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Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 26
Discrete time dynamics (Part V)

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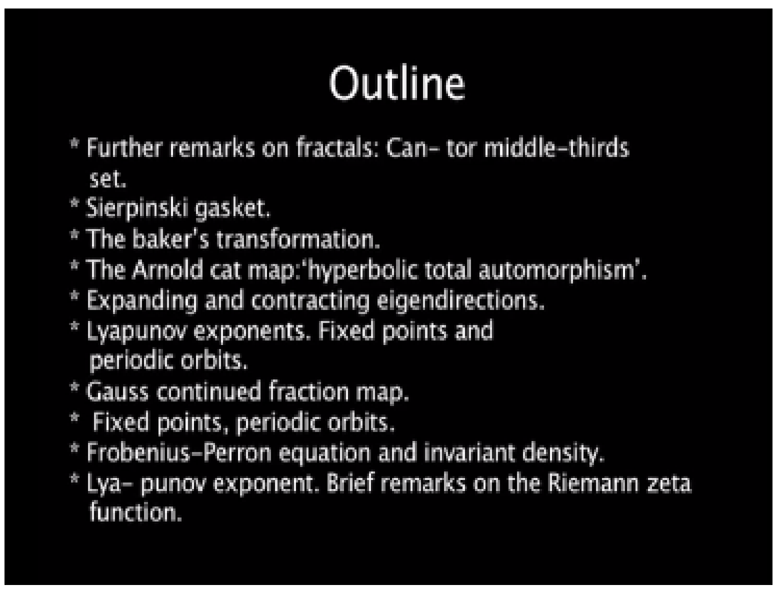

Yeah let us begin with a sort of assortment of topics some of which were left out earlier and some of which complete what we already discussed the first of these has to do with fractals I gave an example of how we could find fractal dimensions for various sets here is another such example suppose I take all numbers between 0 and one on the unit interval.

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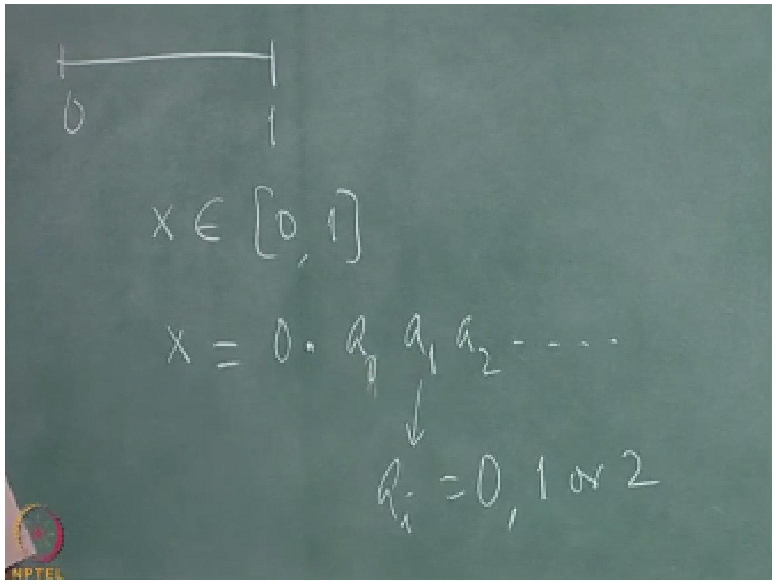

l and I write this these numbers in ternary rather than binary so I write each number say the initial number that I start with or any number at all X element of 0 to one I write this as 0 point A is a_0 a_1 a_2 dot where each of these digits is either 0 or 1 or 2 so I am writing everything down in ternary rather than binary so the allowed digits for each of the ai 0, 1 or 2 so clearly this a_0 is the coefficient of $1/3^1$ this is the coefficient of $1/3^2$ and so on and so forth.

Then the question asked is suppose I consider the set of all such numbers in whose ternary expansion the digit 1 does not appear at all it is clearly a set of numbers and I would like to know what is the fractal dimensionality of this set how would you go about this.

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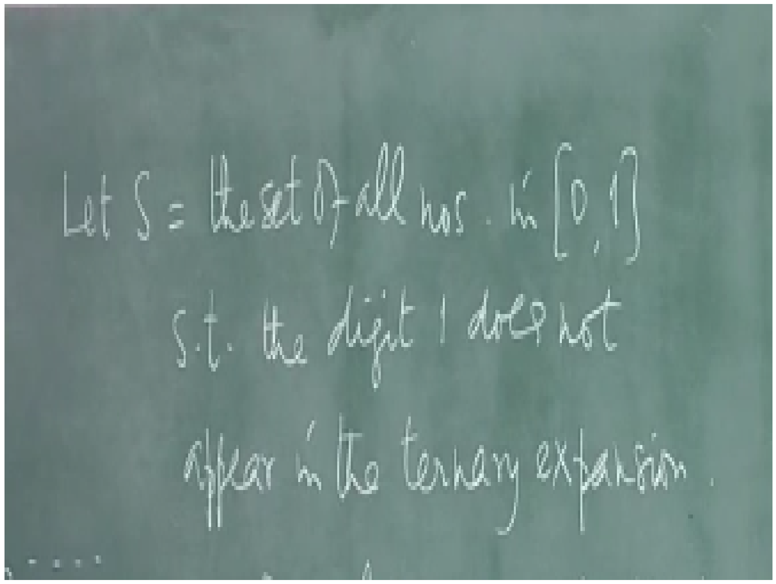

So let S be =the set of all numbers in 0, 1 such that the digit one does not appear in the ternary expansion of these numbers what is the say what is the fractal dimensionality or box counting dimension of the set how would you go about this what would you do so I am not going to permit one to appear here so this should be either 0 or 2 that should be 0 or 2 0 or 2 and so on clearly I am a subset of numbers between 0 and 1 and I asked what is the fractal dimensionality yes so how do you go about it what would you do.

Yes but I would like to know what is the fractal dimensionality of this set how would you go about that suppose a knot suppose a_0 has a 1 there and where would it be on this than this circle where would it be so this is a 0 or a 2 and if it is a 0 this number is < 1/3.

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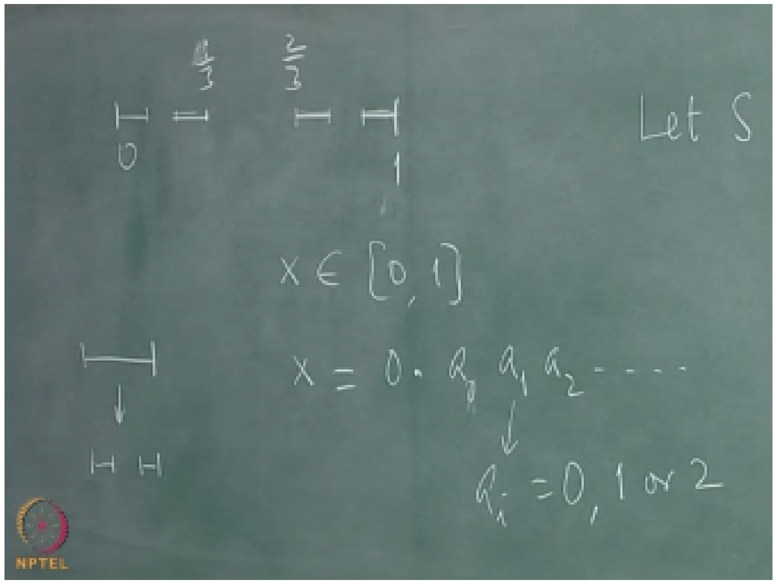

So it is clear that the number would be so here is 0 here is one this is 1/3 that is 2/3 yeah it is exactly the middle third Cantor set and it is not hard to find this because if this digit is 0 it is here if the digit is 2 it is there if the digit is 1 it is in the middle you are not going to allow it ought to be =1 so it is this is canceled and then if you look at a1 then it is again going to be the middle third that is raised to each time so it is going to be here or here and so on.

You keep doing this forever you get the middle thirds Cantor set and what is the fractal dimensionality of the middle thirds Cantor said well at each stage at each stage you take some interval of the proceeding stage you break it up into three parts and retain only the mid the end two parts and each of these is one-third the length of the preceding stage log two/log three.

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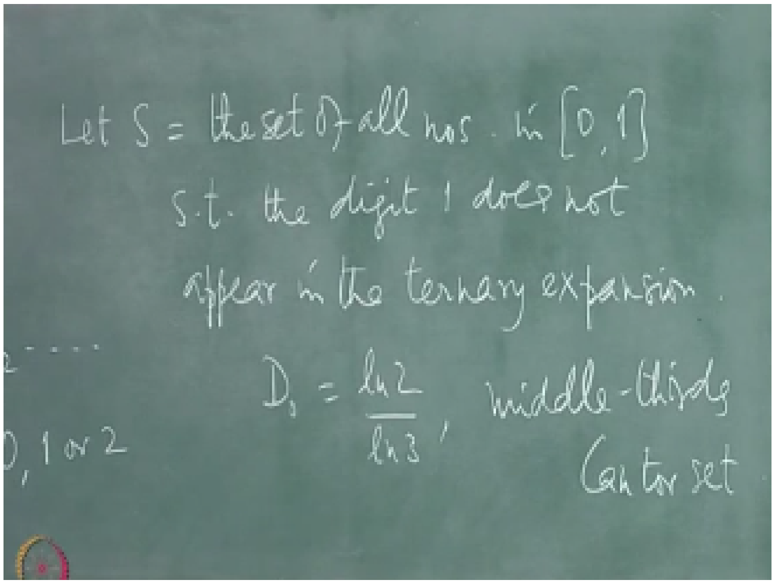

So in this case D0 = log 2/log 3 and it is the middle thirds Cantor set so that is the answer there is another famous regular fractal in two dimensions the many innumerable fractals.

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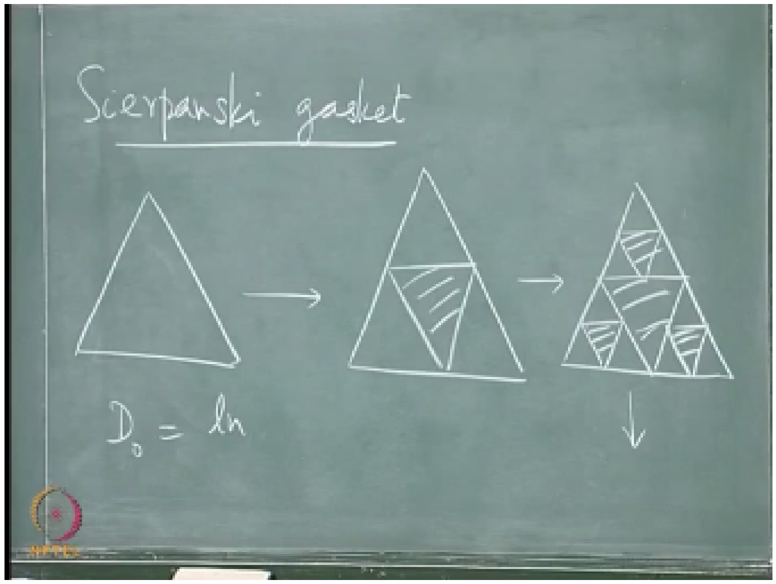

Another one is the so called sierparnski gasket and let me explain what its properties are you start with a unit equilateral triangle in the first stage so this is the and this goes over in the next stage to the following you remove the triangle in the middle so this portion is gone and at the next stage you repeat this process so this is already gone that is a whole and then there is another hole here and another hole here and another hole here and you continue this ad infinitum.

