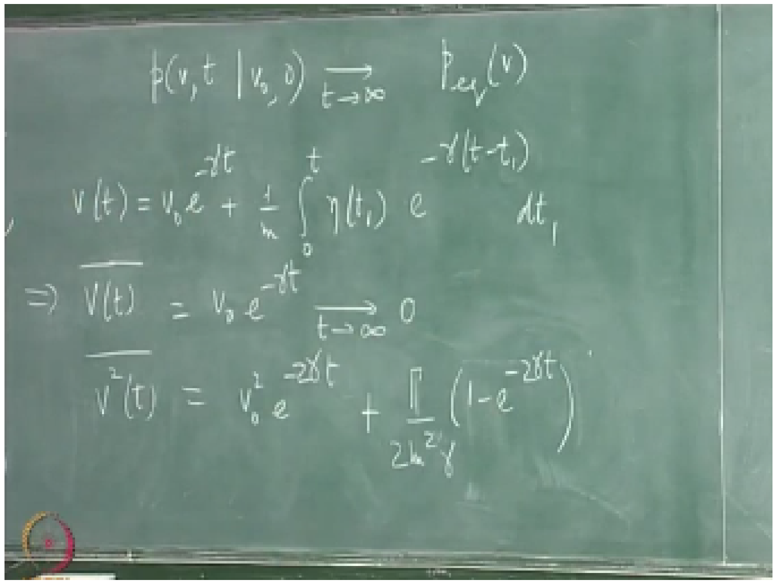

And similarly the other direction plus a 1 over m square integral 0 to t dt 1 0 to t et tu then e to the - γ t - t 1 e to the - γ t - T 2 times the average value of ε of t1 times ε of t2 but that we have a large bomb or we have a model for it in this white noise approximation which is nothing but γ over this x a δ function of t1 -t2 we have to do this slightly more complicated integral but that is not hard because it is the integration region is symmetric both of them run over the 0 to t range and so the region of integration is the square on which you have to integrate on the 45^0 line which is which corresponds to t 1=t2.

So this means you simply replace t2 by t1 everywhere get rid of this and this becomes a t1 it just gives you a factor to hear this goes off and I pull out this integration is gone I pullout e to the - 2 γ T and I have e - 2 γ T 1 and that is e to the 2 γ t 1 - 1 / 2γ right. So this goes away and I get a 2 m^2 γ and I get 1 – e to the – 2 that is it. So where does that get us it says V^2 of T conditional average is γ / 2 m square γ + $V_0{}^2$ - γ over 2 m^2 γ e to the - 2 γ T now we can do two things one of them is to let he go to infinity and see what happens so if you do that this is just γ / 2 m square just a constant of some kind.

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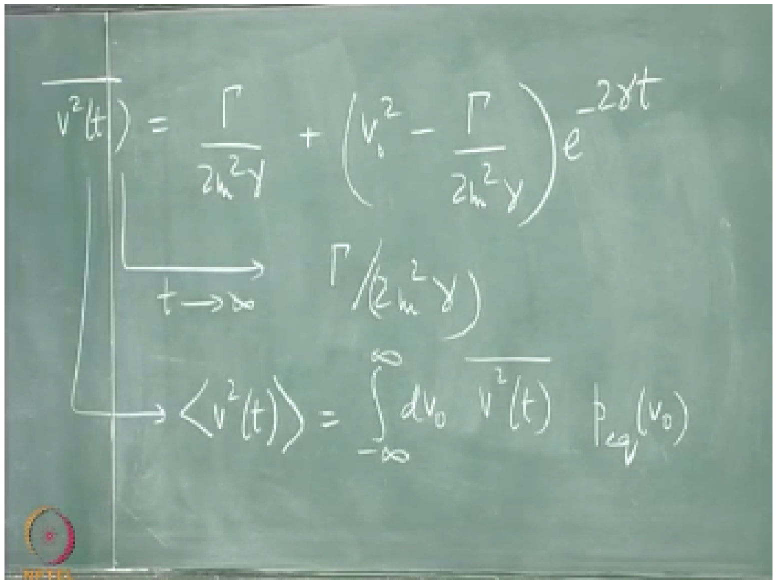

On the other hand if I take this quantity and I do v squared of t that is equal to integral DV 0 V ^{2} of T bar P equilibrium of V0 it could also do that I could also averaged over all values of the initial velocity with the Maxwell in distribution which we assumed is valid at T = 0 what happens here if I do this is just a constant. So this part does not have risen to anything that is a constant does not average to anything what happens to this what is the average value of the initial square of the initial velocity over the Maxwell in distribution that is of course given by the equal partition theorem.

