Indian Institute of Technology Madras
Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 22
Coarse-grained dynamics in phase space (Part II)
Stochastic dynamics (Part I).

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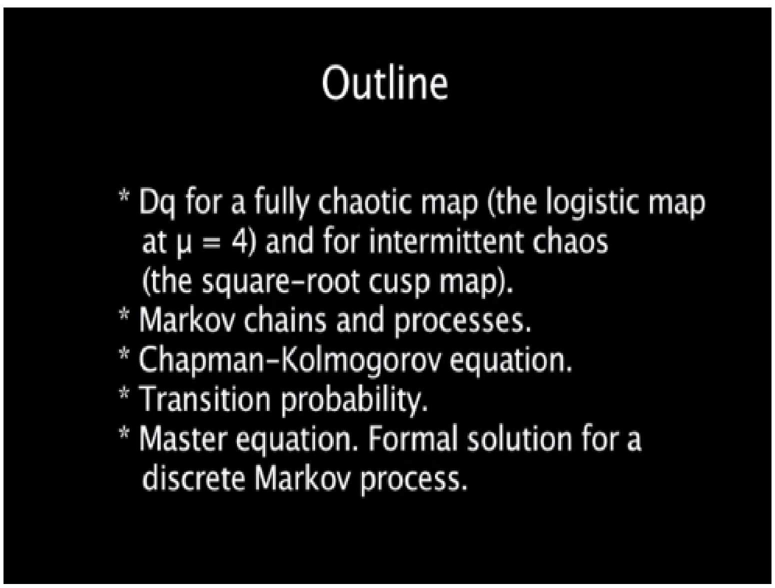

Yeah, so let us go back to where we stopped the last time which was the logistic map.

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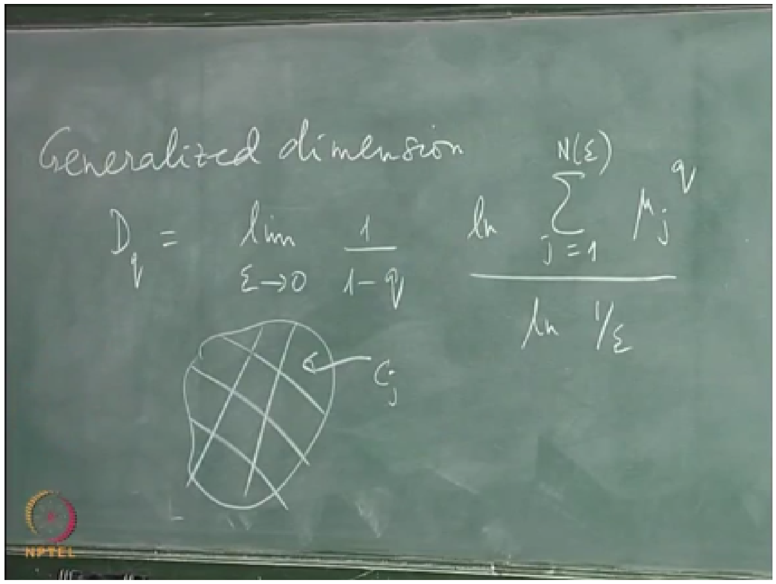

And we discussed the idea of a generalized dimension and to recall it to you, the generalized dimension dq was defined as the limit as ε goes to 0 of 1/1-q, the log of 1 to n/ ε μj^q/log1/ε. This in the case, if you took the full face phase whose measure is unity, normalized unity and to broke it up into cells of various kinds, and this is the Cj, the jth cell and μj is the invariant measure associated with this.

So you begin to see that I have in mind a system which displays chaos which is certainly argotic and which displays chaos, and in which a typical phase trajectory wanders round and round in the face phase or some portion of the face phase like an attractor. And I take that portion and break it up into cells Cj and I associate some letter of the alphabet with each of these cells and I do the symbolic dynamics of the trajectory in the sense that I keep track of at each iteration in which cell the trajectory is to be found and typical trajectories to be found.

And having done that when I compute this generalized dimension here which is some kind of weighted sum over this invariant measure of the J itself, raise to the power q. And this set of number Dq gives me some information on the way the attractor is on the nature of the attractor. In general, there is no reason why this Dq should be integer numbers, q need not to de an integer by itself and Dq some function of q.

And the general statement I made was that Dq is a non-increasing function of q. In other words, as q increases from - ∞ to + ∞ Dq decreases, maybe not monotonically, but certainly non-increasing.

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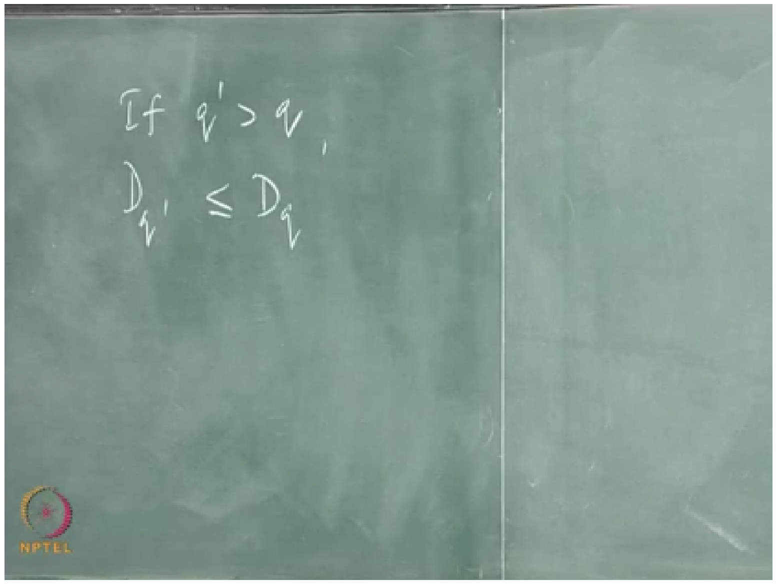

So the specific statement was that if q'>q then Dq'<=Dq so that was he statement I made about these generalized dimensions. The next thing we saw that in the case one dimensional maps in which the density was uniform, you could break into end cells of equal size 1/n, and then it was easy matter to show that Dq = 1, the topological dimension of the attractor. The fractal dimension is same as the topological dimension.