You are still left with a geometrical figure at the end which is riddled with holes so to speak this is called the sierparnski gasket what is the fractal dimensionality of this object you start with this figure the area here is of course this is a two-dimensional filler search so in this case you would expect me not to D0 =log divided by what do you think so do you think so yeah it is equally divided so you are getting rid of this and then you are getting rid of this it is a tie and you are left with these little things which will then get riddled completely by holes.

So what do you think is the fractal dimensionality of this object we start with an area whose topological dimension is 2 so what is the fractal dimensionality here so these are areas mind you what is the area of this figure okay so what is the fractal dimensionality finally will it be > 1 or < 1 so what do you get actually you do not necessarily have to you might get rid of then the entire area so there is no rule saying that you should have something which is > 1 or < anything like that.

So what happens in this case how many pieces are you breaking each one - but you are removing that it is gone well so it is log 3 on top and each of these is 1/4 of the earlier so it is log 3/log 4 which is a number between 0 and 1 well it is not how do you did not say log 4/log 3 okay so this is log 3/log 4 okay - take and so on you can construct this in higher dimensions I leave it to you as an exercise to tell me what the fractal dimensionality is if you did this in D dimensions D spatial dimensions the next stage.

For example you take a tetrahedron which is a simplex in 3-dimensional Euclidean space you take a tetrahedron and remove the middle by exactly the same construction and you are left with tetrahedron at the corners of the original tetrahedron and you can generalize this to D dimensions so I leave you to tell me what D0 is in the case of a d-dimensional sierpanski gasket in D Euclidean dimensions you start with the tetrahedron that will be the next thing this would be the simplex four-sided figure it is a simple and now remove the middle the solid object in the middle and you are left with four tetrahedron at the ends the shape is self-similar that is at each stage you started with the triangle it becomes smaller triangles.

So you start with a tetrahedron it becomes smaller tetrahedron so you chop off the middle such that you have a tetrahedron at the corner here the corner there at the corner there yeah what is the equation to a simplex in n dimensions what is the equation to this tetrahedron what would you say is the equation to this it is a convex figure it is called a simplex in D dimensions and now you have to generalize the idea of this object it is a rigid object there 2 D dimensions as you go higher.

Okay I leave this as an exercise to you and we will give the solution subsequently okay so these are all constructs artificial constructs made in order to illustrate various properties of fractal sets with the middle third fractals Cantor set itself you can generate a multi fractal if you calculate I there if you said well I just go through this construction at each stage and I associate a uniform measure with all these pieces then all the generalized dimensions of this object would turn out to be just log 2/log 2 nothing would happen.

But if I start associating different weights with these objects different probability measures then in principle I can end up with a multi fractal in which $D_q \neq D_0$ for all values of q so this is often used in modeling we would not get back to this right now let me now introduce you to a two dimensional map which has standard properties in the sense that it is hyperbolic displays chaos and it gives you a little inkling of how this chaos arises we already saw two-dimensional the example of a two-dimensional map we saw the Baker transformation.

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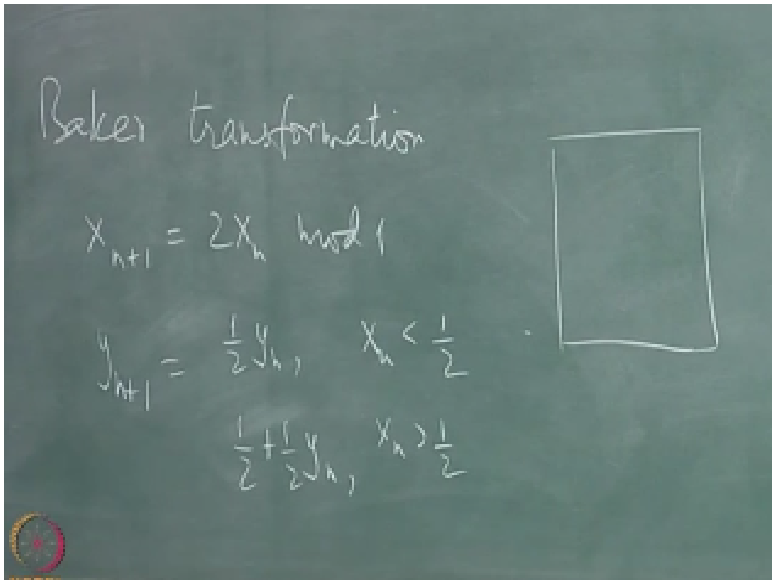

Which you recall was baker's transformation the Baker transformation of the Baker Mac this started with xn and xn + 1 was = 2xn modulo 1 so you have stretched in this direction and the yn + 1 was =1/2 yn if xn was < =1/2 and it was 1/2 + 1/2 yn if xn was > 1/2 in the unit Square and we saw this is an invertible map so the number of pre images for images for any point is =1 we also saw that this map is chaotic.

And that the unit square in the XY plane gets completely scrambled up as you iterate this map and that it has to lay upon of exponents one of which is log 2 and the other is a log 1/2 and that the sum of the Lyapunov exponents was =0 as it should be because this is a measure preserving map and it makes in that sense a Hamiltonian system in a very crude sense now the other famous map which does that is called the Arnold cat map and it is got a fancy name a more rigorous name.

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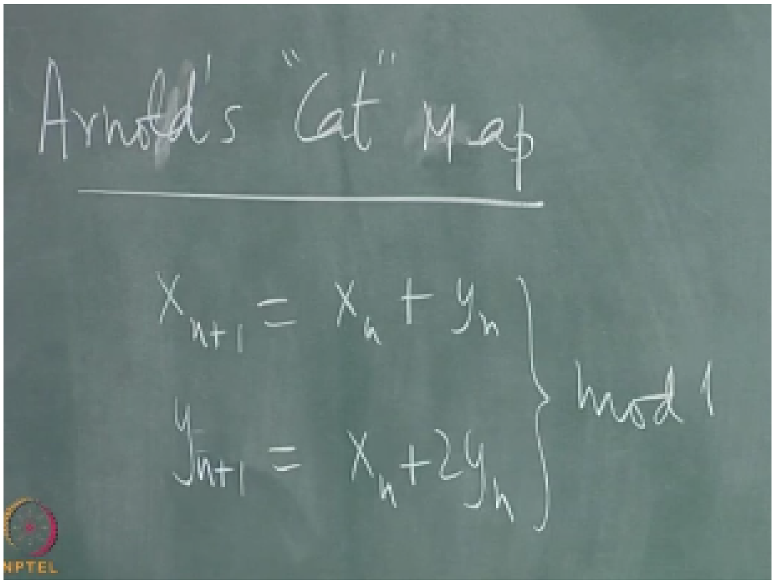

But let me just call it Arnold cat map is as follows since xn + 1 is =xn + yn and yn + 1 =xn + 2yn and both are modulo 1 now let us see what is mapped us in pictures what it will do is the following.

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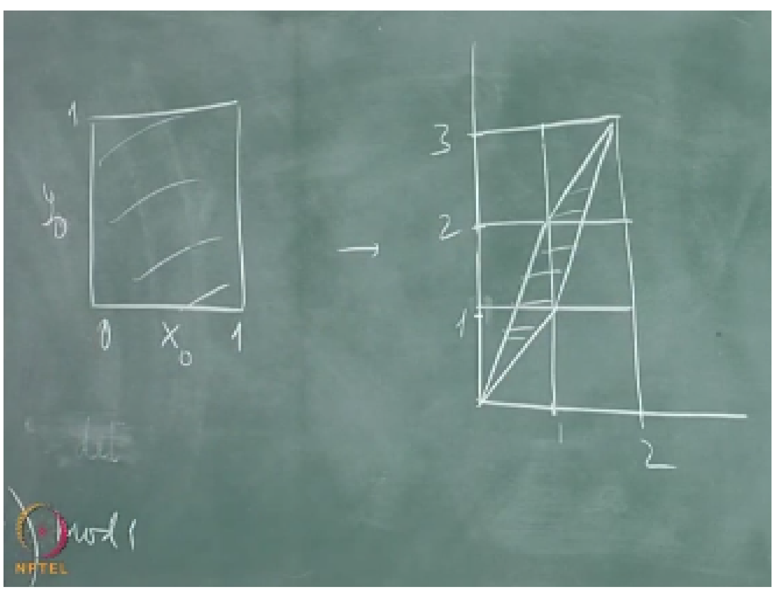

It will start with this thing here X_0 and y_0 so it start with the unit square and each of these variables and then x1 is $x_0 + y_0$ modulo 1 and Y1 is of course $X_0 + 2y_0$ again modulo 1 if you did not have the modulo 1 then what it would do is to do this it will double in the X direction and it would triple in the Y direction not drawn it too well so let me slightly smaller scale so here is one here is two here is here is one here is two here is three.

So what the map does is to take this guy and distort it in this fashion such that the unit square becomes this so this is what the unit square goes into this stage here but you are supposed to do this modulo one so what you do is to cut out all these pieces and put them back into this square what is the total area of this distorted square that just depends on the determinant of this map so if I call this map let us call this map a.

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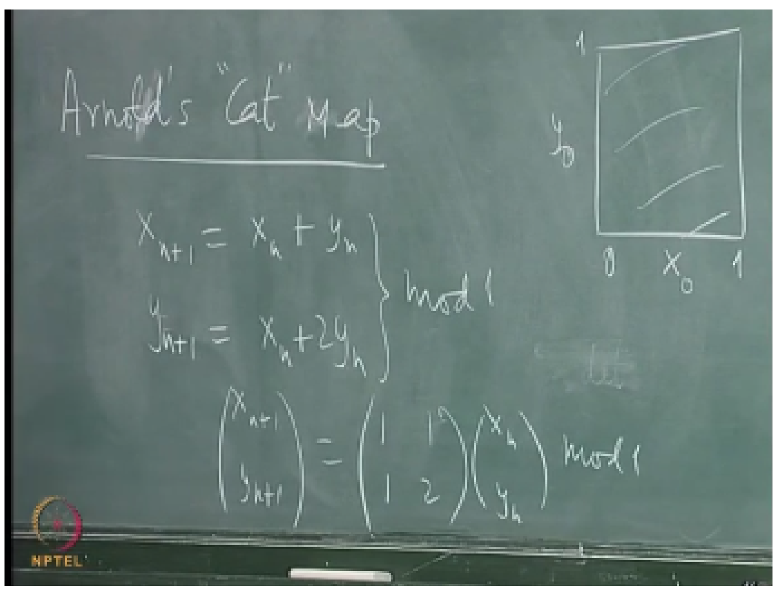

So what I have is xn + 1 yn + 1 is =this matrix 1 1 1 2 times xn yn the whole thing is modulo 1 of course the determinant of this matrix is unity the Volt's measure preserving we guaranteed that the area of this object there is exactly the same as the unit area and if I cut it and put it back here then this map looks like it is on several pieces but I could also pretend this map is on a unit torus right.

