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TOPICS IN NONLINEAR DYNAMICS
Lecture 25
Coarse-grained dynamics
in phase space (Part IV)
Stochastic dynamics (Part IV)
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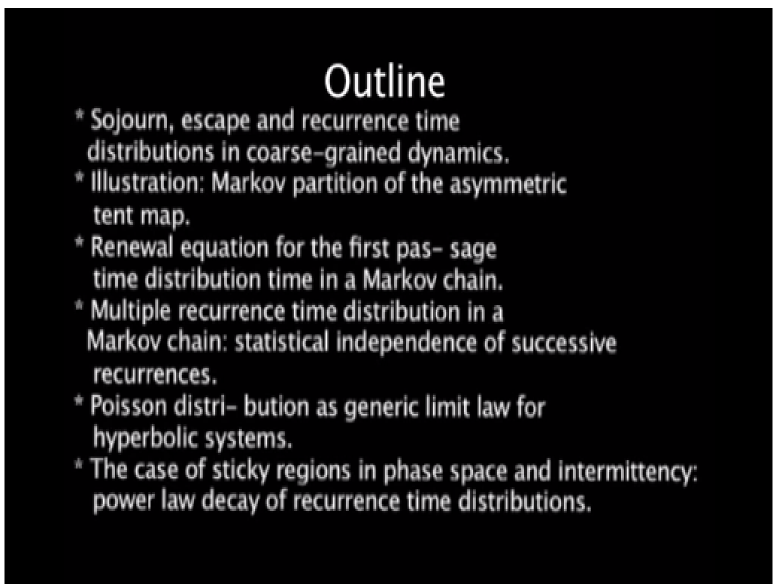

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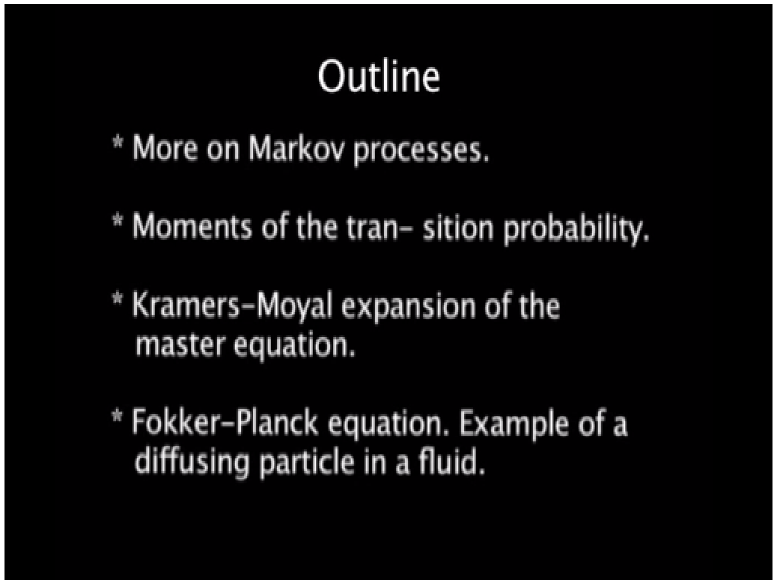

Yeah, let me go over once again some points regarding recurrence that we discussed with regard to maps and coarse-grained maps. And then give an example using the asymmetric tent map which we have already studied in some detail.

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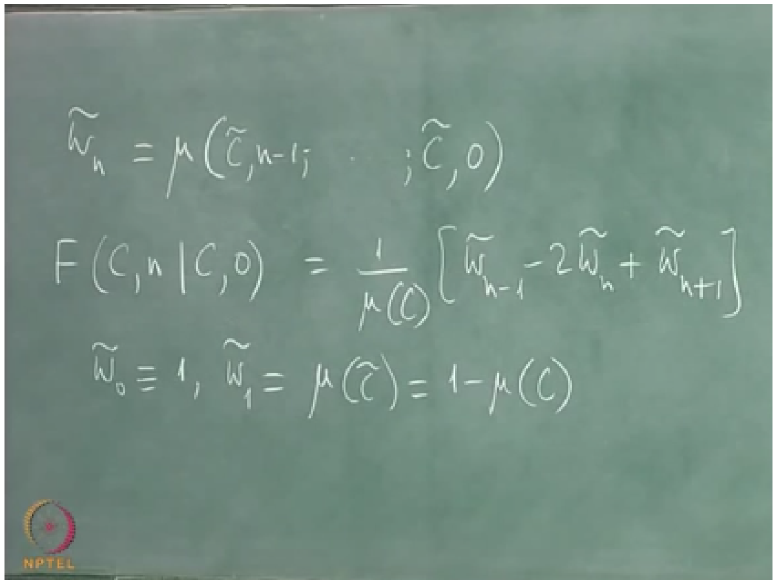

As an illustration of the general principles involved recall that we had a phase space whose total measure was 1 and in this phase space I define cells let us say this is some cells some particular cell C. And then we were looking at the statistics of the recurrence of a typical phase trajectory to this cell in coarse grain dynamics. And just to recall to you what results the sort of results we had proved we discovered that if μ(C) is the invariant measure of this cell.

We asked for the measure of that set of events where a typical trajectory starts in this cell at time 0, and then returns to this cell for the first time at time N, and we call that the recurrence distribution. This we found was given by a certain formula in terms of a quantity Wn which was the measure of all those events where you started in the compliment of cell C at 0 and you remained in this complement till time n-1.

And then going to this cell in terms of this quantity W and ~ we discovered that the probability and now I will use the word probability itself for recurrence namely to return to the cell for the first time. And let me call that f return probability of return for the first time at time n to cell C given that you started and C at time 0, this was what I denoted by are the last time, but this is a more expressive notation it says you started in cell C at time zero and who returned to sell C at time n for the first time without having come there before.

After the sojourn the rest of the time in between in the complement to the cell C. So we are looking at statistics of things which know this start here and end up back there at a specific time in this is the random variable here is N and we are trying to find the distribution in this n. This was given by1 over the measure of the cell C multiplied by Wn - 1 ~ minus twice Wn ~ + Wn + 1, this was the formula we derived on very general considerations.

And there were just two assumptions made in this derivation the first was that there exists an invariant measure that the system has settled down to some stationary probability distribution denoted by μ here, and the second assumption was that the system is ergodic namely if you started anywhere the system walked around everywhere in phase space and came back eventually to this point.

And it does so repeatedly it keeps repeating itself over and over again the system is not periodic that would be trivial kind of recurrence its ergodic and therefore, we speak not of recurrences back to exactly the same phase space point as the starting point, but to some neighborhood of it to some cell in which this initial point is. Notice that this has the structure of a second derivative in time it is a second difference in.

And this itself is like a stay probability, because you start in someplace and you stay there without moving out of it into the cell C. So it is essentially telling you that and this is typical it is a general sort of result that the first derivatives of stay probabilities would give you escape probabilities and the second derivatives would give you return probabilities. So in order to start at C and recur return back to C return to see you have to first leave and then you come back.