So half MV not square average is Kt o half KT, so V_0^2 average is KT / m so therefore this thing here would give you γ / 2 m square γ + k_b^t / m- γ either the - 2 γ that is the result I would like this to be equal to this if the system is in thermal equilibrium I would like the two to be equal I am not using the equip-partition theorem I am simply saying be very careful I am saying in equilibrium statistical mechanics.

The principle of equilibrium statistical mechanics tells me that when a system is in thermal equilibrium then for an ideal gas or a system of non-interacting particles of this kind the average value of kinetic energy in any component in the average value of half M times the square of the velocity any component of the velocity is half Kt I am not do I am not using any equal partition theorem.

I know that the distribution is max value that is an equilibrium distribution it is the canonical Gibbs distribution and I know that from equilibrium statistical mechanics, so that is an input automatically. Now I am asking time dependent questions which are outside the purview of equilibrium statistical mechanics, and therefore I need to have a specific microscopic model and the model that I have chosen is the language of amore but this has to be consistent with the equilibrium statistical mechanics.

So I am not doing anything to it I am simply taking the system in thermal equilibrium focusing on a particular particle at T = 0 and asking time dependent questions of this without changing the thermal equilibrium situation and it is clear these two must be equal because the system hasn't been affected at all what's in thermal equilibrium should remain in thermal equilibrium this quantity should be independent of time and this should be equal to KT over m when is that going to happen if I equate this to this has got to be zero there is no other way the time dependence can we got it.

Moreover the answer has to be KT / m and that magic is obtained simultaneously not only does this vanish if this is equal to that but this coefficient also becomes kt / m, so you have complete consistency provided only that this be equal to that, so what does that tell you finally this is possible it is consistent this large more model is consistent with equilibrium statistical mechanics if and only if consistency if and only if γ over 2m ^{2} little γ is KT over m or if I take it across to the other side is equal to M γ K_b T in the launch of a model.

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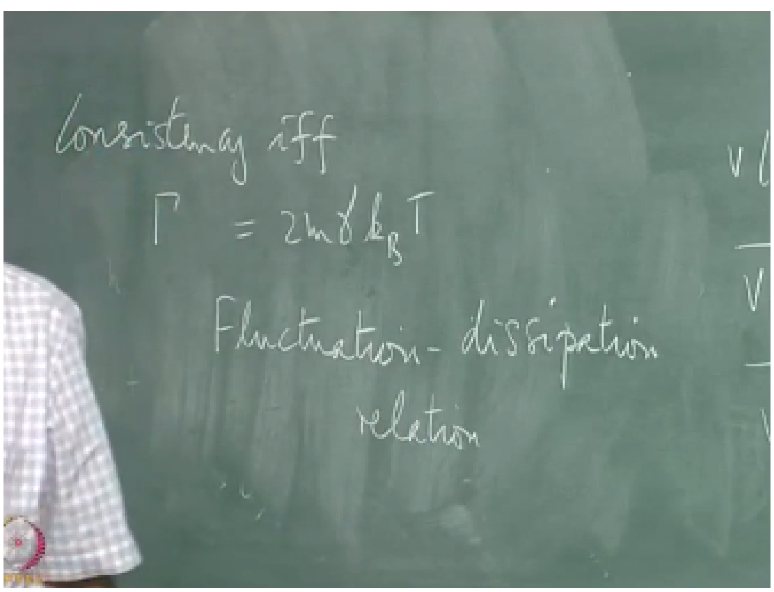

So what have we achieved what has happened here finally this quantity was put in phenomenological as a friction constant in the systemic part of the random force this quantity here was put in as a strength of the random part of the random force truly random part of the force the molecular collisions this measures the strength in some sense, because it figures it is the coefficient of the δ function which is in the autocorrelation, and it says the strength of the fluctuations as measured by this γ the fluctuations in the random force due to molecular collisions must be related to the damping coefficient.