The entire interval was the attractor, but if the invariant density of the measure is not uniform then this is no longer true. And the question was what is it in the case of some of maps we have looked at or which we know what the invariant densities. So let me take that up now and show you what it is for the logistic map at fully chaos.

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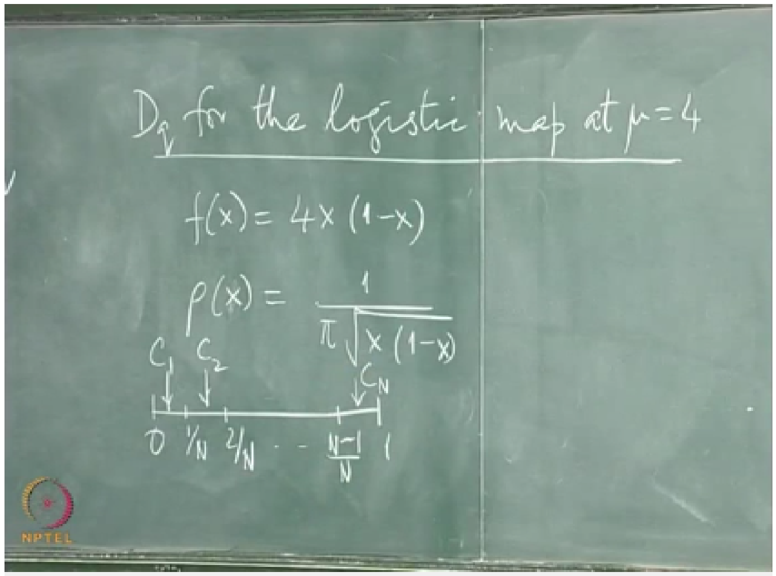

So, this is the next task to compute Dq for the logistic map at μ=4. So if you recall, the map function F(x) = 4x (1-x) and the invariant density in this case, the normalized invariant density was 1/Л√x(1-x) okay. Now, what do we do? Well, here is the interval 0 to 1, I break it into n cells of equal size, so this is 1/n, this is 2/n, and so on and this is cell C1, this is cell C2, and finally you have $n\text{-}1^n$ and this is cell Cn.

And I am going to compute this measure for each of them, take the limit and goes to ∞ and compute what Dq is. So that is our program.

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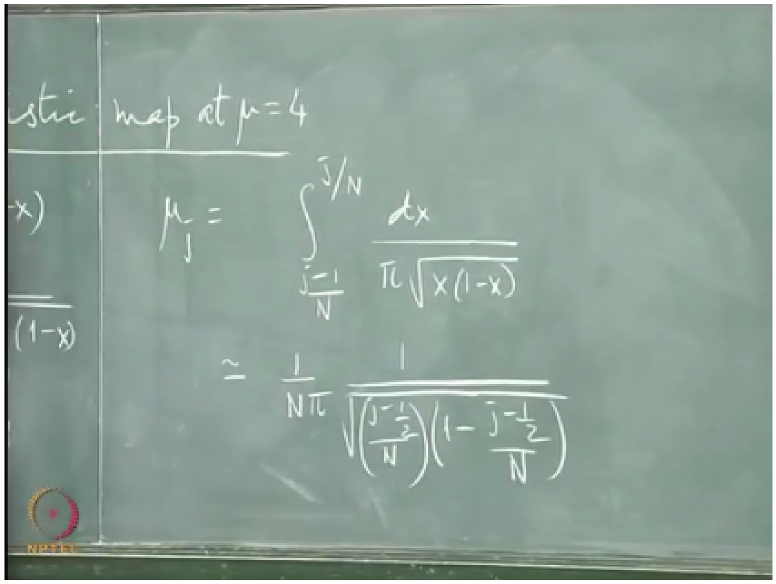

Now what is μj for the j^{th} cell is an integral over Cj but Cj runs from j-1/n to j/n dx ρ(x). And what is that equal to, well this is Л√1-x (1-x) and remember n is going to become very large and the size of each cell is going to become very, very small.

What has this become approximately equal to? It is approximately equal to in this cell in any of these cells, say the j^{th} cell here the valve of that integral as this interval becomes smaller and smaller is the value of the integral at the midpoint multiplied by the length of this interval. And the length of the interval is 1/n, and the value of the integral at the midpoint is 1/nЛ 1/√(j-1/2/n) (1-j-1/2/n), because the cell runs from j-1/n to j/n and the midpoint is J-1/2/n. So that is what μj is, and therefore, Dq can now be written down at least formally.

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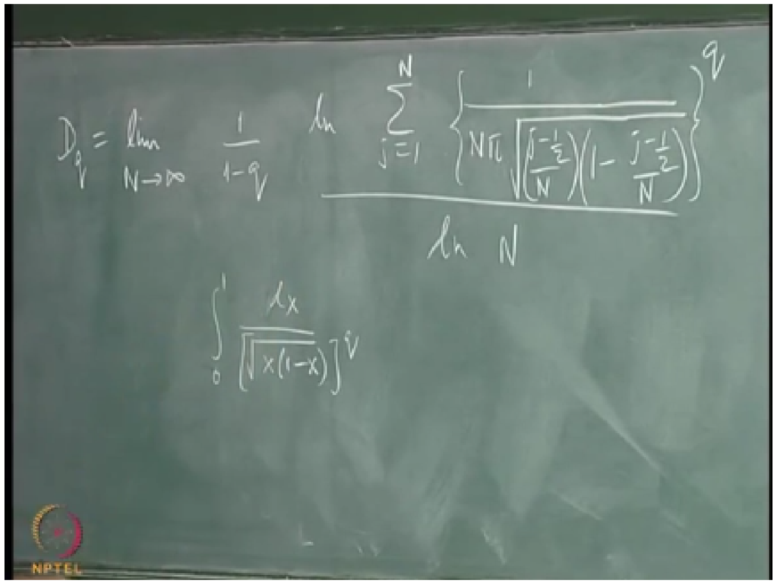

And we have Dq equal to limit n tending to ∞ because ε = 1/n 1/1-q log ∑j=1 to n that mess $1/\text{nЛ}\sqrt{(\text{j-1/2/n})}$ $(\text{1-j-1/2/n})^{q}$/logn, 1/ε is n. This is the quantity whose limit I want. Now this I suggest very strongly you have some from 1 to n, this suggest very strongly to actually have an integral here, you could convert this sum back to an integral once again. And what would this integral be.