So it is pardon me, it is like the second derivative in time is not it this is just the second difference. The first difference in time would be something like Wn ~ - Wn - 1 ~. And the difference of the difference is like a second derivative in time. So it is a second finite difference of course this can be formalized, but intuitively it is clear what is happening you have a certain cell C and you would like to find the statistics of return to this cell for the first time.

Well you have to go out and you stay there for some time and then you come back, and this is like the sojourn outside this cell. So you start here, you escape, you stay out, and then you come back and that is so it is not surprising that exit time distributions or escape time distributions are like the first difference of stay distributions and return distributions or recurrence distributions are like the second derivative in time like the rate of change of exit time distributions or escape time distributions.

So this structure is not surprising post facto after the end by hindsight it is quite physically quite appealing. Now of course remember also that we had defined W ~ = 1 and W1 ~ was simply equal to the set N equal to 1 here it is just the invariant measure of C ~ itself. So $\mu(C\sim) = 1-\mu(C)$ okay. I would like to know apply this problem apply this general formula to a specific problem. And the problem we look at is the tent map a one dimensional map a symmetric tent map because I would like to play with this parameter a make it as small or large IP's.

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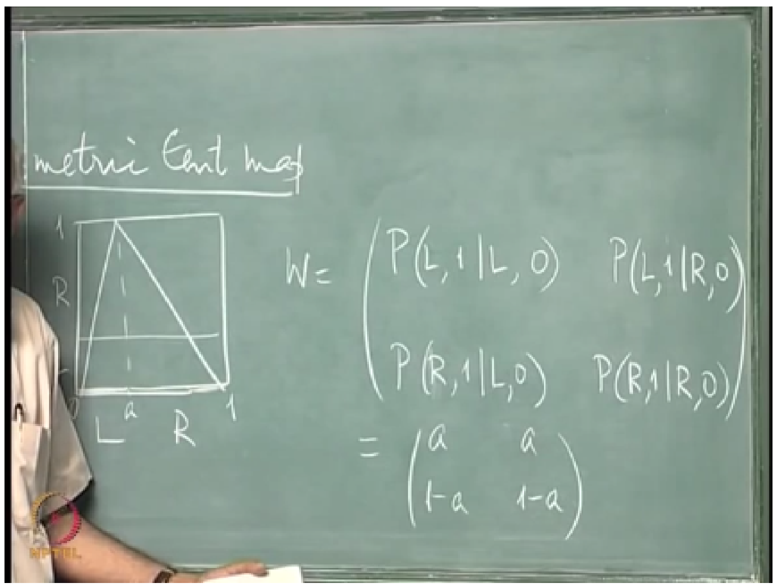

So the example we are going to look at is the asymmetric tent map for which we already discovered that a coarse-grained dynamics into two cells this is a, this is 0, this is 1 and 1 there, the coarse graining into two cells namely a left cell from 0 to a and a right cell from a to 1 this led to a transition matrix W which was the probability that I start and the left cell so this is L that is R and let us complete the diagram by writing an L here and an R there as well.

So I start here and then I moved to I remain in L at 1, the other probability was (L, 1, R, 0) P(R, 1, L, 0) and P(R, 1, R, 0) this is what I call the transition matrix W and this was easily worked out to be equal to a and a here a 1 – A and a 1-a okay. And the claim was that this is a Markov partition; in the sense that this probability that I mean either sell L or R let me just denote these by I, J and so on.

So I am in the cell J at time n given that I start at cell I at 0 so let us put that 0 explicitly here this quantity here if it is a Markov partition then you are guaranteed that this is equal to the JI^{th}

element of the n^{th} power of the one-step transition probability. So this is a one-step in one time step these are the probabilities of jumps, and if I take this matrix raise it to its nth power and then compute the matrix elements corresponding matrix elements that should give you the N step transition probability.

Of course you can independently compute this conditional probability by just writing this out as a joint probability, and then dividing by the absolute probability of starting at cell in cell I at time 0. And that will involve a Δ function which involves the nth iterate of this map asymmetric tend map. So it is a non-trivial problem to compute this nth iterate and then find out what this probability is.

But I leave it to you to verify and this is not difficult to do here, that this probability is indeed given by nothing but the corresponding matrix element of the n^{th} power of this matrix which proves that it is a Markov partition, and this is a consequence of the Markov property. Once you have this we can now start computing what these probabilities are that is very straightforward here.

And what happens in this case and we can in fact compute what this number is, because I just have two cells. Now so C is let us say L and C ~ is R just a part Markov partition into two cells, and we can very easily compute what these probabilities are what is this so joint probability for instance what is this, what is the most interesting property of W as I have written there. What are the Eigen values of W.

Yeah the some of the as you can see the sum of each column is equal to unity what does that tell you, yeah that there is a uniform left eigenvector which is 11 and the eigenvalue is one what is the other Eigen value of this matrix. What is the determinant of this matrix it is zero so what is the other what is the other Eigen value 0 has to be the other Eigen value there is just two Eigen values here.

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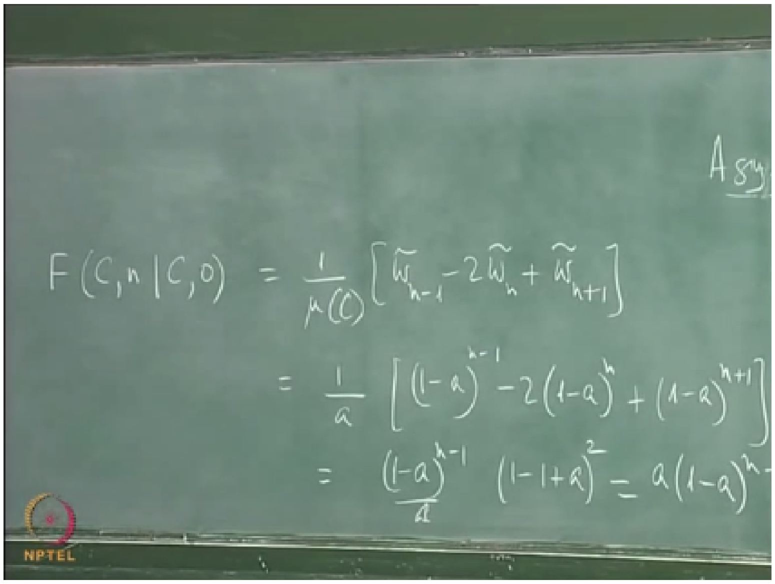

Therefore, it is immediately clear that for this matrix at any rate we have an extremely simple property which is W^2 equal to what is W squared equal to square this matrix 1 and 0 of the eigenvalues. So the characteristic function right 1 and 0 are the eigenvalues of this matrix. So it is clear is λ is an eigenvalue this must be true it is 2 by 2 matrix. So this is the characteristic polynomial of this matrix.