So you cannot have arbitrarily large fluctuations for a given damping coefficient and vice versa the stronger this is the stronger that is to preserve thermal equilibrium and this is a relation is the first example of a very deep relationship it is called the fluctuation dissipation fashion or theorem it goes by many names in this version it is called a Nyquist relation the theory of thermal noise in a resistor you to ground in motion of charge carriers this is precisely the sort of relation you get between the power spectrum of the voltage the random voltage and the resistance that you have here at any fixed temperature it is exactly the same relation.

Now once you put that in now that is input for consistency we are guaranteed now that the system is consistent with the equilibrium statistical mechanics. So once that is taken care of then we can actually go ahead and compute averages we can compute all kinds of averages now, so this is the first crucial point you have to put this in as a condition on the launch of my equation having done that let us now see what P itself would-be how would we get at what the probability distribution itself.

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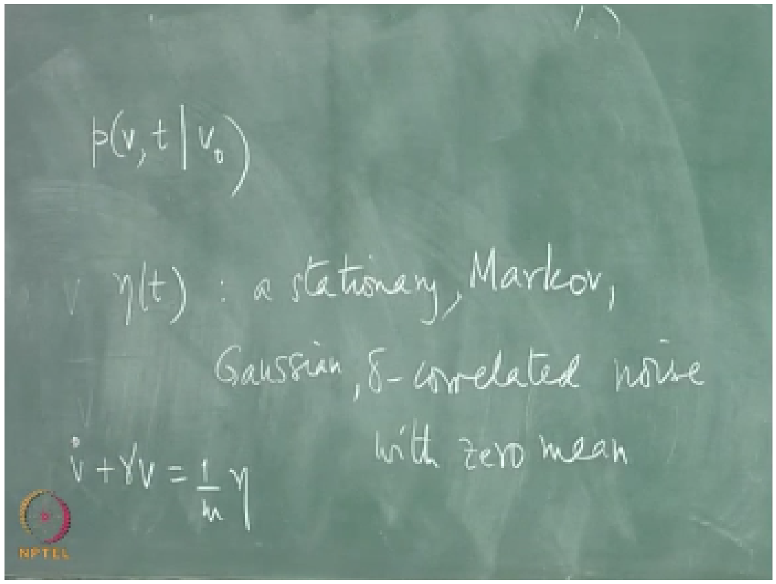

So what is P of VT given a V 0 I will drop the 0 to show the instant of time I drop this and ask what is this guy equal to, well I need an equation for it I need to make some assumptions it turns out that if I make the assumption that the velocity is given by the launch of a mortal the larger equation and γ is what is technically called a stationary δ correlated Gaussian Markov process for a Gaussian white noise then it turns out that this process V as described by the launch of my equation is a Marko process in itself it is not δ correlated it has a specific correlation time which will find out now.

But this conditional probability here satisfies a master equation characteristic of Markov of characteristic of Markov processes in this case a very simple fokker- planck equation but will write the solution down go back and look at the photo plank equation I am going to write the solution down by assuming that 80 of T is a Gaussian white noise so it of T a stationary mark of η correlated with zero mean that is the technical assumption.

So it is reasonable because it reasonable to assume that the statistical properties of the molecular collision force due to molecular collisions does not change with time reasonable to assume that is δ correlated on the time scales we are looking at it is got zero mean that is physically reasonable we are going to make the technical assumption that it is Gaussian namely its probability distribution function the density functions are all normal distribution they are all Gaussian distributions.

Now that is a technical assumption but there are reasons to believe that that is also plausible assumption based on the fact that this γ is a resultant of a very large number of uncorrelated events. So something called the central limit theorem comes to our aid and therefore this is not an implausible assumption it is probably is the most likely thing the fact that it is Markov that is an assumption we want to make an assumption that this is completely uncorrelated it is very short time memory one-step memory that is a technical assumption you could relax it in fact in real fluids you have to relax it a little later.