Well, if I put, if I forget about this factor and then I put 1/n outside, this would be related to the following integral, it would be related to 1/Л apart from the factor Л x $(1\text{-}x)^{q}$ it would related to this integral, do you agree? Because it is a Riemann sum, and I can always write this Riemann sum back as an integral provided the integral converges. And when this is going to converge, when is this integral going to converge.

It is got singularities at the 0 and 1 out here, and it is clear that it is going to converge because there is square root here, q/2 should be less than 1. If it is greater than 1, this is diverges. When q/2 = 1, you have dx/x that is logarithmically divergent. So the integral converges for $q < 2$ and diverges for q greater than or equal to 2.

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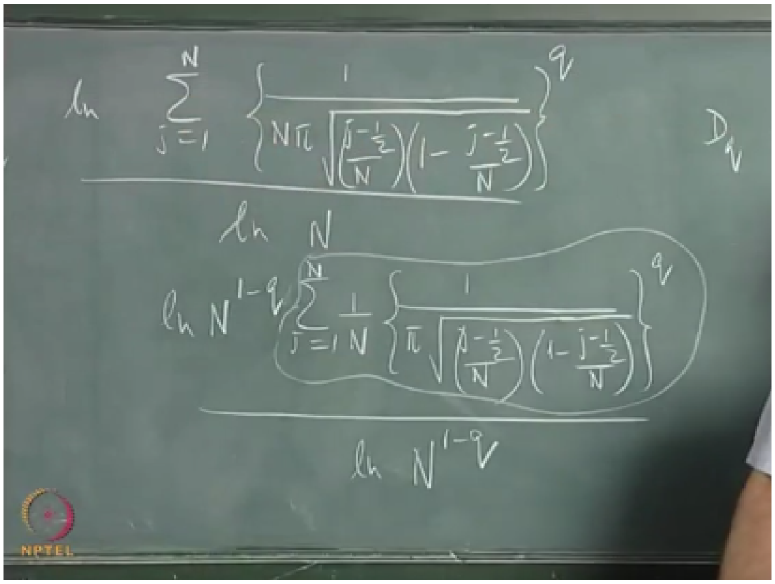

So it immediately means that you have to consider two cases separately, you have to consider case one q < 2 and case two q greater than or equal to 2 separately. For q < 2, so let us look at that first Dq is equal to limit and tends to ∞, 1/1-q log ∑j=1 to n, let get this n out of here and that is n to the –q. And then let me take it out the integral, so logn^{-q} comes out of the integral and then what I have here is 1/Л√(j-1/2/n) (1-j-1/2/n)^{q}.

And it is really saying take the function x (1-x) and take the midpoint of those, the value of the midpoints, and consider this sum, you know, if I divide that sum by one over end, this quantity in the limit n tends to ∞ is exactly equal to the integral from 0 to 1, the whole thing divided by, so I divided by n, so therefore, I better put it back so logn^{1-q}. So please remember, I divided by an n in order to convert this to an integral and I have to put that n back there, divided by, and get rid of this. And this is logn^{1-q} because 1-qlogn is logn^{1-q}, that becomes very, very simple to evaluate.

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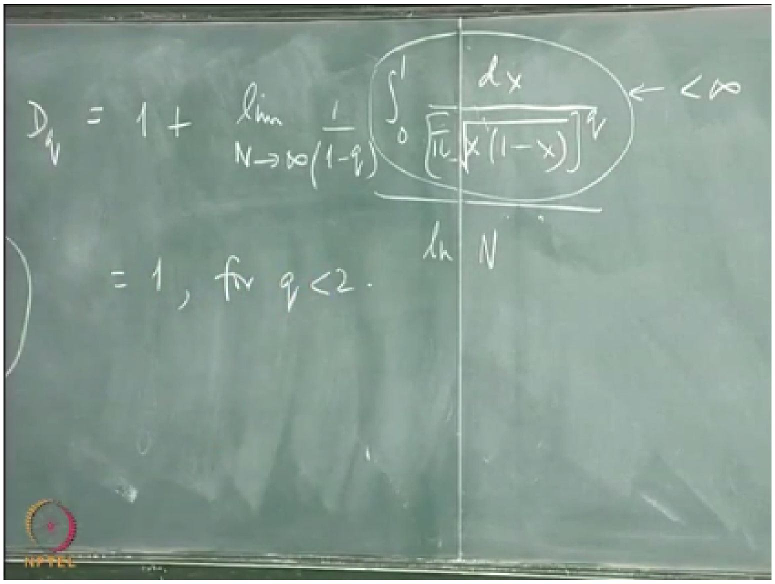

So this quantity is equal to Dq is equal to, well the first time is clear, so log of this times is that when I take this out and it gives just unity, that is very clear. So this 1+ the limit as n tends to ∞ of this sum divided by this, but you see this sum in the limit is simply the integral from the 0 to 1 dx/x (1-x) Л√x $(1\text{-x})^{q}$, that is all it is. That how we got this to start with divided by $\text{logn}^{1\text{-q}}$ and recall with q<2.

So what happens there, what happens to this thing here. So I put a 1/1-q outside and I have a logn. This integral is finite, and therefore when n turns to ∞, this logn in the denominator ensures that this terms goes to 0 completely, because this integral is finite, it converges. So it is immediately clear that this is equal to 1 for q < 2. D0 had better be equal to 1 because that is exactly what the fractal dimensionality of the attractor was and the attractor was unit interval, the full interval.