After all once I say it is periodic I could put this on a torus I could simply say this it is after all copies of this so we could bring it back and put it on a torus and then on a torus in which the circumference in this direction is 1 and the circumference in this direction is also 1 this object whatever object you put here remains continuous you are not tearing anything but the moment you say modulo 1 and you cut and snip and so on it becomes disjoint pieces.

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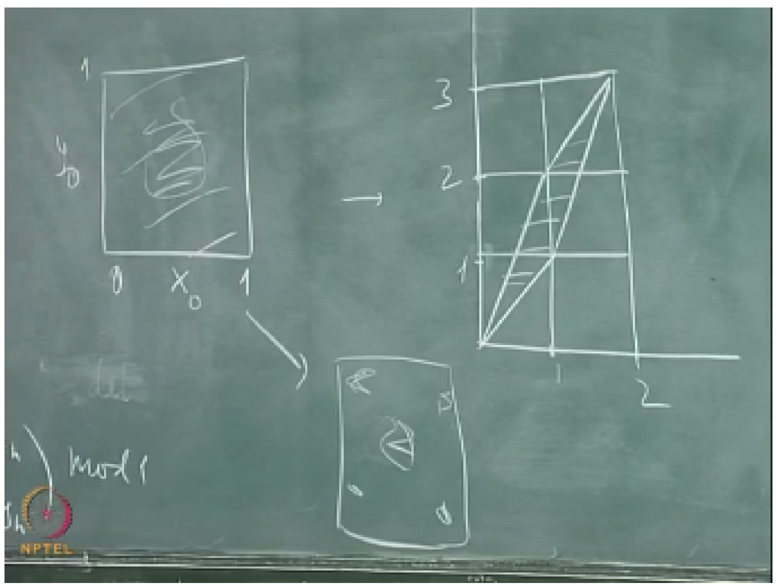

So what will happen is that if you took some object there some area here that area would go over and get scrambled up some of it will be here some would be here some would be here etc and since I am on first through a cat's picture said this is the way the cat gets deformed it scrambled up so what he did was to start with something that looks like this and then that get scrambled up as you go along it became known as the cat map however there is serious purpose to this well let us examine the properties of this map and see what happens.

So my claim is that this map produces chaos it actually scrambles at it up completely there is mixing and there is exponential sensitivity to initial conditions the phase space is bounded as we see it is measure preserving so nothing is lost unit square remains the unit square but nearby points under iteration become go arbitrarily far away they become as big as the system size itself and there is exponential sensitivity like the baker map because you put this because it is hyperbolic and let us see what is meant by that what are the eigen values of this matrix.

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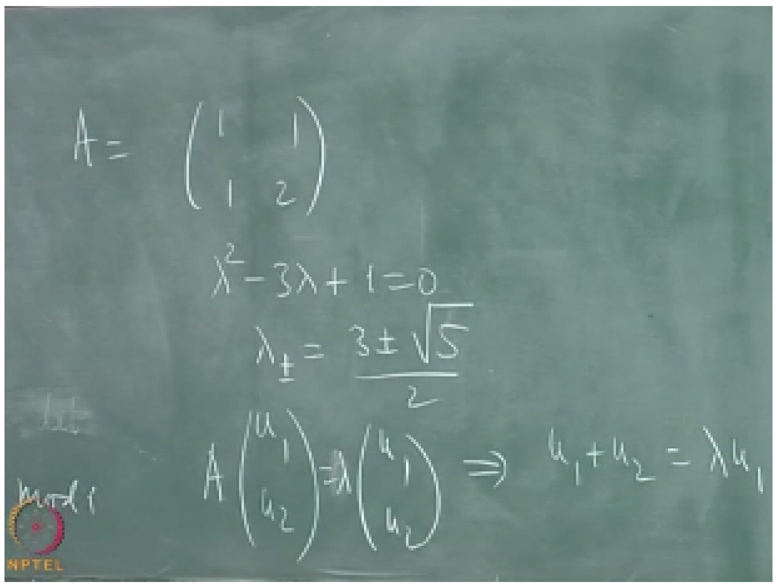

So I have this matrix a which is =1, 1, 1, 2 what are the eigen values of this matrix well it is clear λ - 1 times λ - 2 that is λ^2 - 3 λ + 2 - 1 + 1 =0 so λ + or - is =3 + or - √of so that is a 9 - 4 is a 5/2 those are the eigen values of this matrix we check what the eigenvectors are as well and the eigenvectors would satisfy if u1 u2 is an eigenvector then I apply this and I insist that this be = u1 u2 so this implies that u1 + u2 be =λ times u1 the other equation does not give anything new so it is clear that the eigenvectors or λ + the eigenvector is.

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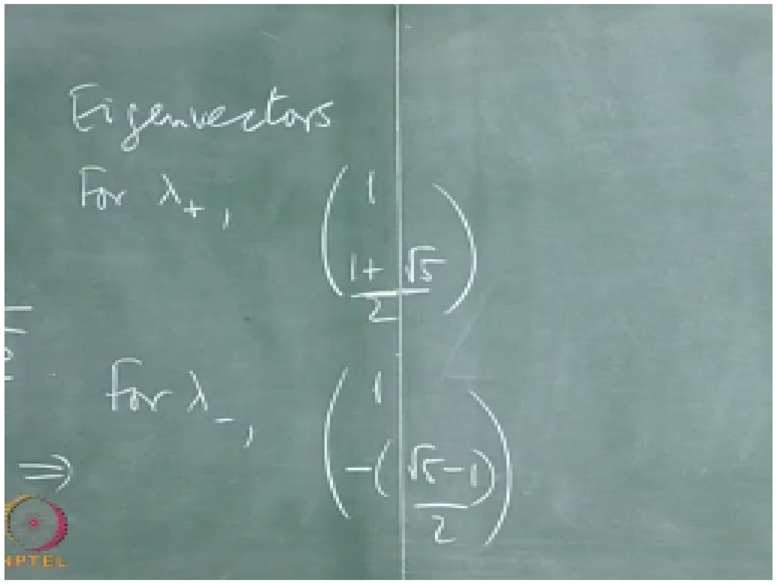

So if i choose u1 to be =1 for instance u2 is λ - 1 times u1 so this is =1 and λ - 1 that is 3 + √ 5 - 2 that is 1 + √ 5 that is the eigen direction and for λ - the eigenvector is 1 and then this is the - sign here 1 - √ 5 but since √ 5 is bigger than 1 let me write this as - √5 - 1 what is the significance of this number 1 + √ 5/2 it is a famous number it is the golden mean it is the golden mean this number is the reciprocal of that number as you can see.

So we can plot what the eigen directions are no and since this is a constant matrix this is a linear map if you forget the modulo 1 since it is constant the Utopian is the same everywhere at all points and therefore this is what controls the expansion or contraction at every point it is clear that there is an expansion in one direction and a contraction in another direction if you go back to the original picture that I drew it is quite evident.

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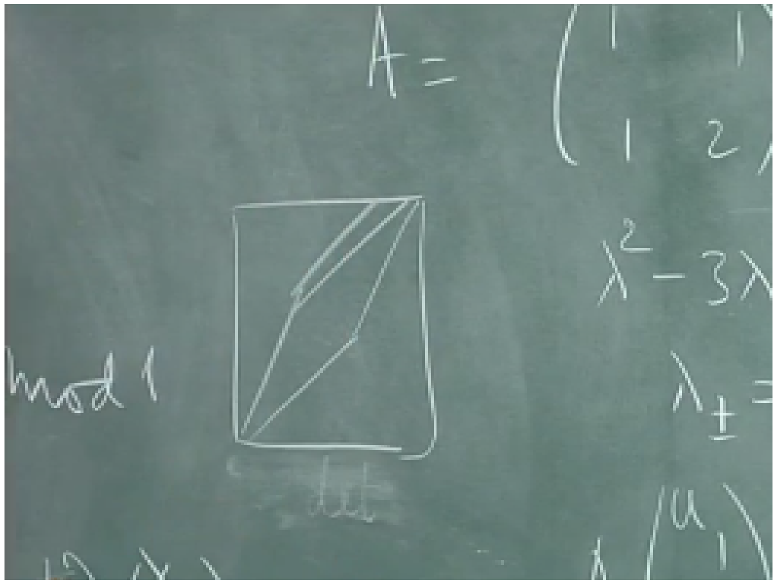

That if this gets shifted out and then you end up with the sort of stretching in one direction in this fashion if part of that parallelogram was cut and put here so there is a sort of stretching in this direction there is a contraction in the opposite direction so if you quantify that these are the eigen directions on which you have the stretching and the contraction now we can easily check out what it looks like.

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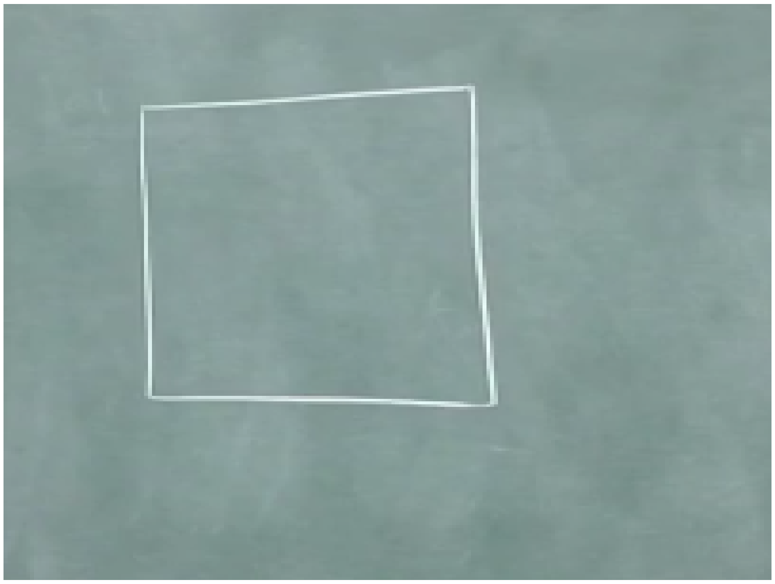

So here is the unit square and let us look at this eigen direction in what direction is this relative to this what is this direction this is the X direction and that is the Y direction this corresponds to X and that corresponds to Y so vector unit vector with these as components a vector with these as the X and y components would make an angle with the x axis which is the tan inverse of 1 + √ 5/2 / 1 this is about 1.68 or something like that 1 + √ 5/2 we want the tan inverse of that.