But that is the simplest assumption to start with given this and given the large my equation which says V dot + γ V = 1 / m and η has these statistical properties it turns out that we is also Markov it is also stationary it is not δ correlated it has a finite correlation time which in this case will turn out the γ inverse but it also turns out to be Gaussian so it will turn out that the Gaussian property remains it is very robust under this equation.

So the driving variable is Gaussian distributed, so it is the driven variable if you grant me that then I could actually write down this thing here this conditional probability density function because you know that a Gaussian process a Gaussian random variable is determined its distribution is determined by two parameters the mean value and the variance those are the only two things you need so let us compute those quantities here we can actually write down what these quantities are because to start with I am going to require the following on this p.

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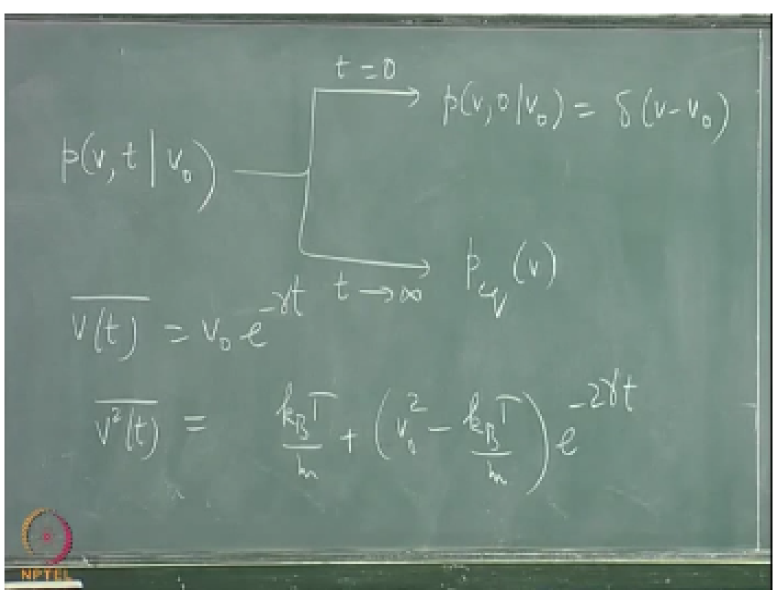

I am going to write it down by hand side so at T equal to zero what is this density function it says this is the probability that the velocity is some v at time T given that it was v 0 at T =0, so what is the initial condition on this guy what does that imply for this it is going to be 0 unless v equal to v-0 it is a δ function it is going to be normalized to unity. So it is a Dirac δ function and what would you expect this to become as T tends to infinity on the probability density function the max value nothing is happening.

So I would expect this to become P equilibrium of V must check out that my solution in fact does so, and then if we have knowledge of what the mean value is and what the variance is we can actually solve the problem completely but we do have that knowledge because I know that V of T is V 0 eto the - γ T and I also know that V^2 of T is equal to that thing there this quantity here with for consistency γ / 2 m^2 γ replaced by kt / n.

So this is kb t /m plus v 0^2 - 4 m e to the -what is the variance of this process then therefore we squared of t – V of T whole square this is equal to the variance of V of D what is that equal to all I have to do is to subtract this out here and then you begin to see you see immediately that it is equal to KBT / m into 1 - 2 because this term cancels against this right. Now if I make the assumption that is Gaussian.

Then we can actually write the solution down the normalized solution down what is the Gaussian what is the Gauss in probability density function look like remember that if the mean is μ and the variance is σ and then the standard deviation is σ the variance is σ^2 then the distribution density function is 2 π σ^2 e to the – the variables i - the mean whole squared divided by 2 σ^2 that is the Gaussian normalized Gaussian probability density function.