So it immediately implies that W^2-W0 or this must be equal to that yeah therefore, it implies at all powers of this matrix, this matrix is idempotent all its powers are itself. Independent of A whatever be a the matrix on raising it to any power gives the same matrix for all n, because that at once tells you that the probability of being in L at time n starting at L at time 0 is just the nth power it is just the 11 element of this raised to the n which is the same thing as that so this is equal to A etcetera.

So it is extremely simple in this case and we can compute these numbers what is W and ~ then, you can write this in terms of these probabilities and work it out etcetera. So but you can see this almost intuitively 0 to a is L in this case are this is our cell C let us say. So this is the C and this is C ~ that is for illustration take I see I take C ~ to the left cell and C ~ to be the right and then what is Wn ~ are going to work out to be in this problem.

This corresponds mine due to the complement so you have to be in our and it is a stay probability in R it is 1-a to a certain power so this thing is equal to $(1-a)^n$. So you are going to start at zero and up to n -1 you are going to stay there it is just this. So what is the second difference, this thing in this problem first of all $1/\mu(C)$ that is just a, because the uniform the invariant measure in invariant density is uniform.

So the measure of the cell on the left is just a, the length of that interval multiplied by $1 - a$ to the power $n - 1 - 2(1 - a^n) + (1-a^{n+1})$ that gives you there is something I have left out, that is fine. So this is equal to $(1 - a)^{n-1}/a$ and multiplied by $(1 - a)^2$. So it is an extremely simple formula and it tells you something interesting, it says the probability of return to this cell C at time n for the first time starting at cell C at 0, the first the recurrence probability is given by this geometric factor it is a times $(1-a)^{n-1}$.

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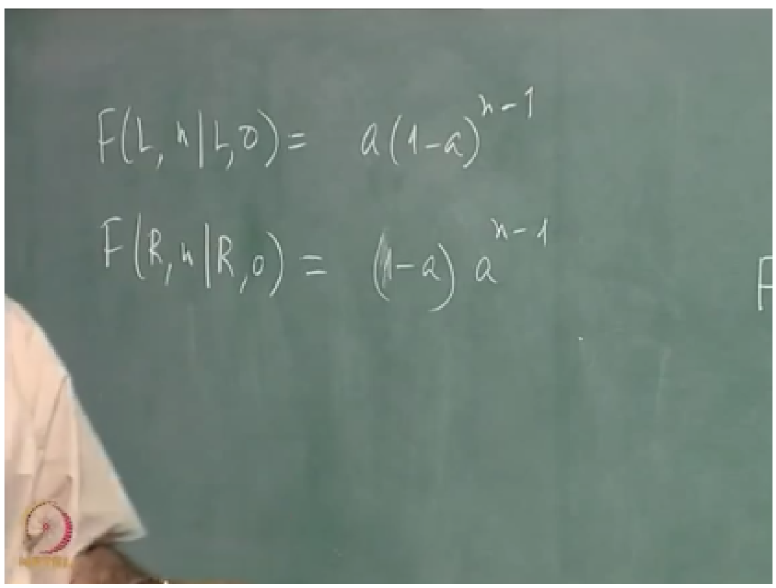

Similarly we can ask for recurrence so the right so let us go back let us write this F(L, n, L, 0) = a times $(1 - a)$ to the $n - 1$. And similarly F(R, n, R, 0), but this is not hard to guess what this is going to be yeah a n-1 is just interchange between left and right. So this is equal to $(1-a)a^{n-1}$. And now let us check normalization because the system is ergodic we have to be sure that there is a recurrence that is going to happen.

So if I sum this over n I better get one and indeed I do as you can see if I sum from 1 to ∞ yeah, the invariant density is one yes, oh no, no, no, this is not a question of no that is not true. The chance of it coming back the probability that said L given this does not diminish this is still equal to a, it is still equal to a. And what is this correspond to this is equal to the probability that it will start in L at zero you come back to L for the first time at time n for the first time.

And of course you will keep coming back over and over again. So this probability summed over n must give you unity, if the return is sure event we know it is a proper random variable and indeed so, because if I sum this from $n=\infty$ this is just a geometric series. So it is a/1-1a which is unity.

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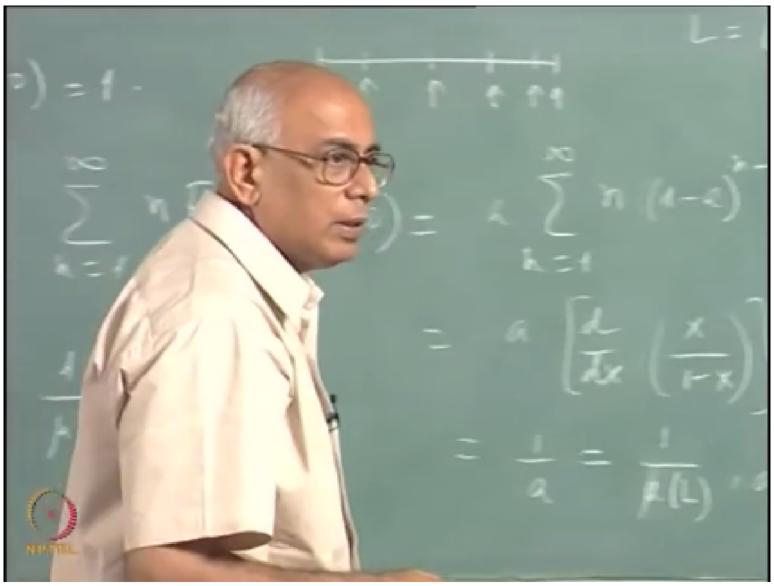

So it is certainly normalized, this is certainly true summation $n = 1$ to ∞ $F(L, n, L, 0) = 1$ as required. What is the mean time of return what is the mean time that it takes to get back n L say to the left this is equal to a summation $n = 1$ to ∞ n times $F(L, n, L, 0)$ now what does this work out to. So it is a summation $n = 1$ to ∞ n times $(1 - a)^{n-1}$ so what is that equal to it is just an arithmetic or geometric series so it is nx to the n-1.

So it is just the derivative of x^n right, so it is a times d/dx if I have x^n here let me differentiate it from 1 to ∞ the sum is x/1-x and I have to set x=1-a. So what does this give you if I differentiate this I get 1-x, so it is $1\text{-}x+x/(1\text{-}x)^2$, so that is $1/(1\text{-}x)^2$ so that is $1/a^2= 1/a$ right. This is $1/(1\text{-}x)^2$ and put x=1/a. So I get $1/a^2$ and I multiply by 8 that is 1/a. And indeed that is exactly what it should be because we know that this must be equal to the measure 1 over the measure as required.

The smaller a is it is intuitively clear that the longer the mean time of return for the first time is going to be. So that was the reason for my choosing a here rather than ½, so that I can tune this and you can see immediately that it takes longer and longer if they become smaller and smaller. The mean time I have written to a specific point which has set up which is the set of measure 0 will diverge and that is exactly what is happening.