So that is some angle is > 45 degrees and some crazy angle it is 58 degrees or something like that this is the direction and the other guy it is easy to check that these two slopes the product of the two slopes is - 1 so there are right angles to each other this transverse in this direction and that is going to be something like this not hard to check that this corresponds to an expanding direction and the other corresponds to the contracting direction how do you come to that conclusion how do you do this after all if I iterated this map n times I get a^n acting on this x_0 y_0 modulo 1.

Right now how do I conclude that there is chaos in this system and that things actually exponentially separate out in at least one direction it is the λs that control it as you can see and just like in the case of the one dimensional example of the Bernoulli shift where I know that xn was 2 to the nX_0 x modulo 1 and the Lyapunov exponent became log 2 because that factor 2 to the power n.

If I take the log of it and divide by the initial separation I end up with log 2 as the answer if I divide by n and take the limit as n goes to infinity I get the Lyapunov exponent so that is exactly what we have to do here except there are two Lyapunov exponents and what are these two what are the Lyapunov exponents for this map no the log of these things the log of this because you see if I solve this map it is evident.

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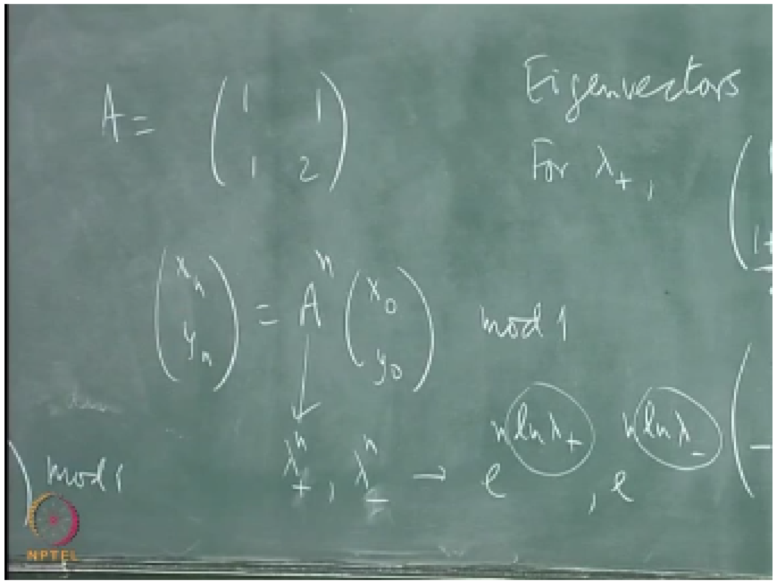

That I am going to get something like xn yn $=A^n$ x_0 y_0 modulo 1 always that is putting it back in the unit square but this guy here if I took this matrix and diagonalize this matrix and wrote elements down the diagonal elements down it would be controlled by the behavior of λ 1 to the n and $\lambda 2^n$ multiplied by two guys each other which guy can do the diagonalization of this matrix and then eventually the Lyapunov exponent will be governed will be given by log λ1 and log λ2 I call them λ + and - so that means so these things could be written as $e^{n \log \lambda}$ + e to the n log λ - and then this number here is a Lyapunov exponent.

I am sorry I use the same symbol λ for the eigen value a should have probably used μ here and then λ for the Lyapunov exponent.

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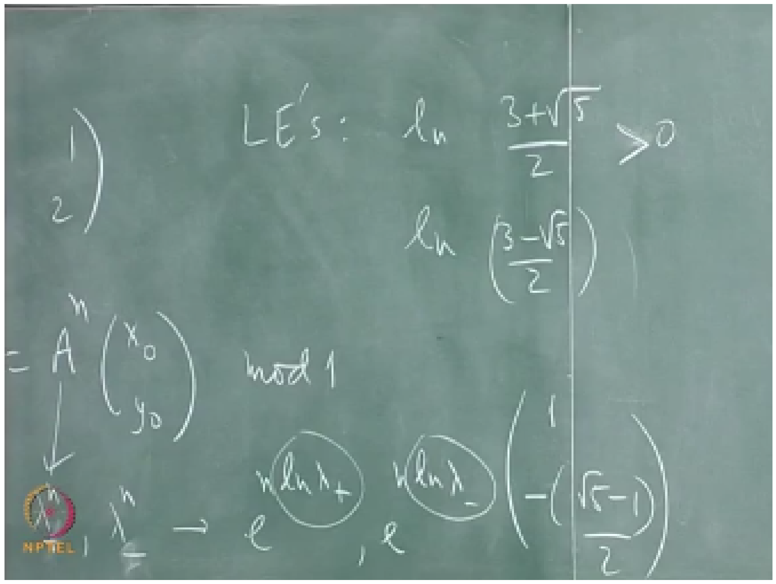

So that is the reason for the confusion now what is log λ + yeah so that yeah yep enough exponent log 1 + 3 sorry 3 + √ 5/2 is this positive or negative it is positive exactly and the other one is log 3 - √ 5/2 and what is that it is negative because this is the number < 1 therefore it is the contracting direction and we have also found the eigen directions so this is negative this direction contracts in this direction expands this point here is like a homo clinic point because that is exactly the point if i put this on a torus it will keep coming round and round so let us pretend that we are doing that so since this is periodic modulo 1 it means that once you go out here you are back.

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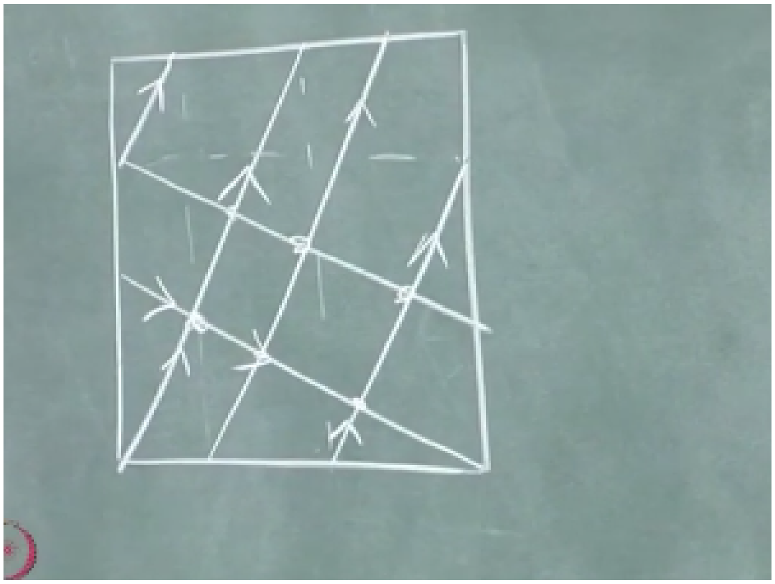

Here you start here and then this is what the unstable manifold is going to look like so here you are unstable and you keep going round and round so once you are there you are going to be injected back here and then parallel to this you want to go off similarly here you are going to go off like this and then you are going to come here go off like that and this will keep happening all the time so there is at every point there is a transversal intersection there is a transverse intersection.

So what is happening is that you are on a torus and along almost every point on the torus you end up with except periodic unstable periodic points which we have not yet discovered you have a contracting direction and an expanding direction they are orthogonal to each other so it is as if you have got saddle points everywhere that is exactly what happened in the case of the Bernoulli shift as well it is as if you have got separate races everywhere because you have got these unstable periodic orbits at the rational things get thrown out on both sides and that is exactly what happens here now because it is on a torus and because it Maps the torus to the torus it is called a total automorphism.

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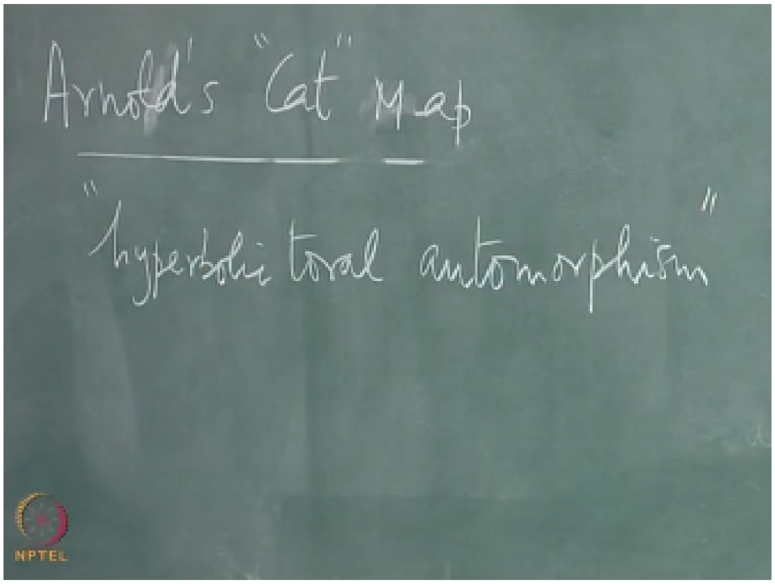

And because it is hyperbolic no Center manifold here it is called a hyperbolic total automorphism and the name cat map is easier to handle but there is not the only one any number of such maps could be constructed to put the integers down here such that the determinant is + one and you end up with such a automorphism it is a paradigm of chaos it is exponentially unstable almost everywhere and it is got an even dimensional phase space its measure preserving so this was Arnold's example trying to mimic what happens when chaos sets in a genuine Hamiltonian system.

And it happens because of the hype what is called the homo clinic tangle and this gives you a sort of toy or caricature of what happens in such a system that you have this expanding and contracting directions everywhere and that the measure is preserved but it is chaotic with this layer for no exponent let us try to find what the periodic points are where are the periodic where are the fixed points of this map and where are the periodic points of this map how do we do that well I need to know when this map leaves a point unchanged and this is not.