So D0 was any case equal to one and now discovered the Dq for all q < 2 = 1. If you had a uniform density like in the Bernoulli shift or in the 10^{th} map that fully developed chaos then Dq is equal to 1 for all q. But here for q<2 we can assert it is 1, but we have to reexamine the problem for q > 2. So now let us do the next step, that is go back ask what happen if q > 2. We have to go back to this.

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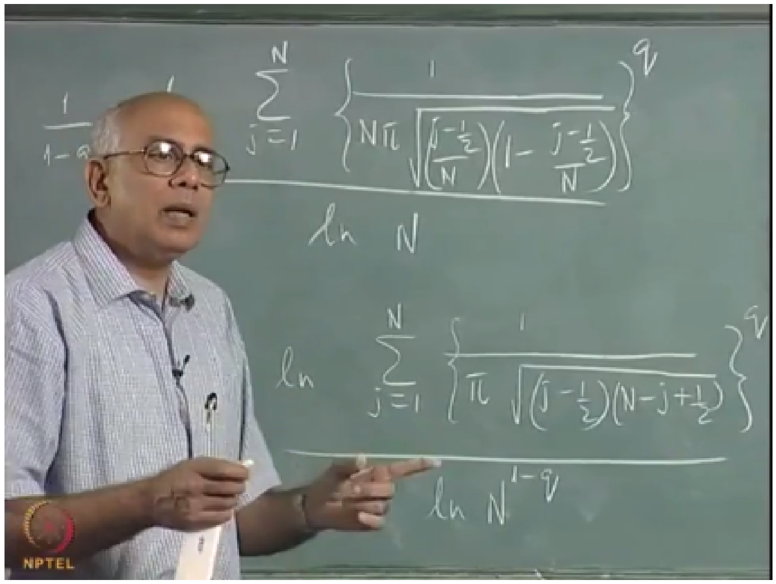

And then we end with, so this is the second part q greater than or equal to 2 when we cannot convert this summation into an integration, because the integral does not convert, but the sum is completely finite. So what do we do in this case, this case Dq is equal to limit n tends to ∞, and the denominator of course you are gonna get log n to the 1-q, so let us put that here. But in the numerator you have an n to the q which comes out there.

But then you cannot convert the rest of it into an integral, so it not much used doing that. On the other hand let us simply multiple through by n and this n goes away, because you get an n^2 with a square root that cancels this, and in the numerator that myself a little more space this thing becomes limit and tends to ∞ log ∑ j =1 to n, so let us multiple through by n, so we have a 1/ЛI√(j-1/2) (n-j+1/2)^q. And the n^2 cancels out with this n here divided by logn^{1-q} I have to deal with this.

Now what sort of sum is this? It is clear that for j, j near 1, now on this end, the lower end, these terms the denominator is going like $1/n^{q/2}$, so those terms goes to 0, the terms for j, we are going to have j n very, very large, so the terms for j are go to 0 like 1/n to the q/2, because this factor is going to dominate, and these are finite small numbers. On the other hand, at the other end of the sum, for j near n, this is going to be some small number and you are going to have an n here.

So once again the terms are going go like $1/n^{q/2}$, the terms in the middle near n/2 for example, both these factors are going to be of order n, because this is going to be like n/2 that is going to be like an n/2. And therefore, those terms are going to disappear like $1/n^q$ because there is square root of n^2. So the conclusion is, you have an infinite sum, a large number of terms for very large n, the terms are, the lower end are going to disappear like 1/n to the q/2, terms at the upper end are also going to disappear like 1/n to the q/2, and terms in the middle are going to disappear faster like 1/n to the q itself.

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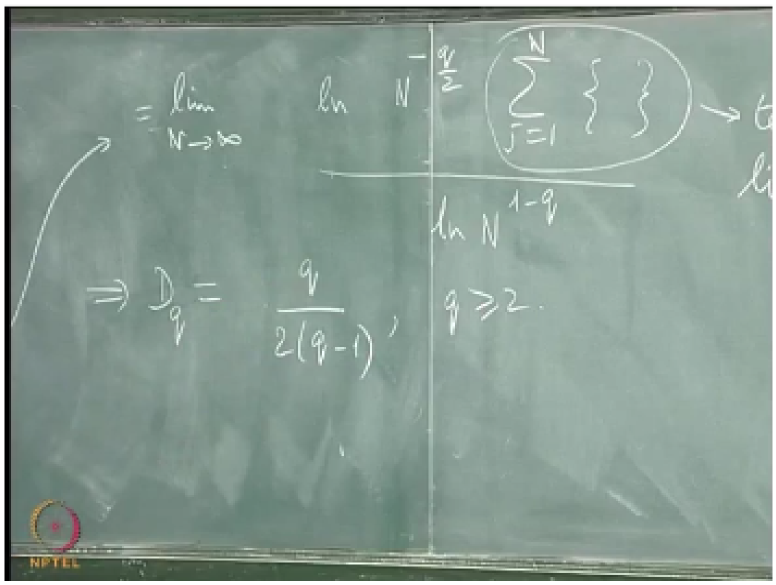

So this whole thing could be written in the following form, the whole thing can be written as equal to limit, n tends to ∞, one can make this much more regular, but this is as simple as a way of understanding it as any $\log n^{-q/2} \sum$ j=1 to n times something or the other which will give you in the limit the finite sum, because this is the rate at which every term vanish, it is on the ones in the middle are going to go 0 even faster.

But certainly there is lot of terms of this kind, contributions of this kind divided by $\log n^{1-q}$, where this sum tends to finite, not infinite, finite, non negative quantity, finite limit, as n tends to ∞. Having pulled out the dominant end, this is what the rest of it is and that immediately implies, that this implies D_q is =, since this case it is done in the denominator that cancels against this in this term, this finite so that limit will be 0 when I take the log of this sum.

It will go to some 0 because of the log n in the denominator and this becomes = q/2(q-1), because the - minus sign - Q/2 / 1-q that is all and this is the answer for Q ≥ 2. What happens at 2, this 2 cancels and you have Dq = 1. So there is continuity in this Dq and then it decreases beyond that.