And similarly nR=1/μR which is 1/1-a okay. But then once you have this distribution you can do more you can find the variance of the time of return or for a first offer occurrence or first return, and then all its statistics completely what sort of distribution is this, this is some kind of geometric distribution, I could also write this as a^{n-1} log1-a so since log1-a is a negative number, you can see that is exponentially decaying in time.

And this is very typical of these processes whenever you have a hyperbolic system, whenever you have no stable fixed points nothing I mean the map has got a non slope greater than unity everywhere this thing is said to be uniformly high it is supposed to be hyperbolic expanding everywhere at all points. Then it is a characteristic of such systems that the distribution of recurrence is going to decay exponentially in time.

And that is exactly what this does, because I could write this as a^{-n}log1/1-a, and it is an exponential decay. We can do more we can go further and do this, we can even find limit loss and let us see what I mean by that. You could ask alright if it comes back once will it keep coming back that is the first question, and if so what is the statistics of such successive returns. And the other question you could ask is I said this is a mark of partition.

And I left it to you to verify that the probability that you start in cell I at time 0 and you return to it and you are back there in cell I at time N or you are in some other cell J at time n it is just the ji^{th} matrix element of W to n the one-step probability to the power n. What is the guarantee that this is Markov one explicit verification, but do we have any general rules. Well in this case if the map is piecewise linear in one dimensional maps of this kind if the map is piecewise linear as this is.

And if your cells in phase space are such that the boundary points of these cells are either fixed points everything is unstable of course, or around periodic orbits if they are on periodic orbits or their pre images of periodic points, then the partition is guaranteed to be Markov that is true in this case, but me yeah where are the boundary points for this where are the end points for this partitioning 0 is 1, a is 1 and 1 is 1 right.

And the partition was simply L = 0 to a and R = 1 a so no, something like this. Now zero is already an unstable fixed point of this map and where does the point 1 the point a go, where does this point go, if I start at this point after one iteration you get to this and I am here and in the next iteration I am at 0 which is an unstable fixed point. So the point a is a pre image of the unstable fixed point at zero.

And the statement is if you take us an interval of this kind and you partition it not necessarily into two if you partition it and you call this cell c1, c2, c3, c4, and c5, then if these points these points which are the endpoints of the cells the boundaries between the cells, if they are either unstable periodic points, or the part of unstable periodic orbits, or they are pre images of such points.

Namely after a few iterations they fall into such points then you can prove that the partition is a Markov partition of course, you can refine this partition further I can make this smaller and smaller I could look at points which fall into a after one iteration and so on and so forth. And if I partition it further in this fashion it is called refining the Markov partition you guaranteed that the partitioning stay is Markov.

So it is useful piece of information to try, it is useful to try and partition a phase space into a Markov partition, because then you can use the machinery of Markov partition of Markov change in order to analyze the dynamics as we did here indirectly in some sense. Because once you have a Markov process you have another property which I did not mention a renewal property which is the following.

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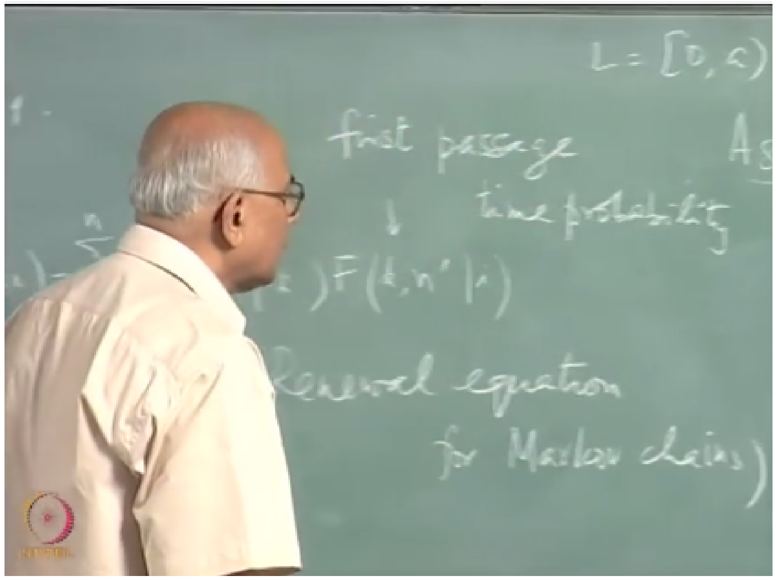

If I want to compute the probability of reaching some cell J some state J of a Markov chain starting at some state I at 0 let us assume it is a stationary Markov chain. And this quantity is equal to we can write a chain condition for it as follows so you start at cell Л at 0 move to a cell K for instance at time n' for the first time. And then start and cell K and move in the remaining time.

So if it is a stationary distribution, but stationary we only that K n – n' to state J P and this is equal to summation over n' = 1 to n. That is a renewal condition it says start here at zero move to some set some other state K for the first time at time n' and then you start at that state K and in the remaining time you end up at the state J. And then you sum over all the intermediate times here.

And this is the first passage first time you pass from I to J, so first passage time probability. So you should not have reach this K from I ever before until time n' and then in the remaining time you do this and you sum over all the n' and you get this probability. Now the theory of Markov processes gives you a formula for this, because you tell me the one-step transition probability I can find the N step transition probability by just raising that to some power n right.

Now how do I find these quantities those are precisely the quantities we want in recurrences the first passage them the first return probabilities. So I would like to have in our case what happens if I start at C, and come back to C here. So I would like to put I in K equal to each other. What does this suggest to you, is it possible for me to find this quantity this set of quantities given something for this set of quantities what does it suggest to you, think of n as time.

And then this is a function of T' and it is summed over T' and you have T – T' here. So what does it suggest, how would you solve this sort of equation, it is like an integral over T' right. So we have something which says let me just use P here you have P(t) it is going like ∫dt' up to T from zero because you can discrete steps if you make the steps small enough times P(t-t)', and then this is some F(t)'.

So what does this suggest how do you solve for F(t)' in an equation pardon me, it is convolution exactly you take Laplace transforms. So that is the natural way to solve this problem. Therefore, if you took Laplace transforms here in this case in discrete-time then you have a formula for this. And that is in this case it would lead exactly to the formula which we wrote down under general considerations here, the second difference term that we wrote down.

So you have such a renewal equation this is called a renewal equation it is an example of a renewal equation for a Markov chain. It is a very convenient way of finding first passage time distributions and that is exactly what we are interested in the first return time distribution. So this is one more way of doing it instead of going through the general formulas that we could have done it this way as well in this case because the partitioning was a Markov partition yeah.

Yes, that is exactly the statement I made there is a general way of proving that if you have a piecewise linear chaotic map, then if you Markov partition the interval if you partition the interval such that the boundary points are either on periodic unstable periodic orbits or the pre images of such points you are guaranteed that the partitioning is Markov, this can be established on general terms.