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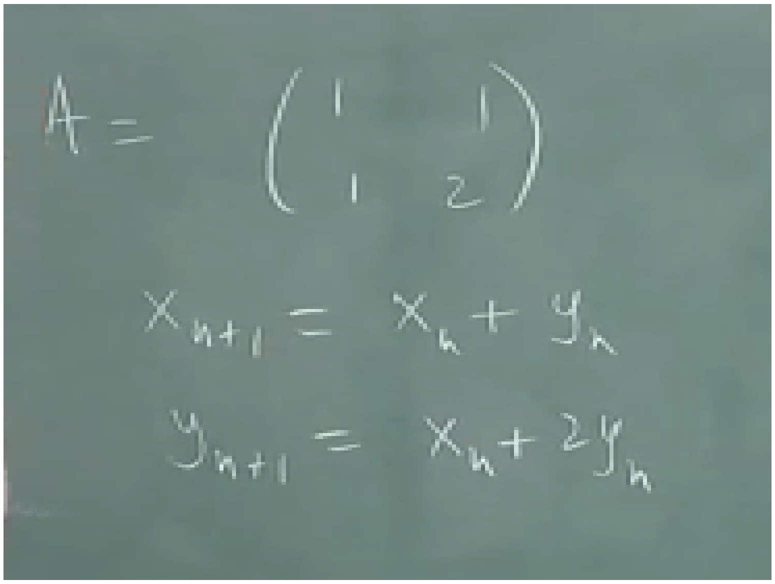

So trivial to do but let us see what to do about it now notice that under iteration since we have xn + 1 is =xn + yn before you do the modulo 1 if I start with these numbers between 0 and 1 and I go to xn + 1 the maximum that xn can increase by is one it cannot increase by more than one similarly y n + 1 is xn + 2yn what is the maximum by which yn could increase before you do the modulo 1 2 as we saw in the picture right you could add a 2 to the unit square and then you had a square of rectangle of height 3 and with 2.

So it is clear that before you do the modulo 1 the largest that X can increase by is unity and the maximum that Y can increase by is 2 therefore if you have fixed points if you have a fixed point and what does it imply any fixed point of this map of the nth iterate of this map.

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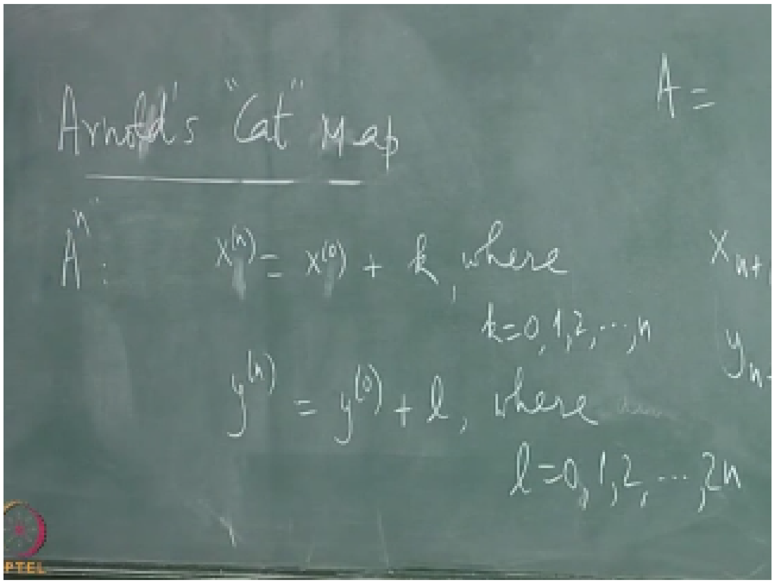

So I take A^n and I ask what about the fixed points of A^n when n is 1 I get the fixed points of the map when n is 2 I get the period 2 cycles of this map and so on and so forth so I am not trying to find all the periodic points and all the fixed points of this map so it is evident that if x x_0 for example x_0 y_0 is a fixed point of this object then it says Xn must be $=X_0$ + some integer K which goes away in doing the modulus bad notation to use it like this suppose X_0 is a fixed point then after n iterations it will do this where k can be 0 for 1 or 2 up to n it cannot be more than n.

Because after n iterations a point cannot increase by more than n since in one iteration it cannot increase by more than 1 before I do the modulus before I do the modulo trick similarly y n $=y_0$ + l if y superscript 0 is a fixed point of A^n and so is x_0 if x_0 y_0 is a pair is a fixed point of the map A^n then you must have after n iterations this must go to this + integer where n =0, 1, 2 up to 2n because it could go right up to n and then in the modulo 1 it goes away that integer part goes away now this will help us find where are the fixed points of all of the map and its iterates. So let us look at the map and let us look at the map itself.

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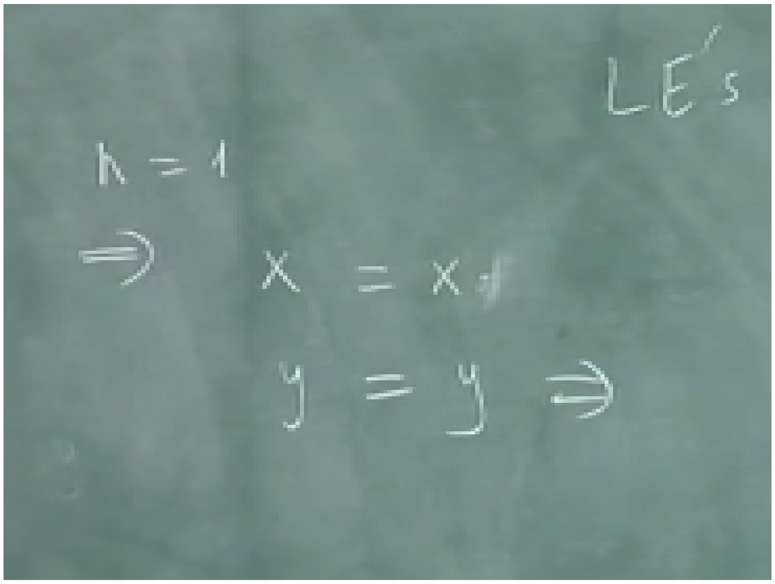

So n =1 implies that any fixed point must satisfy x =x + k where K can be either 0 or 1 but now you I impose the modulo 1 condition so what is the only possibility here 0 so it must satisfy this similarly y = on the other side y + and the only thing you can have here is a 0 implies under iteration what is the only fixed point now this is the only fixed point the origin is the only fixed point of this map nothing else now I leave you to do the higher iterates so do the first you trade a^2 first and so why do not you discover some rational points would also be allowed and then you do a^3 and so on.

And it will turn out that all rational points are fixed points of this map or of its iterates and this is not trivial to prove not very difficult either you can show that all rational points either are fixed points of the map the only one is the origin or lie on periodic orbits like in the case of the Bernoulli shift now the rational are dense everywhere on the unit square the pair of rational dense everywhere on the unit square and those points would lie on periodic orbits.

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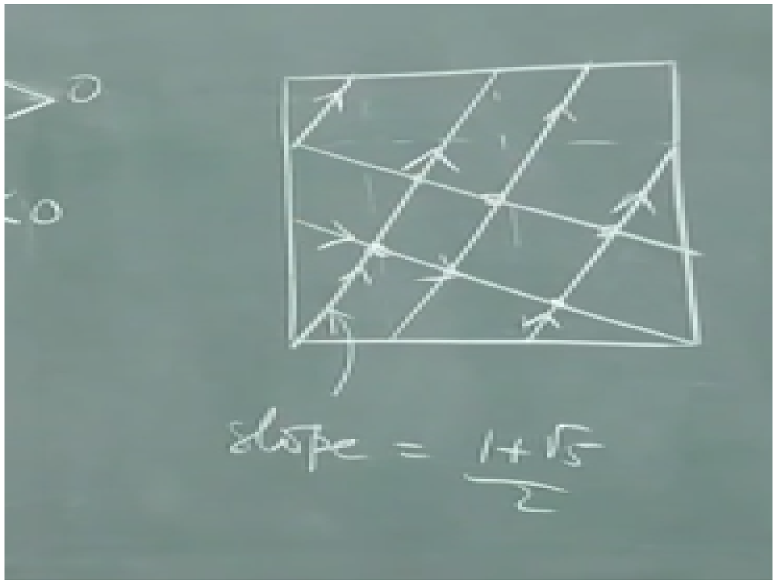

But tell me what happens to this line this line here let us look at this guy here the slope of this line 1 + √ 5/2, 1.618 or something like that some in a rational number what is special about this line what sort of points can line on this line yeah both x and y cannot be rational right so no rational pair can line on this line nor can it line on all its subsequent iterates or on this so these points will not pass through any pair of points any points whose coordinates are rational and yet they would densely fill up the entire square.

But they miss all the rational points that is what a chaotic trajectory would do typical chaotic trajectory will do but all the rational points are left out point behind me the fixed points are all rational points so a line like this is the unstable manifold and this is the stable manifold this line does not pass through any pair of points both of which are rational and nor does any of its subsequent iterates.

So this line here comes back and starts off here and then it after it goes here it goes here and then it comes here and goes here and so on this thing will not have any point on it both coordinates of which are rational and that would be generic that is typical so such lines this line it the measure would be the total measure of this area itself and the rational points form a set of measure 0 there the unstable periodic orbits and in its neighborhood you have these expanding.

And contracting directions so it is fairly it is a fairly complicated picture but it is like a paradigm of chaos Hamiltonian chaos in this case because it is measure presumably we saw a little bit about what rational do and let me give you another illustration of a map a simpler map where you have this some properties of the rational will emerge do that I am not sure if I discuss this earlier you go back to a one dimensional map where you have some interesting properties did I discuss the continued fraction map at all the Gauss map let me do that now.

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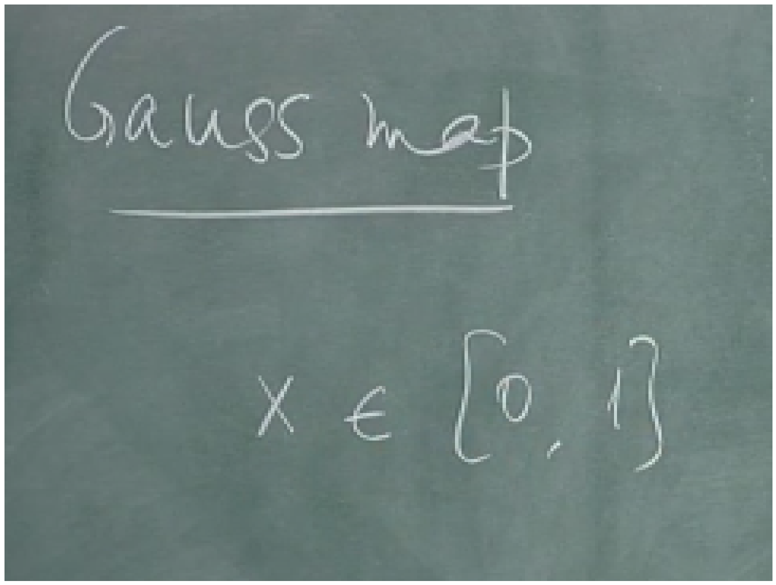

So it gives you an illustration of the continued fraction map this question was actually this map was first considered I do not know the exact history of it but Gauss did very important work on this map and it is the following question in fact the question that Gauss was interested in was the following he said suppose you take a number between 0 and 1 so I take an element X element of 0 to 1 somewhere between 0 and 1 and I take its reciprocal that is the number bigger than 1 I throw away the integer part keep the fraction I take its reciprocal I throw away the integer part and keep its fraction.