So if you sketch this Dq, this is what it called and it looks like, the generalized dimension for this problem.

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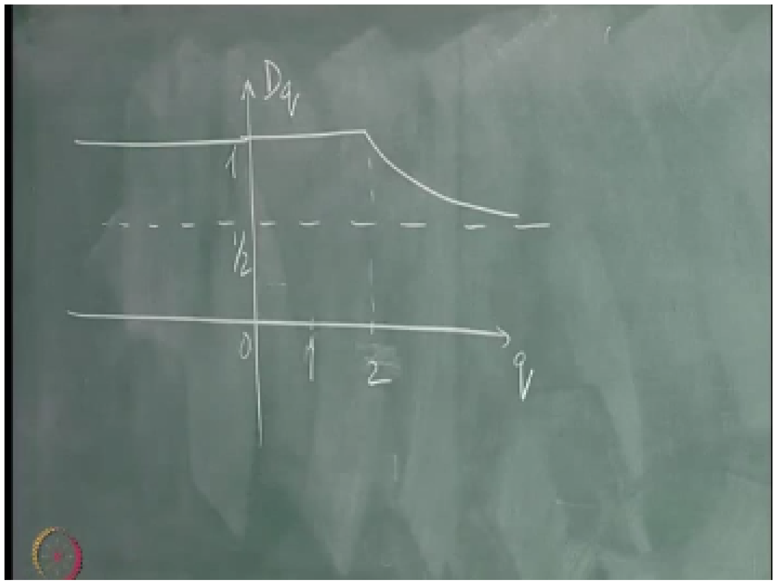

Here is Q and here is D_Q, it started with the value unity and this is the origin, here is 1 and here is 2 for example, kept there until this, this is one on this side, it is not the scale on the vertical access, is one there and then it drops at this point and as apolitically tends to ½ it is easy to see and this is what it looks like. So this is the spectrum of generalize dimension for the logistic map at fully developed chaos.

The set of numbers in the D_Q is something like them moments of these invariant measures, waited appropriately. So it is the same infinite set of numbers so it is actually a function in this case and gives you information about the distribution itself. Any for which DQ is the function of Q and not just a constant is just a fractal, it means there are many other dimensionalities buried in this system and in this tractor and this kind of thing is called multifactor and DQ is sometimes called multifactor set of exponents or whatever.

It is related to other exponents, it is related to other functions which also characterize this distribution, so the whole thing is an effort to actually find based on the score screen dynamics, what the nature of the attractor is. It is like finding that moments of distribution if you like and then reconstructing the distribution, the probability distribution itself so that is really what is being done.

Let us take another example where you have similar kinds of behavior and then we will try to draw a general lesson from it and this was the map in which you had intermittency. So if you recall we had the square root custom map.

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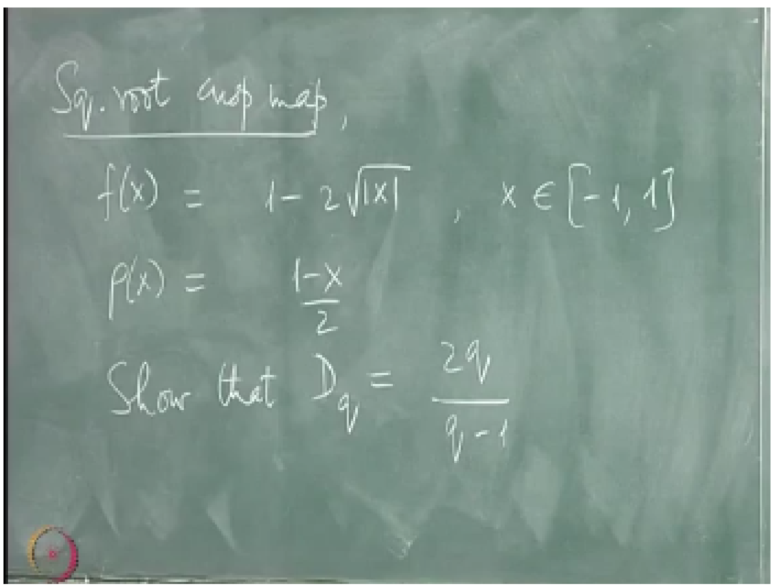

Which was f(x)= 1-2√x where X was an element of the interval - 1 to + 1 and the invariant density in this case because you had a marginally unstable fixed point that is - 1 the invariant measure density in this case was a linear function, it was 1 - x / 2, so it piled out near the left hand side and tapered off to 0 at x equal + 1 and you have to do exactly the same thing as before. Once again, you expect this sort of phase transition "at a point like this" at some value of Q where would that be.

Because ultimately what you are trying to do is to find an integral of $\rho(x)^q$ and to do that you have this - 1 to 1 and when would this diverge. As long as this is finite, you could convert the summation into integration and do the problem very easily. When would it diverge? It is clear if Q becomes negative, it start diverging here < - 1, it diverges because DX/1-X is logarithmically divergent and to any higher power in the denominator and it diverges.

So here you expect that at Q = -1 you have and after that and beyond that if remains a constant essentially and indeed that is true and I request you to go through exactly the same steps of before and show this is an exercise show that this problem Dq = 2q/q-1, what is this look like if I sketch this and what does it becomes and it becomes 1 and thereafter. Strictly speaking, I should have written this.

So here is -1 and beyond this point it remains 1 in this case and at Q is = - ∞ it hit the value 2 and therefore, 2 here and in fact it starts like this and goes on. So this is the way D_Q looks at this case. So it is immediately clear that if you have some kind of polynomial in ρ of x and rational function in ρ x then wherever you have the roots of the numerator and the roots of the denominator as in the other case you have points of this kind, whether slope of DQ changes abruptly in fact, it is not hard to show and I would like you to show this general result.