Once you have established that the partitioning is Markov, then you use this the machinery of Markov chains to solve the problem you do not have to go back to the dynamics the actual map itself. All I have to do is to compute the transition probability in the transition matrix the W for such a situation. And everything else is given in terms of W yeah, yes, not that you can solve it, but a great deal of information about the system is being is obtained.

Once you have did not manage to find a Markov partition, but there is no guarantee that for higher dimensional maps, other maps non piecewise linear maps etc it is trivial to find such a state a partition not true. So it goes one way is that if you have a Markov partition then you are in good shape, this is what it implies. But how to find such a partition is not known in general. So it is not that we chanced upon the Markov partition here.

It is just that I know that this map the tent map is symmetric or asymmetric it is a sort of paradigm of chaos in a certain sense it is like the Bernoulli shift, it is a paradigm of chaos. And for such systems then we can see that a great deal of information can be obtained. Now most of the rigorous results that are proved in ergodic theory have to do with systems which are formally something called axiom A systems.

Namely at every point in the phase space first of all there is an invariant measure, and at every point in phase space almost everywhere you have an expanding direction, you have something which is where you have I should say in more technical terms where the stable and unstable manifolds intersect transversally they are not tangential to each other. So these are technicalities mathematical technicalities.

But what we are trying to do is at a much lower level is simply look at some very, very simple low dimensional models, one-dimensional models specifically. And use a little bit of the machinery of Markov chains to show that the actual dynamics can be removed can be you know, you can cover it up, you can subsume it in the dynamics of a coarse grain Markov partition. So this is our modest aim here okay.

Yeah absolutely, absolutely yeah, yes let me rephrase what he is saying, he is saying if I refine this partition further and further which I can by taking this map taking the pre images of this point and writing it in terms of more cells, then in principle I can go to a limit where I can perhaps even mimic the actual dynamics itself yes, yes that is eventually that is of course the eventual aim in principle to do this.

But it is not true in practice it is not easily achievable in practice except in model systems of this kind, but if you can prove something is true in principle then at least you have some idea that the results you get by studying this kind of analysis by this kind of analysis are reliable in some sense. So this is at that level no more than that. In fact let us answer this question directly what happens if I have successive recurrences, what does it look like can I gather some general information from this.

And let us look at that and ask what is the probability we have seen the probability that you come back for the first time at time N, we will ask the following question now.

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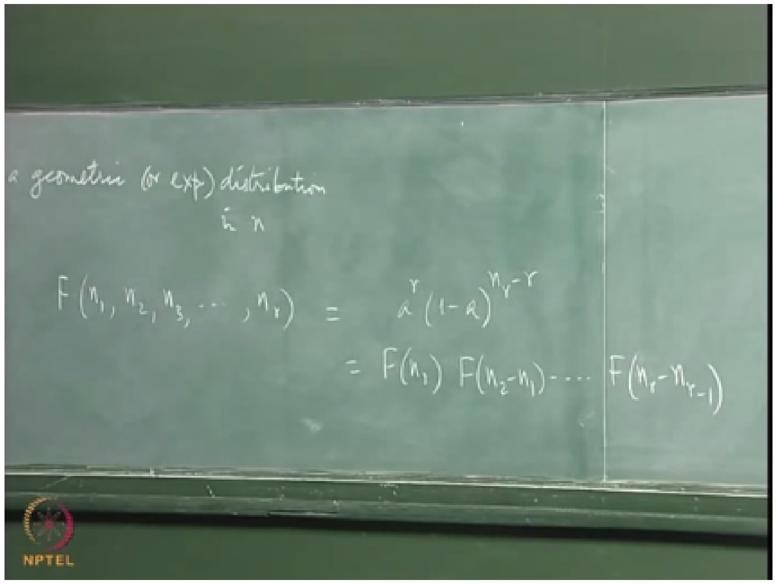

So we keep this in mind that this is an exponential distribution geometric or exponential okay. So let us ask the following question what is the probability that I returned if I started L at time 0 that I returned to L for the first time at some time n1, so let me call that N 1 just for ease of notation let me say this is the first this is the probability of return to L at time n1. And then after that the next return is at some time n2.

And the return after that the first time is at n3. So this is now asking for the probability of successive returns to L or references to L at these instance of time there are such recurrences. And I can ask what is the probability of this how do I compute this, well this is not hard to do because once we have this machinery that this thing is acting exactly like a stationary Markov chain, and it is not hard to show that this thing here is again or precisely the same point same kind.

It is a times $(1 – a)^{nr-r}$ multiplied by a^r I think. And it is not hard to see that this is equal to F(n1) times F(n) and call it F(n), F(n2-n1) … F(nr-nr-1). So simply factors in this fashion and what does that suggest to you, what does it suggest to you if I say that the probability of a recurrence in the sense we have defined at the instance n1, n2, n3 … nr the probability of R of these recurrences at these specific times is just the probability of a recurrence at time n1 multiplied by a probability of a recurrence in a time interval n2 - n1… up to here.

What does that suggest to you about these successive recurrences they are independent of each other they are independent statistically independent that is the reason these probabilities simply factor in this fashion. So that is telling you how random this system is in a sense that this has completely factored out. And it is just a function of those time intervals nothing more, so it is telling you that in these systems successive recurrences are statistically independent of each other.

Not going to be always true, but in this instance. And so exactly it is essentially due to the lack of memory in the Markov chain, in the Markov process, well in this case the Markov chain that is essentially what it is due to. You could ask what is the probability given a certain amount of time n what is the probability that I have R recurrences.
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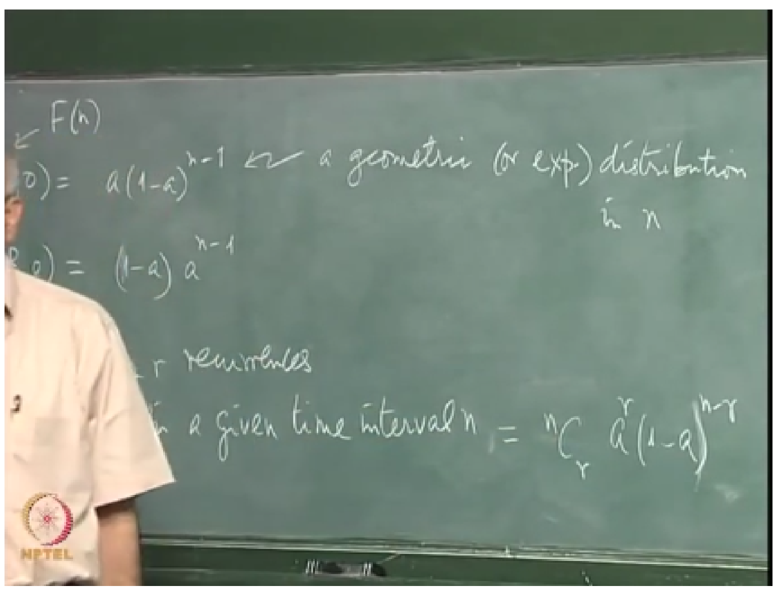

Probability of R recurrences in a given time interval what is that given, what is that equal to, what would that be. Well we already have a problem an expression there, so what would this be right. So you have given this, now you are asking what is the probability of R recurrences appearing. So what would that be, this is already suggesting what the answer is going to be. Absolutely it s $a^r(1-a)^{n-r}$.