And then the question was if I took a typical number and kept doing this and iterating this procedure what is the probability that the number I end up with after a very large number of iterations is < some given X that is it so the question was I start with a typical number between 0 and 1 take the reciprocal throw/the integer part and keep iterating this procedure and ask why what is the probability that the answer I have is < some prescribed threshold X which is between 0 and 1 and he found that this answer was very interesting indeed it was proportional to log 1 + x.

This probability is proportional to log 1 + X and this excited him so much that he wrote a letter to Lagrange saying that he had found the solution to this problem and what he did essentially now terminology is the following let us see what this map is.

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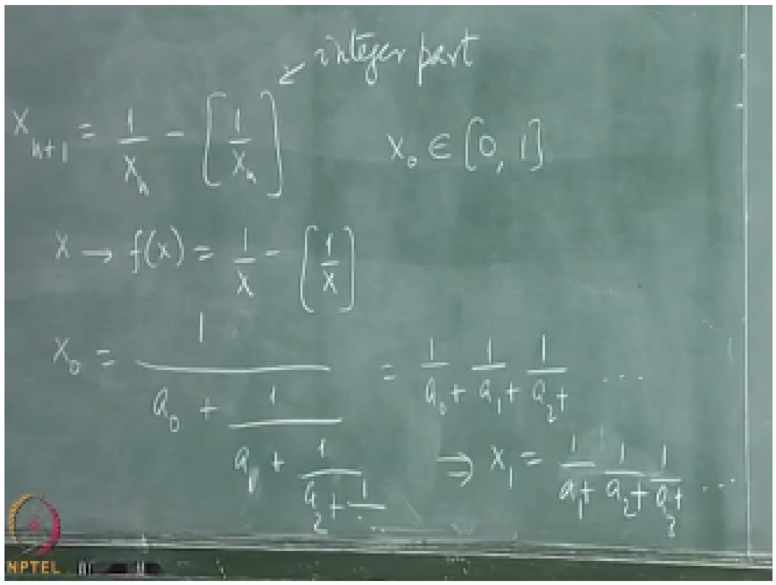

So you start with the map which says that at the n + 1 stage you take the reciprocal of xn and throw away the integer part so it is - 1/xn and this guy here is the integer part so I use the square bracket to denote the integer part and now the question is what it is this map caotic if so what is the Lyapunov exponent what is the invariant density of this map etc this is the question we have to answer what is the figure of this map look like what does the graph of this map look like.

So if I write x goes to f of x is 1/x - this what does this graph look like what does this graph look like but it is already going to be mod 1. So I start with x_0 you start with that so I start with 0 and 1 between 0 and 1 I take the reciprocal this is bigger than 1 I throw away whatever this is integer and I still have something between 0 and 1 and I keep doing this so it remains between 0 and 1 automatically same thing right same thing okay so this tells you explicitly what you are doing it will become clear why I wrote it in this fashion in a minute.

What does the graph look like in fact why make a mystery of it let me tell you why I wrote it in this fashion instead of writing it as a binary number a binary decimal which is what we did for the Bernoulli shift I can also write it as a continued fraction? So I can also write x_0 is =1/a number bigger than 1 and in standard form any continued fraction can be written as an integer +

1/once again an integer + 1/for a and in a simple notation this is called 1/a $_0$ + this is notation for a continued fraction.

So that you do not have to write this crazy diagonal everywhere so these are integers positive integers what is X 1 we take its reciprocal then you have an integer + a fraction and you throw this integer away and therefore you have this right.

So this immediately implies that X 1 =1/1/a1 + a2 + so you have lost information about this a 0 it is clearly not invertible do you think the map is chaotic do you think it has fixed point it has fixed points once we draw this graph it will become clear it has fixed points but do you think these fixed points would be stable no they would not because the slope of this map this is some number it is constant each time it is a different constant the slope of this map is one/x ^{2} the - sign and the mod of that is 1/x ^{2} and since x is < 1 this is always positive.

So it is clear that this map has no stable fixed points and its iterates are going to get only worse so it is a candidate for chaos once again they would of course be points which lie on periodic cycles there would be points which lie which are fixed points of this map and so on what would the fixed points look like for that we should draw this graph it is clear that if I draw just 1/X then if I draw y =1/X that is a rectangular hyperbola which passes through the point 1, 1 but it actually does some crazy thing like this and goes off and I start with a number between 0 and 1 here and the graph is always bigger than 1 but I cut this and put it back right.

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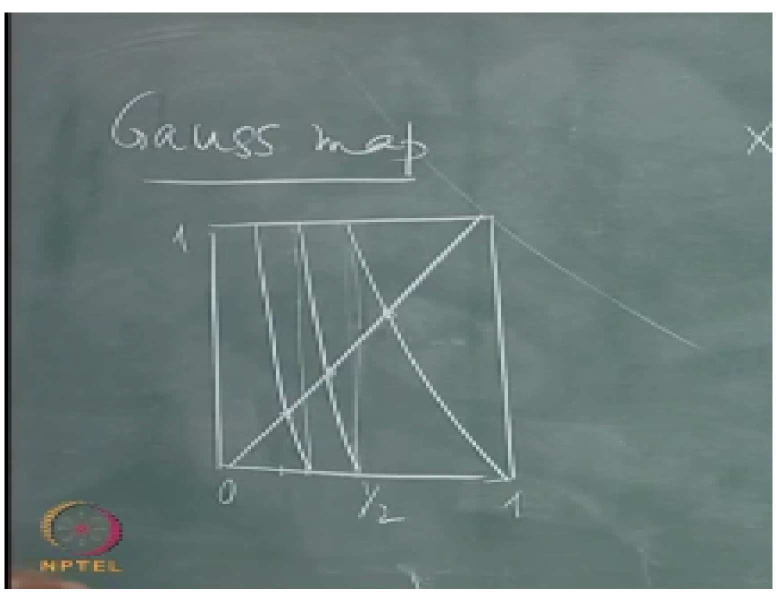

So as long as the initial point is to the right of a ½ this is ½ all I have to do is to subtract 1 to bring this piece and put it back here so this map looks like this if the number is between 1/3 and 1/2 then I have to subtract 2 and bring it back and put it down here so the map looks like this little slip a steeper slope if it is between 1/4 and 1/3 then I have to subtract 3 and therefore this map starts looking like this and it is getting steeper and steeper as you go down it has an infinite number of branches.

Right and all of them have slopes in magnitude much bigger than one as you go to the left and therefore all fixed points are unstable but what are the fixed points there are these intersections these are all the fixed points what is this point correspond to so I would like to know what the fixed point there is for this mat this is the largest fixed point of the map here and on this branch the map function f of x =1/x - 1 because that is what i subtracted.

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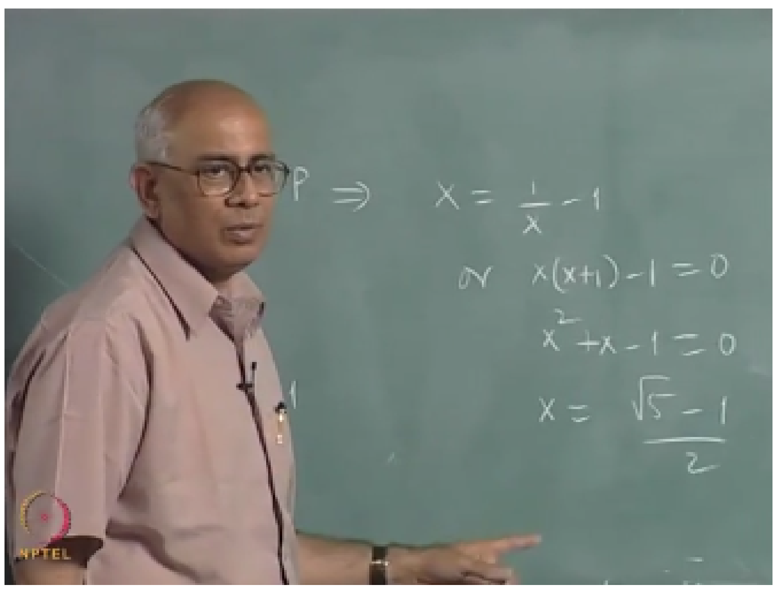

So this fixed point implies that x is =1/x - 1 or x times x + 1 - 1 =0 x^2 + x - 1 =0 or x =- 1 + or - √1 + 4 is 5/2 and you want the number to be between 0 and 1 so the fixed point in fact is √5 - 1/2 so that is the only acceptable solution that is the golden mean once again what is its continued fraction representation and that tells you why this number is the most irrational number of irrational numbers exactly it is just one all the way through you see what is happening here is that you want this number to satisfy.,

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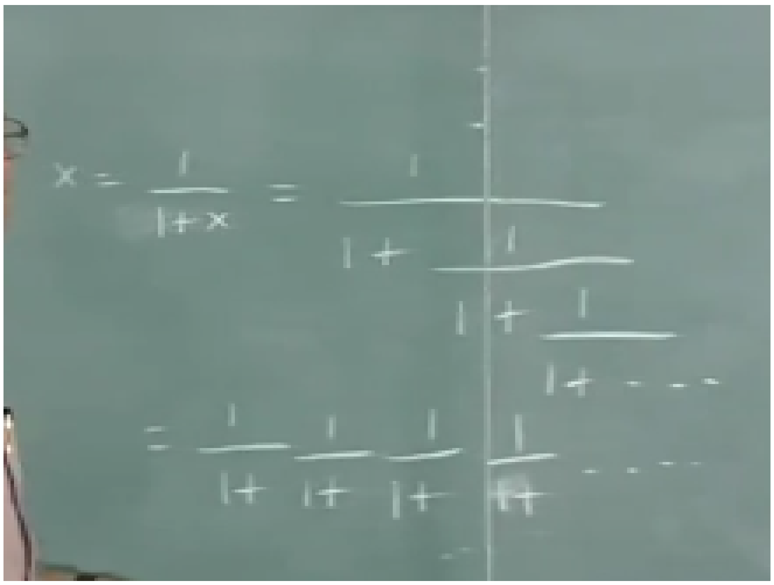

1/x is 1/x + 1 or x =1/x + 1 or 1 + x but now I substitute for x here as 1/1 + x etc and keep going so this is =1/1 + 1/1 + 1 in other words all the A I's are 1 and now you see why it is so terrible to approximate this number by a rational number because if one of these A I's was very large compared to 1 you could truncate the number at that point and give a very good approximation to the number this is what generate when he took π and did this and gave a continued fraction expansion of this such that the third or fourth of these guys was 293.

Which when compared to a fraction was much larger and therefore he got an accurate estimate of π to some very large number of decimal places by writing a very compact continued fraction expansion so π all the transcendental although irrational is not one of these very rational numbers quote-unquote which is in the sense that rational approximants are very difficult to find very poor rational approximants approximate this very poorly because these numbers are all comparable to each other they are all equal in fact.