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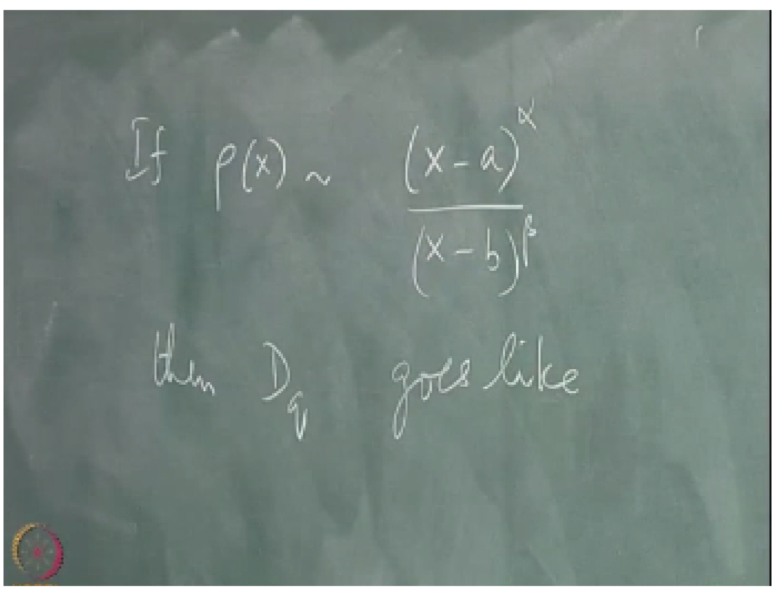

If ρ of x for example is proportional to apart from constant say $(x- a)^{\alpha}$/ x-b at the two ends for the example in the interval A to B suppose it have singularities at the end of the range in which the map is defined when DQ has changes of slope at points which depend on α and β on both sides and it looks like as follows. Goes like the following draft, let me draw it in pictures and then

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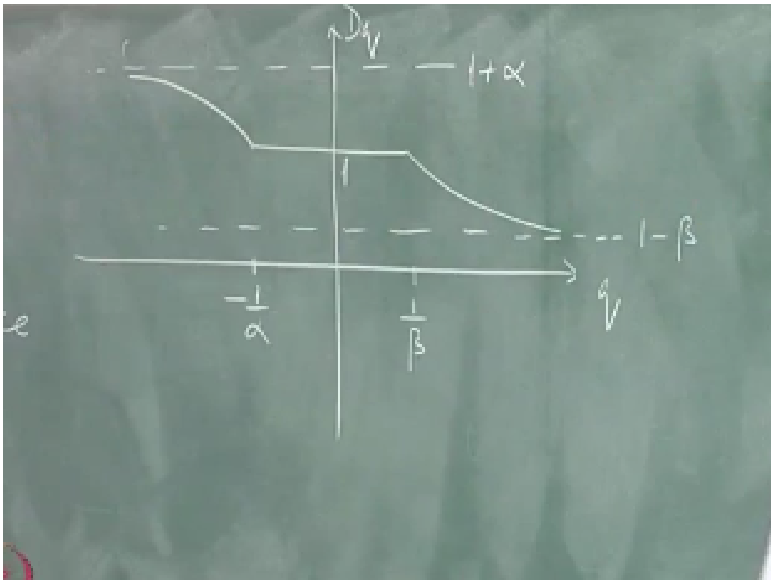

By the way α and β > 0, positive integers so you have this, and that is DQ and this is - 1 / α, this is 1 / β and DQ start with a value and at this point there is phase transitions, it remains like this and then it tapers off to a limiting value which is 1 - β, because β has got to be <1 otherwise it is not integral and out here, it is like this, this is a 1+ α, so D - ∞ tends to 1 + α, D + ∞ tends to 1 - β and in between it hits the value 1 and it looks like that.

If you have more products here in between more 0s and more places where it diverges, then it will show up right there, all those points were then general show, discontinuities in the slop of DQ? So this gives you some idea of what the generalized dimensions look like. We will come across some other example and more in high dimensional factors. But I urge you to complete these derivations.

It is very simple, all you have to do is, is to use exactly the technique I use for the logistic map, whenever you can convert things to an integral, you should do so and then the limit is easy to take. Otherwise, you must retain the sum and find out what the dominant behavior of the sum is. You can do this very regularly but what I did there was essentially right.

Now let us go back a little bit and ask, okay we have some kind of course graining but can we relate the dynamics of the score screen dynamical systems, can this relate this symbolic dynamics to some random process itself and it turns out the answer is in many cases you can. We saw what the idea of a generating partition was. I stated that if you divided up the face phase into cells and kept track of the symbolic dynamics of the trajectory moving between these cells.

Then if you can reconstruct the actual dynamics by looking at the sequences of letter then it is a generating partition, not all clear if every system has generating partitions, but there are other cases where the partition has other kinds of interesting properties. In other words, the jump of the system between these cells looks exactly like a random process called a mark of chain and let us sees what that is.

For that I need to introduce few concepts in probability, especially in mark of chains and not clear how much background I should assume here but let us take our usual attitude and spend few minutes as in the digression on mark of process or mark of chains.

So this is the digression, mark of chains or mark of process depending on other continuous times or in discreet time, let us does the following. So let me assume that I have a random variable X and it changes with time and time could be discreet or continuous, does not matter. But suppose that this variable generate a time series as I go along. Then how do you specific the probabilities or distributions of this variable.

Well, one of them would be to ask what's the probability and if it is a continuous variable then of course I must talk about the probability density otherwise I talk about probability itself and will use these two words interchangeably for the purposes of this little digression. I could ask, what is the probability that it got some value X at some time T and if the time is discreet then it is N.

Let me for the moment use continuous time and go back to discreet time little later. So, this will given the probability or the probability density that the variable has the X at the time T or it is in range from X to X + DX at time T. Let me call it P1 with some function nonnegative function, but this is not enough. I also need to know what the probability at this variable is has said a value x2 at t2 and x1 at time t1.

Let us stick to the standard conversion and write earlier times to the right and later times to the left. So, you need to know this joint probability as well, two time probability. You also need to know three time probability that is x3 t3, x2, t2, x1 t1 and so, each of these have general different functions and to specify the random variable completely, I need to know all the joint probabilities.