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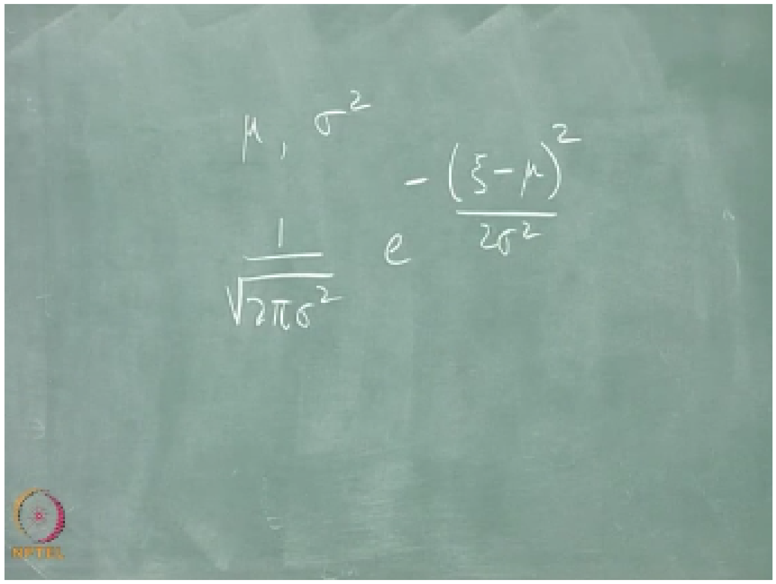

So we copy that out and therefore we can assert now that p of VT given V 0 must be = 1 / √ 2 π σ^2 but σ^2 is this, so this is M over 2 pi K_B^T x 1 - e to the -2 γ t to the power half times the exponential of - is I - μ whole square. So that becomes V - V_0 e to the - γ T whole squared that is the mean value which is a function of time there is an M there / 2 $K_B^{T.}$ Now of course it is tricky to show that at t = 0 as T goes to 0 you end up with the δ function but it does so because as you can see as T goes to 0drop this is V - V 0 whole square but this guy goes to zero out here and this fellow also.

So you have a zero from this whole exponential and then you have something which blows up from here and the result is a δ function in the limit although it is not immediately obvious as T goes to infinity indeed it goes to the Maxwell in distribution because this part goes away you are left with this here and this part goes away that goes away and indeed it goes to this. So this is certainly valid and what will this distribution look like if I draw a picture it would do the following.

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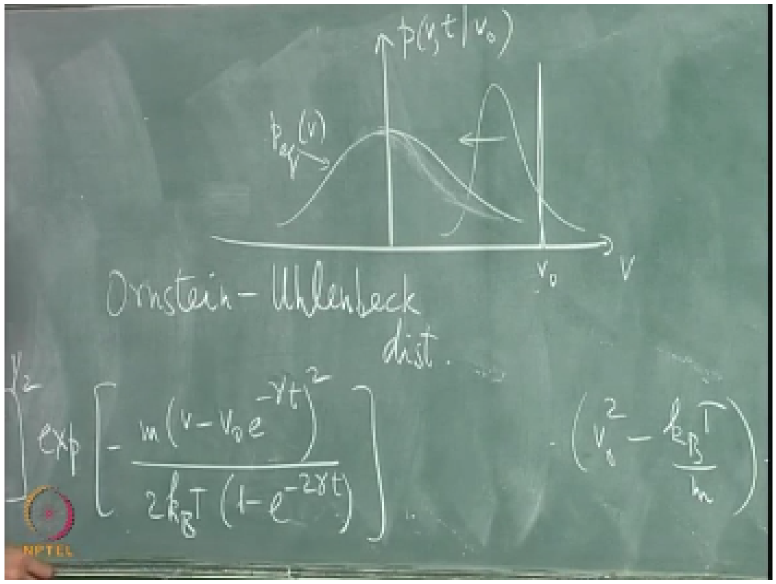

So as a function of V if I plot p of VT for a given V 0 = 0 it is a δ function at V 0 it is a spike and then as T increases this broadens out the average value this thing is asymmetric distribution about its mean which is at V 0 e to the -γ T that slowly drips. So you have something it looks like this and then slowly drips and eventually at t equal to infinity it becomes the max value and that is a symmetric distribution. So the peak broadens and moves to the left drifts to the left and gradually hits this and the variance changes with time like this in this fashion.