And there are small as they can be 1 what is this number what do you think it is yeah it is exactly it is just 1/2 + 1/2 + etcetera because that number satisfies this so this is the silver mean this one is the golden mean there is a silver mean.

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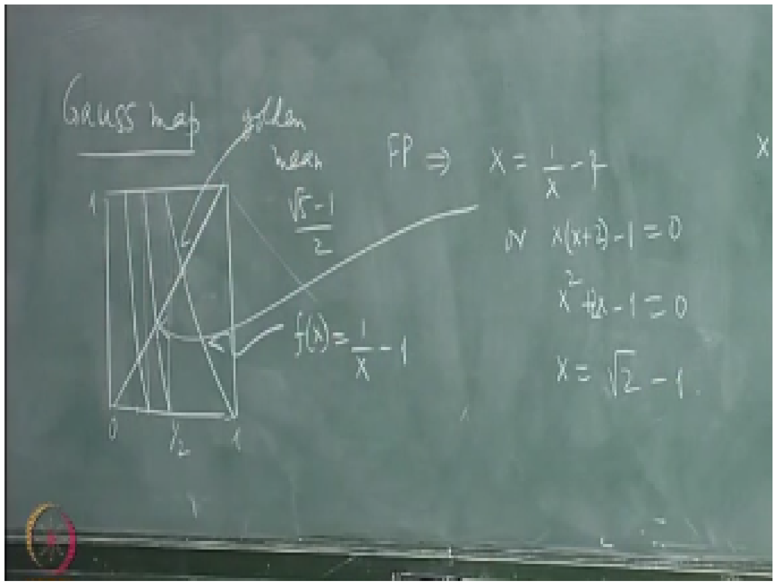

What is the value of this number what is this guy here satisfies this so let us solve this so it is X into X + 2 - 1 is 0 and the route that we want is this is 2√2 so the route that we want is √2 -1 for guessing what this is what is the continued fraction expansion of this one with all that integers are 3 so much for the fixed points and it is clear that all of them are unstable the slope is > one so much for the fixed points what about period two cycles of this map you would want it to flip back and forth it is quite clear that when you wrote that when you write the continued fraction expansion you would want it to flip back and forth right.

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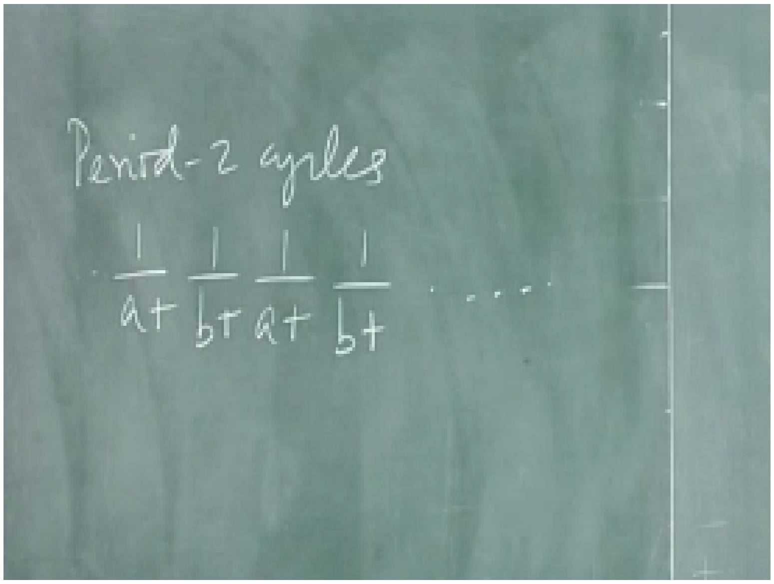

So period two cycles are all numbers whose continued fraction reads like 1/a + 1/b + 1/a + so perhaps you would have 1/1 + 1/2 + 1/1 + 1/2 + etc that would be the leading the way first the largest period to cycle it flips back and forth with 1/2 + 1/1 + etc so you see the continued fraction expansion of the periodic orbits of this map have a very interesting simple expanding expansions and that is why it is also called the continued fraction map that is exactly what it is just as the Bernoulli shift was the binary shift if you like the Bernoulli map was a binary shift.

Similarly here this is the continued fraction map now let us try and find its invariant density what was Gauss's result here and how did he get it and it is Lyapunov exponent well the invariant measure would satisfy the Frobenius Perron equation.

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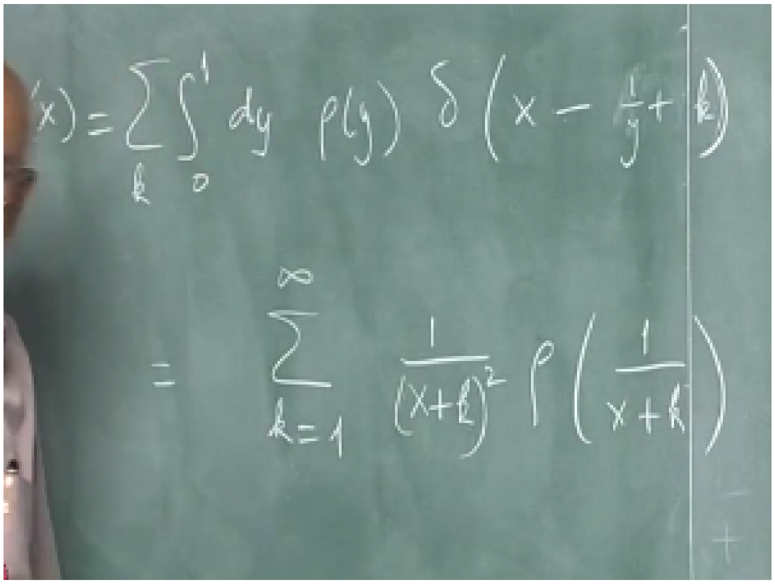

So that would be ρ(x) =an integral 0 to 1 dy times ρ of Y times the *δ* function of x - f (y) but what is this f (y) yeah this thing here would depend on where you are right but this f (y) in general would be something like 1/y - some integer K and all K is from 1 to infinity would contribute corresponding to all these branches so if I did that I end up with a Σ from k =1 to infinity of ρ of well x + k is y so y is 1/x + k so 1/x + k that goes in here right this is 1/y + k and as a Σ/k / the Jacobian of the transformation.

They probably know the transformation when I take the modulus it is just a 1/this guy here so what would it give you 1/x + k the voles quite because it is divided by 1/1/x + k the ρ^2 so this is just x + k so let us write it properly $1/(x + k)^2$ ρ of 1/x + k now this does not look like something which is going to solve very easily it is a functional equation which is completely crazy right but this is what gauss solved in effect we solve this.

So since we have by hindsight we have gauss to support us let us write the solution down and see whether it works or not so the solution that he had I pointed out that he said it is proportional the probability that the final that the iterate after a long time is < a certain number x is proportional to log 1 + x that is the cumulative probability therefore the probability density is the derivative of log 1 + x.

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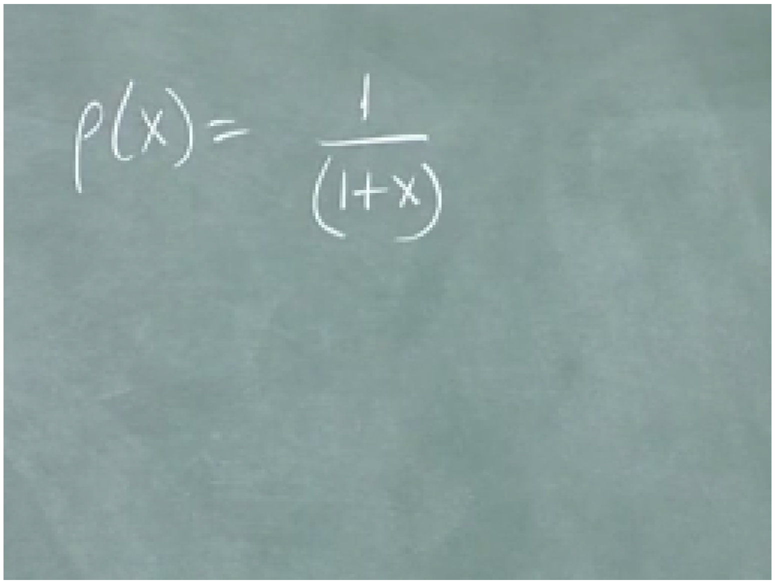

So it is 1/1 + x but you must normalize this to unity between 0 and 1 if I integrate this I get log 1 + x right and I put in the limits at the upper limit I get log 2 the lower limit I get log 1 which is 0 so the normalization constant is 1/log 2 let us put that in write our exact expression this is the normalized density and the question is whether this satisfies it or not so the question we are asking is the log 2 cancels out on both sides it is a homogeneous equation. So the question we are asking is 1/1 + x question mark =k =1 to infinity.

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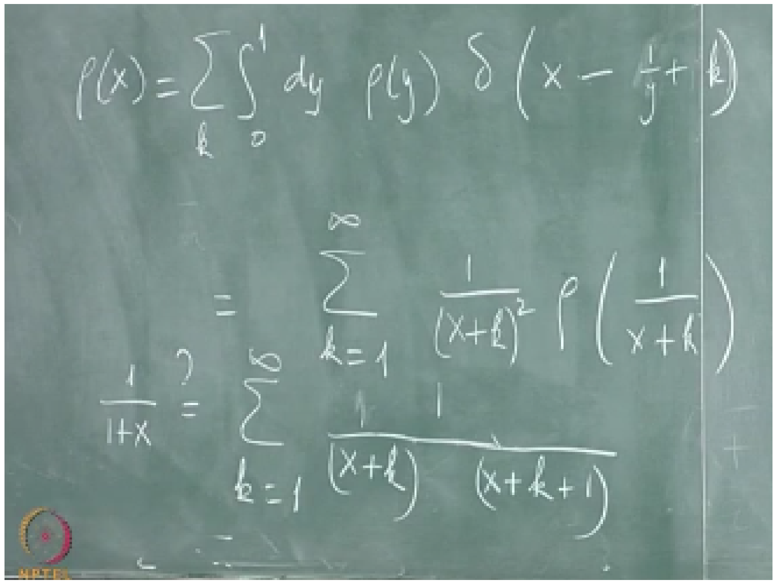

$1/(x + k)^2$ 1/k + I am sorry 1/1 + 1/x + k so you can see that this becomes x + k on top and this becomes x + k + 1 and one of these factors cancels and you get this that is a convergent sum because it is going like $1/k^2$ so it converges faster than 1/k so the question is this true this is a je some what do you do next do partial fractions you do partial fractions.