For arbitrary, so you need an infinite number of probability distributions, joint probability distributions to specify the random variable completely to find all possible co-relations in the random variable. However, by definition Pn(Xn, Tn Xn – 1, Tn -1..) by definition , this is = P Xn Tn given that at time, T1 - 1 the variable had this value Xn-2 , Tn - 2 up to the last one.

Multiplied by the probability that you had all the conditions here that you had Xn -1 Tn -1 and this of course is an n time arguments and therefore let me call this again n and let us call this n-1. This is a different object than this probability. This is a joint probability and this is conditional probability. So it simply says that if you have sequencing time.

T1 <T2 <T3 etc then the probability that you had X1 T1 is by definition = the probability that you had all these earlier events happening at the time is indicated multiplied by the conditional probability that if given these events have occurred, the next step the value is xn. So this almost intuitively obvious, well, therefore you can start reducing things because now you could reduce this to a product of an n-1P but the last one occurs given all the earlier once occur.

And therefore in some sense you can reduce this entire to a product of conditional probabilities till you hit this last one. That is what you can do in general. That is about all you can do in general. For a mark of process, the memory is short and it is specified by saying order mark of process means that this conditional probability is independent of all these earlier events and depends only on the immediately preceding event.

So it is like an one step memory that is it, so if I call this the present state, the future, the next step is determined by the present and not by the past history. So for a mark of process, this whole thing get’s whipped out and you have just this, multiplied of course. So the statement is that the conditional probability, so let me state that again precisely and then we come back and write out our joint probabilities.

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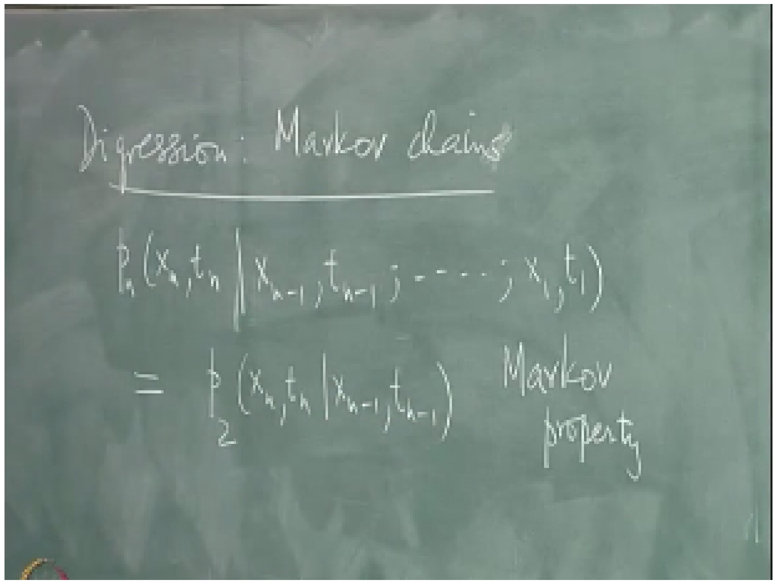

The statement is pxxntn given xn-1tn-1 occurred is identically equal to some function and want of better notation just me call it p once second p of xn,tn,xn-1,tn-1 there are only two time arguments here. So let me just call it P2, it is function with two times arguments and this is the mark of property. It is called first order Markov. If it depends on the preceding two steps then it is called as second order Markov and so on but we are going to restrict ourselves to first order Markov processes.

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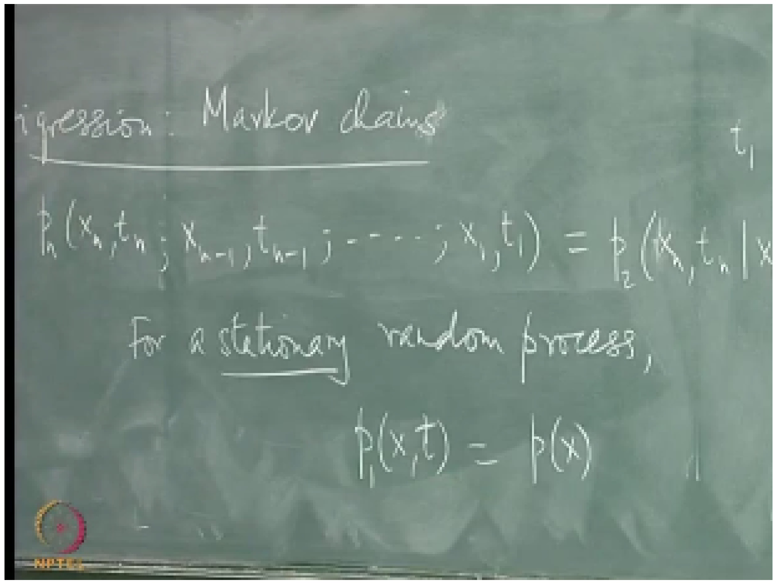

For continuous time, these are probability densities rather than probabilities. Variable here in two different time steps, so I do not see any problem there. All we need is this, so at any instance that is true. This is true. His question is, if you have discreet time then I do not have the t here this n-1 instance and this is the nth instance and then you can understand this but his question is suppose time flows continuously, then what are these things, how close to this time can that be

Arbitrarily close, still true. That is a good question. We will come back to this and see what it implies. It implies certain renewal of the process as we go along. So, you can ask, okay on the time access if something happened here and something happens there, then we are saying the conditional probability depends on what happened here at this point but something else could have happened in between. Yes, indeed that is true.

Therefore, this quantity is going to obey some kind of chain condition, which will propagate you from here to there and then from there to there. We are going to see what that is. But yes, this is a good point. Continuous time, this is actually true as long as tn-1>tn that is true but what is the implication of this. It is an enormous implication, because it immediately implies that you can take the general n time probability and write it as a factor of conditional probabilities of just two time arguments right away.