But you do not care in what sequence this is going to happen right. So there is an nr, that is the binomial symbol where nCr. It is the binomial distribution and now we could ask now final question, we could say suppose the time becomes very long n tends to infinity and let us say a becomes smaller and smaller. What do you think happens to this distribution, it becomes a Poisson distribution.

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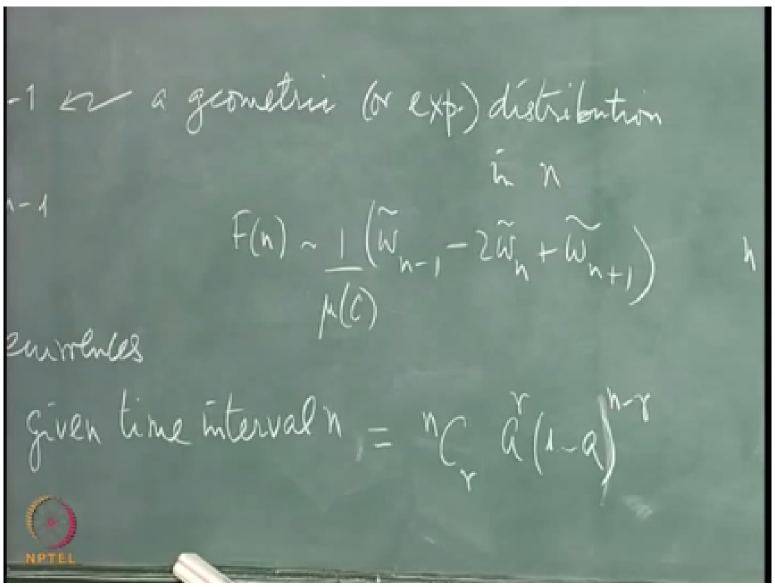

So in a limit in which n tends to ∞, a tends to 0, this distribution nCr $a^r(1\text{-}a)^{n-r}$, n becomes ∞, a goes to 0 such that the product of n times a is finite, this tends to the Poisson distribution r. So in fact let us make it such that limit na = λ say, so what does this go to then, in that case this goes to $\lambda^r/r!\ e^{-\lambda}$, the Poisson. You should not be surprised at this, because the first one the first recurrence as you can see is essentially a geometric distribution; it is the first term in that.

And then successive recurrences would be passed on distributed because they are independent. So this is the reason for saying that if you have an ergodic dynamical system in which there is chaos for example and you are on the attractor or on the invariant set with some invariant measure, then if you take a sufficiently small cell in this system and ask what is the distribution of recurrences to this cell that is passed on distributed this much.

And this is generic, this is typical behavior, it changes drastically once you have stickiness in the system like we looked at intermittency, then of course you can intuitively see this whole thing will go out of the window, because the system would tend to stay in a cell where there is a marginal fixed point, if it is just marginally unstable. And then the recurrence time distributions would be very different from this exponential behavior.

In fact it would start leaking like power loss various power laws here the limit distributions are not possible anymore they could become Gaussians, they could become other stable distributions and so on. And this is a characteristic way in which you detect the existence of such intermittency or such stickiness in the dynamics, because after all everything depends finally on the behavior if you look at what we wrote for F(n) this went like this sequence Wn ~ -2Wn~+Wn+1~.

And there was a $1/\mu(C)$ we are looking at the long time behavior of this in this simple case it was not e^{-n} times something or the other we are asking what this does. So you can easily see what would you say if Wn ~ went like $1/n^2$ for example, what would this go like for large n, it is like the second derivative right. So it should go like $1/n^4$ in that case. On the other hand if this meant like a 1/n, then this goes like a $1/n^3$ and so on.

So you immediately have the possibility of very slow decay, then of course you are not guaranteed that you have a Markov chain, or you have Markov partition or anything like that you have a general formula, and it could be as correlated as you like which is what happens in the case of intermittency or in the case of systems which do not which is not hyperbolic everywhere, which are the more typical cases which in practice would be the typical cases.

So the final comment one can make on this is that while we have these beautiful generic properties for classes of systems dynamical systems, in practice in real life in a given dynamical system you do not have this beautiful mathematical behavior, the system is not uniformly hyperbolic everywhere in general. And then you would have different kinds of recurrence statistics, you would have slow decays, you would have power laws and so on which will give you some indication of what kind of sticky points are there in the phase space.

So this is where I would like to end this little discussion of recurrence statistics per se. And of this kind of coarse-grained dynamics we look at some other examples subsequently. And now we move on to other topics we talked a little bit about Markov chains, and we defined some of the properties of Markov chains, in particular we looked at Markov processes in continuous-time itself more than Markov chains.

But then we revert it to Markov chains when we came to this course plane dynamics, because it wasn't discrete time here. But essentially the property the essential property you need to remember that is that it is one step memory just the preceding step nothing more than that.

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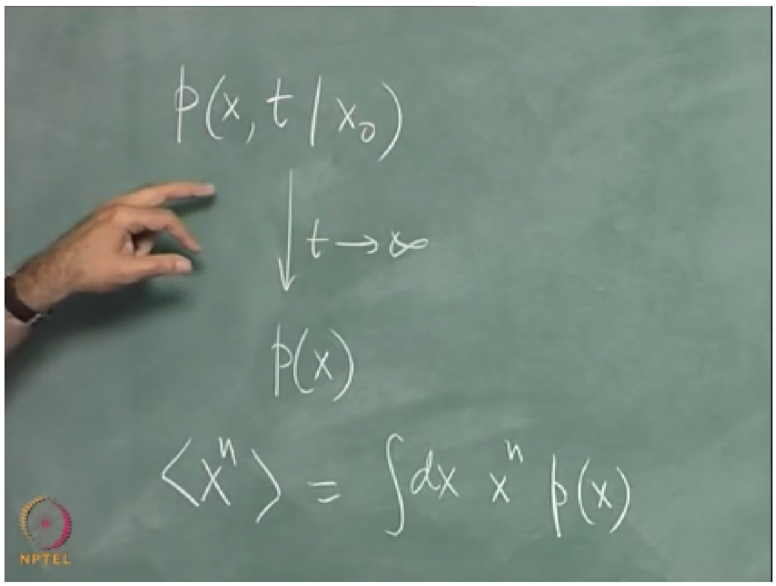

And then the statement I made was that the probability density that the variable is between x and x+dx say in a continuous case state space case starting at some x0 this conditional probability for a stationary Markov process determines everything, it determines the process completely. If you take the limit of this as T tends to ∞, then this is T tends to ∞ tends to a stationary probability distribution function P(x) which gives you the mean value, the mean square value all the moments which are all independent of time.