So Varian starts at zero which is δ function and gradually reaches the value kt / m this distribution function this is called the on-screen on Steam Over beck distribution we will take it from this point tomorrow show you what happens next, but what we want to emphasize here is that this stochastic differential equation that I wrote down the launch my equation is equivalent to a Markov process it describes a Markov process whose probability density function conditional density function is the onset in olinbike distribution we will generalize that lesson to other stochastic equations.

But this is in the physical context of a particle diffusing we will make further consequently we will look at further consequences of this meanwhile I would like to check out the following and that is not hard to do it is the same integral that we did earlier.

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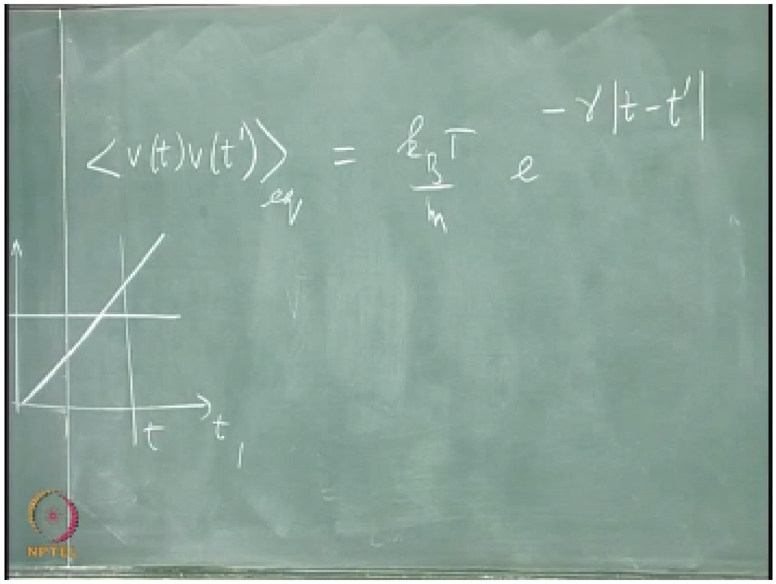

But now check out the fact that V of T V of T' in equilibrium you could ask what this quantity is we looked at V^2 of T in equilibrium we took the same V of T and squared it but now you should take two different instants of time of this kind and ask what happens now you could do this by finding first the average value of V of T V of T' and then averaging over initial conditions or by letting t go to infinity T' go to infinity such that t – T'is finite either way to do this and the result I would urge you to do it both ways write the integral down and then you have now an integration over t1 and t2 which would be unsymmetrical.

So if this is t1 and that is d2 perhaps t1 goes up to t but t2 goes up to t prime and your δ function constraint due to the white noise would be along this line, so what would contribute you can replace get rid of one of the integrations but the other integral is constrained to be up to here and not up to here. So you can easily see that the smaller of t and t' is going to be there is going to be the region of integration for the second integral whichever is left the lesser of the two.

So it turns out if you work this out this will turn out to be KT / m that is the value at T = 0 times e to the power - well you can almost guess what is going to happen this is an equilibrium correlation function for a stationary random process, so it must be a symmetric function of the time difference and there is only one constant of time here in this problem absolutely, so it is just e to the- camera modulus t - t to be that, so check that out.

So the honestly Nolan my process is exponentially correlated it's not δ correlated, so by putting in the lingerie model what if done is started with a white noise which has zero correlation time but the driven or output variable has a finite correlation time it's built in to the system. So it says even though your noise may be a zero correlation it is really instantaneous relations the output variable could slow down could actually have a finite memory and it does in this case.

So it is it acts like a paradigm for random processes of this kind we take it from this point here and I like to for example look at the ordinary diffusion equation in the same language and ask what happens there we'll do that next we look at the displacement we have not done that yet at all we know that when these particles move around they diffuse. So I would like to look at the position variable and see what happens in the position here from the same model and ask whether we can draw similar conclusions for this and connect it up with a diffusion equation and then more general stochastic equations this is what I will do tomorrow, so it stop here.

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