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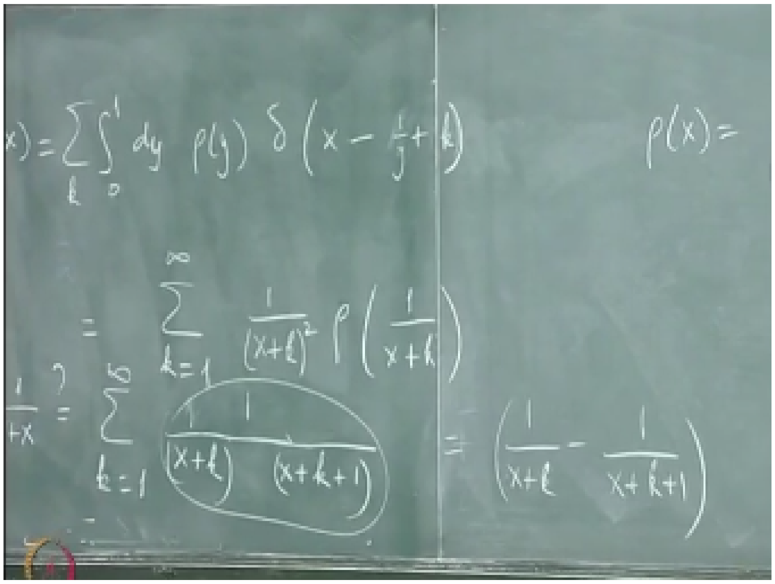

So this guy here is indeed =1/x + k - 1/x + k + 1 and of course terms the whole thing telescopes and only the first term would contribute and it end up with justice only the k =1 term the left-hand side survives this one survives and you get indeed 1/x + 1 so this is the solution and we get we were assured that once you find a solution its unique so this is the solution of course please be careful this sum here this sum here is strictly =Σ k =1 to infinity that I should not write it as a difference of two sums because each of them diverges.

So as long as I keep the Σ outside and the bracket inside the sum that is fine this is completely quotient and indeed it is true so the elementary yeah I do not know how Gauss did it I am sorry I should have read the history of this I do not know how Gauss did it I will look it up I do not know how he did it but I know he was sufficiently excited to actually write a letter and see he is found the solution I do not even remember whether he wrote the letter to Laplace or through Lagrange or whatever but certainly sent a letter to somebody in France know this so now that we have ρ (x)we can find the Lyapunov exponent of this map.

Now what is the Lyapunov exponent physically measuring it is in some sense measuring the average the log of the average slope of this map it is log mod f prime of x so the average stretch factor it is measuring now the slope here at this point at x =1 is 1 in magnitude and everywhere else it is bigger than 1 and it is in fact tending to infinity as you come here so the question is what is the average slope but it is got to be weighted with that factor out there.

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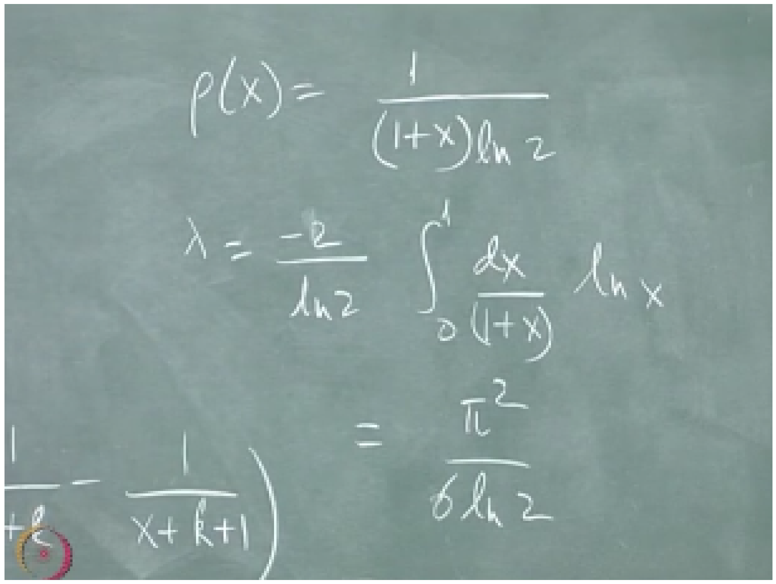
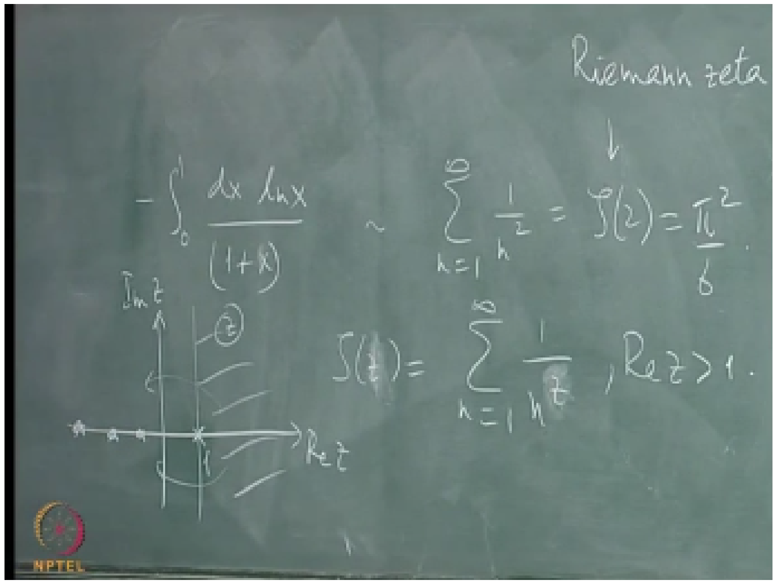


So when you do that you get λ =1/log 2 and integral from 0 to 1 dx/1 + x times log mod f'(x) but f'(x) is - 1/x^2 so this is =log 1/x^2 which is - 2 log x this is negative but so is that so the Lyapunov exponent is positive now what do we do with that integral we can always look it up we know it exists it is not singular because the log is an integral singularity this is very harmless the rest of it is not going to vanish at all.

So the integral exists there is no difficulty about it but what do you think is a value of this integral unfortunately because there is a sort of mixture there is there is a rational function here and there is a logarithm here it cannot be done by elementary means it is not a trivial integral you cannot do this by elementary means.

So little harder work is involved it is a logarithmic integral it gets related to the ξ function of two and the answer here it turns out to be = π^2/6 log because this integral let me write that down here. (Refer Slide Time: 01:02:08)



It is 0 to 1 dx log x - x 1 + x and do this in several ways and show that this is indeed would not happen one of which would be to expand this in a binomial series and try to collect terms and so on the other would be to try to expand the log but neither case this gets related to the following sum which is 0 of 2 and that is $\pi^2/6$ so that is how the $\pi^2/6$ arises and this is defined this is called the Riemann zeta function it is off argument to and it is finite it is not hard to show that this number is $\pi^2/6$ it can be done in several ways the simplest of which would involve into the contour integration.

So since we have not do not want to get into that at the moment but let me define what the zeta function is it is useful information ζ of K is defined as Σn =1 to infinity $1/n^k$ and to start with you could ask what it is for positive integer values of k please notice if I put k =1 it diverges it logarithmically diverges so k is 2, 3, 4 etcetera etc and for 2 it happens to be $\pi^2/6$ for 3 it is fairly complicated before it is π to the 4/90 or something like that and so on so simple expressions are available whenever this K is an even integer when it is an odd integer it is no such simple expression is available.

But in we know what it is numerically this is called the Riemann zeta function because you can actually define it with any argument here I could put an X here and it is clear that if X is positive and > 1 then this converges even if X is irrational in fact since it is only the positive real part of X that matters you could make it a complex variable and all you need is the real part of ζ is > 1 it is an analytic function in the sense of analytic functions of a complex variable of this argument Z and in the Z plane it is defined by this infinite series to the right of rail Z =1 it stops converging when Z is =1.

In fact that series does not make sense anywhere on this line or to the left of it but it turns out this function has meaning even to the left of it has a singularity at Z =1 it has a simple pole with residue =1 and everywhere else it is analytic so in fact you can define ζ of Z_0 by this infinite series not by this series but by a different representation even to the left of it everywhere the most important function if you like in analysis but what we have here is ζ of 2 which incident coincidentally happens to be related to the Lyapunov exponent of the continued attraction.

Now there are deep connections here there are reasons why this is so and since I digress so far let me also mention that the most famous the most important problem in mathematics is the following is the establishing of the Riemann hypothesis which says that this function actually has 0s some points values of Z it actually vanishes it is proper definition to the left is done by what is called analytic continuation that vanishes at some points it turns out you can trivially show that it vanishes at all the even negative integers - 2 - 4 etcetera.

It vanishes at all those points trivially it also vanishes at points on a line perpendicular to the exit of the real axis parallel to the imaginary axis cutting the real axis at the point 1/2 so on this line on which Z is =half + Iy it vanishes at a large number of points and it is got some symmetry properties on this it vanishes at an infinite number of points on this line and the Riemann hypothesis one form of it says that other than these trivial 0s all the other 0s of the zeta function are on this line.

And this has stood 150 years of there is a challenge nobody has proved this so far what is known is that there are an infinite number of 0s of a zeta function on this line what is also known is that all non-trivial 0s with probability one line on this line you may wonder where probability comes in here but it does but what is not known is a proof it says there are no other 0s anywhere else so people have numerically tried to find all the 0s on this they found the first 13 billion 0,s as of last October.

They are all on this line yeah now you could ask what is the measure of the 0 is here what is the distribution of the 0’s here that distribution is related to the prime number theorem which says that the number of primes < a number x is of the order of x/log X so the proof of the Riemann hypothesis on the number field on the complex number the numbers would involve would immediately solve a very large number of outstanding problems.

So much so that it is taken to be a truth and all those proofs rest on the truth of the Riemann hypothesis but this is so far defied everybody this is again the Riemann hypothesis looks very specific to a particular function and so on and so forth but it is much more general than that and it is got generalizations so it is in fact the most important problem in mathematics now what is even more intriguing is that the distribution of 0s here is turns out to be related to the distribution of energy levels for quantum mechanical systems which are classically chaotic.

Now we do not fully understand the connections so there are incredibly deep number theoretic connections between Zeta function analytic number theory chaotic dynamics and quantum mechanics so much of this is still not known movie there lot of open problems but certainly cracking any of these problems would be a major step so I did not want to get into this but incidentally there are all sorts of other relationships which lead to these things apart from the continued attraction map leading to this.

For example I give you two numbers and I ask you what is the relative probability that they are co-prime with each other but they are not two positive integers and I ask what is the probability that they have a common divisor or they do not have a common divisor and so on that is related to $6/\pi^2$ okay.

So and that is not hard to prove either fairly elementary proof for that so there are all sorts of very interesting and deep connections but what is intriguing is there are connections with real life with your actual dynamics dynamical systems and the full reasons are not totally clear yet okay I can stop here today.

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