So in general for example the n^{th} moment of this would not depend on time this would be equal to dx x^n P(x), but all two time and higher time probabilities joint probabilities are all given by this function in terms of this function. Then the question is what do you write for this equation what kind of information or what kind of equation does this quantities satisfy. And we saw that this satisfies under suitable conditions.

It satisfies a chain condition to start with the Chapman-Kolmogorov equation, but in addition under suitable conditions we saw that this thing satisfied a master equation let us write this master equation now.

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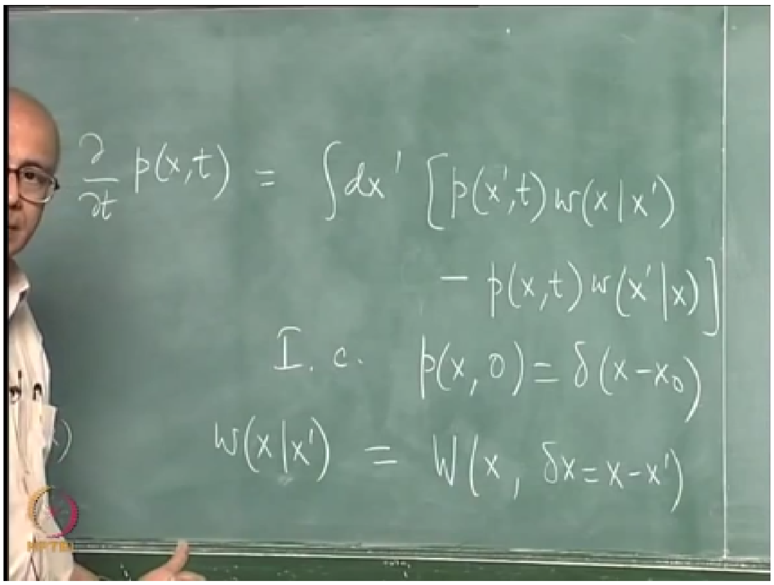

So I said that $\Delta/\Delta t$ P(x, t) let me for the moment drop this x0 will impose it as an initial condition an arbitrary initial condition. So this thing here satisfies an equation of the form dx' P(x', t) multiplied by a transition probability density per unit time which was x' – P(x, t). This was the master equation we wrote down. Now I did not mention ways of solving this master equation it is not trivial, because it is an integral differential equation as you can see.

And then the initial condition on this P(x0) could for instance be some Δ(x-x0) that takes care of this conditional probability okay. Now how do we solve this kind of equations not trivial you have to know what this kernel is like this transition density is like, but there is one case in which you can reduce this to a differential equation which is not generally easier to solve in some sense then different difference equation then an integral equation of this kind.

Well suppose you say W(x, x') which we call is the probability density per unit time, but if you start in the state x' you reach a value between x and x + dx per unit this is the transition probability density per unit time. This quantity here what could this be a function of in general it is a function of the starting state and the end state as well. But we could say well suppose this is a function of I write this as some function of the starting state x' and the jump x-x'.

So let us write that as Δx which is x-x', it is a function of this and a function of that in general. Well it is equivalent to saying it is also a function of x and x-x' entirely equivalent. And now let us assume that as a function of this jump here all the moments of this W exist.

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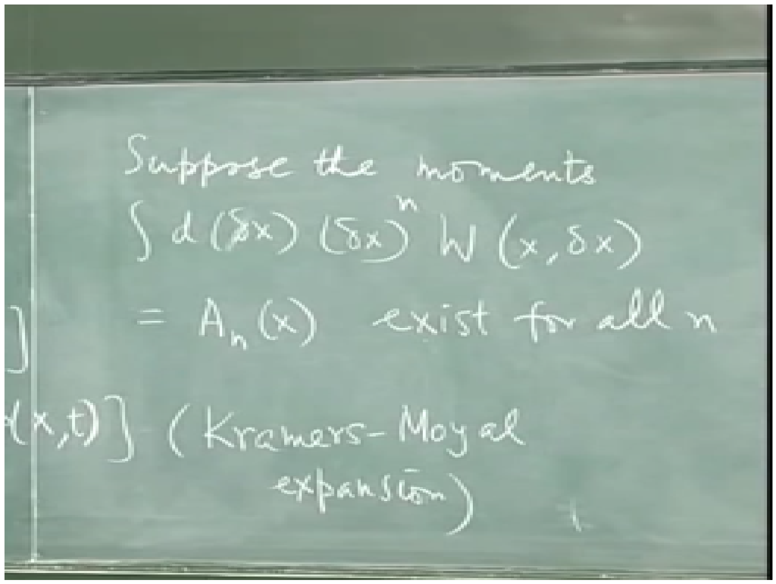

So let us assume that all these quantities $\int d(\Delta x)\,(\Delta x)^n\, W(x, \Delta x)$, suppose you assume that these moments exist let us give them some names and they would of course be functions of x. Suppose for all non-negative integer N. Then under those conditions you can reduce this equation here this equation here implies $\Delta/\Delta T$ P(x, t) becomes in a differential equation. So all we have to do is to substitute for that here in here, and then there is an integration which is doable.

And you end up with this is equal to the following a summation from n = 1 to ∞ unfortunately it is again an infinite sum here you have to pay that price $-\Delta/\Delta x^n\, A_n(x)\, P(x, t)$, you have to worry about convergence and so on I am going to slur over those points, but you can show that this is equivalent to this differential equation. But it is an infinite order differential equation, because you have their partial derivatives with respect to the state variable x of all orders.

This expansion from this master equation this is called the Kramers - Moyal expansion you will see in a minute that you are actually familiar with this expansion at least part of it. Now let us make a further assumption these moments what would you expect physically for any process in general grounds, you would say well this here is a function of a starting state for example and the jump.

And the higher and higher moments of this could be expected to get smaller and smaller. Therefore, you could say suppose these ends as n increases becomes smaller and smaller numerically, then I could perhaps truncate this equation at some stage I could approximate this equation at some stage maybe the first stage second stage and so on. It turns out that there is a systematic way to do this, and turns out that the most common truncation is at the second stage.

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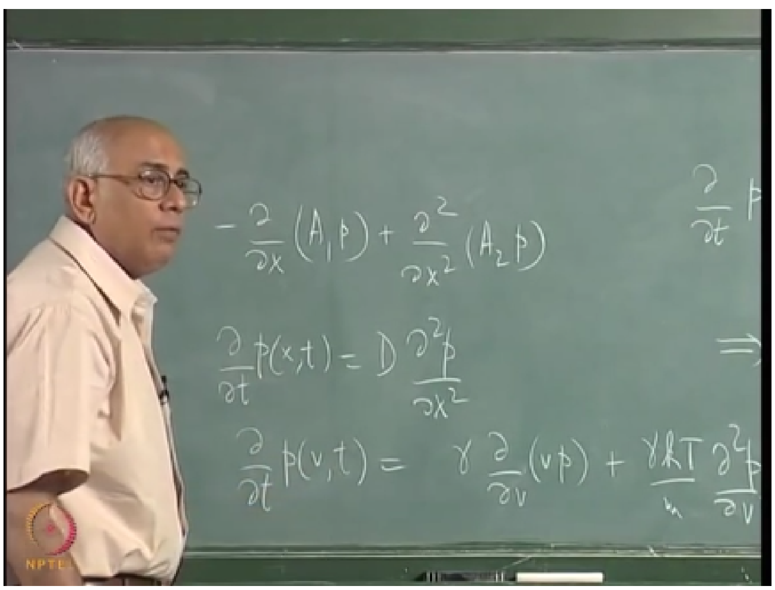

So you end up with an equation which reads $\Delta/\Delta t$ P = $-\Delta/\Delta x$ (A1, P) + (d2/dx2) (A2, P) okay. This is typical of what you get for such Markov processes and this is called the Fokker-Planck equation or the forward Kolmogorov equation there are several names for this equation. It is a first order in time and second order in the state variable different partial differential equation.

Now you are familiar with one example of such an equation what is that which is that equation by the way this is called the drift term, nd that is called the diffusion term.

And the ordinary diffusion equation that you write down for P is an example of such a case situation. That equation if you recall was this, and that corresponding to the case where you had no drift here in a free diffusion of some kind and this was a constant that came out, and then you had this equation. So they begin to suspect that Brownian motion which leads to this diffusion equation is a Markov process indeed it is it is called a wiener process.

And it is a special case of a more general situation. So processes where the master equation can be reduced from that level to this level they are called in the mathematics literature they are called diffusion processes as a generalization of this original diffusion equation. And you could ask where do these equal can I write an equation for x itself, yes you can write what is called a stochastic differential equation for x which is equivalent to this master equation for the corresponding probability density.

If I look at the motion of particles in this room at some constant temperature T, and I assume these are classical particles undergoing random collisions with each other nothing else, then it turns out you can write such an equation for the probability distribution of the velocity of the particle. And that equation looks like this $\Delta/\Delta T$ $P(v, t)$ for one component of the velocity any Cartesian component can be written as γ $\Delta/\Delta v$ (vp) + /n P/ $\Delta v2$.

I will explain what γ is in a minute, but this is Boltzmann's constant there is the mass of a particle and this the absolute temperature. And this is what this distribution is for the probability density that the velocity of a particle is between d and dv at a time T. You have to give me some initial condition, and the initial condition on this problem would be an arbitrary one.

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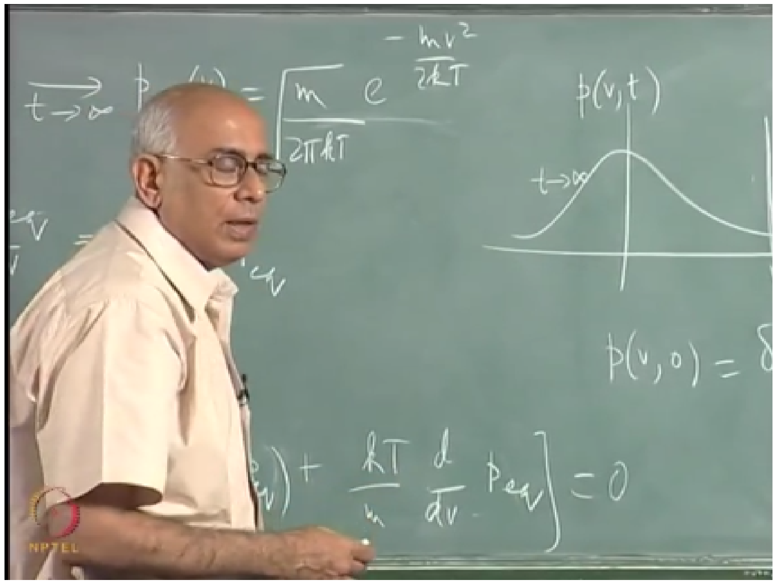

So you could start by saying that P(v, 0) = Δ(v-v0) for example and I can plot P as a P(v, t) which is the solution to this equation. It is a Markov process, it is a diffusion process, and it obeys a master equation of this kind here. Now what would you say is the distribution at T=0 well it is some Δ function at v0, and what is the distribution as time goes on this γ here is the friction constant in the system it is related to the viscosity of the medium.

What would you say happens as time goes on at T tending to ∞, what would you say would happen to this. Well again remember that the probability distribution is v, t starting with some v0 which is what we have written this with that initial condition what do you expect will happen to this as T tends to ∞. Well I have this gas in equilibrium at some absolute temperature T, and now I look at some particle and suppose it is velocity at T=0 is v0 and let time evolve.

And I ask what is its probability density point, but what would you expect as T tends to ∞, it would certainly lose memory of v0 what do you think will become pardon me, what do you think will happen to P(v, t) it should go to the Maxwell distribution, it should go to the Maxwell in equilibrium distribution once again right. And what would that be, P equilibrium of V, but what should this what is this distribution this is one Cartesian component of the velocity. So this is equal to e^{-mv^2}/2kt, e to the minus the energy over KT for a free particle the energy is $1/2mv^2$ and it should be normalized so there is an √m/2ЛktM factor which normalizes this distribution.

So I would expect that this distribution at T = 0 goes into this at T tends to ∞, so this is T=0 and it should drift back and go into this. Let us check if that is true that is easily done, because that distribution at T goes to ∞ is independent of time, and I can write this as a total derivative for the equilibrium distribution. So let us do that quickly I expect 0 equal to this now let us convert this to d/dv and make this d2/dv2.

And this is P equilibrium (vp) equilibrium now the γ cancels out we could take out d/dv and write it in this fashion. So this is 0, it implies this thing as a constant, what can this constant be well I know this P equilibrium must have finite moments. Therefore as V tends to ∞ it must vanish faster than any power of V or inverse power of V. So as T tends as V tends to ∞ plus or minus infinity I want this to go to 0, and I want this to go to 0, so the constant is 0.

Because it must be zero at $V = \infty$ and independent of V therefore the only constant you can have a 0. Well that is an trivial equation to solve it says dP equilibrium over dv. So it says dP equilibrium over dv = -mv/kt equilibrium, what is the solution to that this is the solution that is the solution. So indeed this is happening this is tailored to happen. So we have an example of such a Markov process namely the molecules in this room.

Their velocity represents a Markov process a diffusion process over on top of it. And it is in continuous time. So that is the simplest physical example direct physical example of a Markov process. Under suitable assumptions we have made a large number of assumptions here, this is a specific model of randomness we assume but it is a very satisfactory one to us.

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