Because you can then write this quantity, this n4 to be equal to first the joint probability P2 that you had xn tn given xn-1tn-1 and the rest of conditional probability, this dependence is gone, multiplied by the n-1 times joint probability of this set of events but that could be written once again as the p2 of xn-1 tn-1 givenxn-2tn-2 and then n-2 times join probability but that can again be factored out an so on till you end with p2 of x2 t2 x1t1 multiplied by the single time probability of x^1 and t^1.

So far a mark of process or mark of chain, if these discrete, this factorization is possible Now what is that mean, I practical terms, it means that if you give me the function P2 and the function P1, the job is done. All joint distribution are known, in terms of just two probability distributions. The two-time conditional distribution and the one time probability itself so that is the remarkable simplification that occurs for a mark of process.

Now we are going to make a further assumption. We are going to assume that the statistical properties of this random variable do not change with that. In other words that the variable is a stationary random process But the actual statistic does not change as a function of time. And therefore you can applies that you can shift all the time arguments by the same constant and nothing is going to be change. What is that imply. It says that this function here is independent of time.

Because it is independent of the time argument. You can subtract out that you want completely. In other words, if you ask for what is the average value of x or the mean square value or any moment of x this should be independent of time by took the n moment or k th moment of this, this by definition is dx if it is a continues variable. Pxt x^n if it is a continues process the set of values of x is continues it is this integral otherwise it is a some issue. But the T dependent appears here and this is going to statistically all the moments are independent of time, the only way that can happen is for this distribution does not depend on time.

Therefore, for a stationary process you have a further simplification. For a stationary random process T of x, p, p1, x = PX no t dependence. And no confusion should arise I am going to drop this one. Just call it px. It is a probability if this x takes on a discrete other value, it's a probability density x to x1 continues one. It is the stationary description. And what happens to p2 of xp, x1 say x0 p0. So the probability given that you had the value x is not at time p not the probability to have x and time t is independent of t note, subtract out the t note from both sides.

Therefore, this is equal to p2 of x, p – p0, x0 and 0. You could call that the origin of time which was simplicity of notation let me simply write this is again p use of notation px p-p0 x1. Another words I would not even bother to write down the fact that this is a zero shifted. I use the same symbol p here. I should not really I should call this p21 call that p1. But you know what is happening here. So where is the time argument here. There is no time here at all in the single time probability and there is a single time argument here.

What therefore happens to this business here this entire density well its clear this becomes tn-tn-1, xn-1 and you can erase this in little bit use of notation write this, because it is really depended only on this time difference all the way up to here p x2, t2-t1, x1, p x1. So the matter simply enormously. In other words, If you give me two functions give a px and you give me of xt x0 the job is done.

There is one time depended probability or density or one time independent. You have one more property which you expert that is the following you would expect you have establish this but you would expect, they might took this conditional density or probability which says given that this happen, given this is the initial condition, this is a value of time t given that this happen t=0. What happens to this a t10 ∞What would you expect would happen to this probability as t10 to ∞

I would expect if things are reasonable and the system is sufficiently random I would expect that this probability would become independent of x0. It should forget the initial condition altogether and I would expect therefore that you have limit t10 to ∞ pxt x0 = px itself should become the stationary distribution independent of the starting point.

Let us a consistency condition report on the system. If the dynamic as enough sufficient mixing it turns out, this is always going to be the case. So really you have just one function which you have to determine this. Because once I do that I take its limit as t10 to ∞ and I know this. And therefore this entire for arbitrary n joint probabilities are all determined by a single function of a single time argument which is this and what is the initial condition on this. If it is a density if this x is a continues variable then it is immediately clear.

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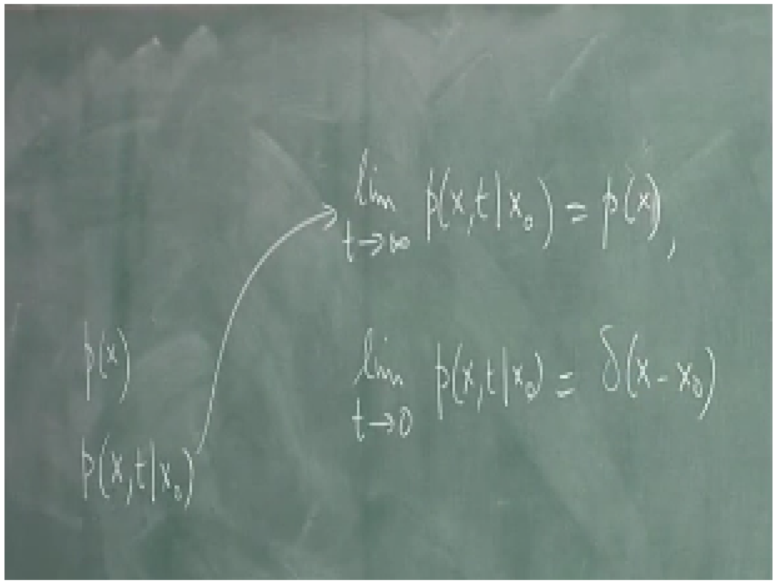

But in addition to this in here therefore have limit t times to ∞ pxt x0 = tx0, px and you also have limit t10 to 0 P of xt, x0 = one. It should have support only had x0, not p of x0. This is the conditional density in x. It is one x=x0 in some sense, but if it is a continues variable and it is normalizable then what would it be. What is the continuous analog of density which is measure 1 Δ 1. That should be Δ of x-x0. That is the only thing that contribute in everything else is zero.

So you have an initial condition, you have a somatic limit and the entire processes determine by this single function. And this quantity for which one generally write the equations down, revolution equations are master equations are whatever you called it, it is written down for this, because after all in any physical problem its only conditional density that is you can model. You cannot model absolute probability you can only model conditional probabilities. Given the certain events are happening you can now talk about the future.

But you cannot talk about any absolute probability. There is no way of writing equations for these things. So the equations write for Markova of process for this kind are called master equations. Several of them but depending some other multiply equations, some other for differential equations and so on so. But there is the very elaborate theory of mark of processes. Which tells you what this things happen, how these things evolve.

Now let us go to the specifics and let us see how to write this down. So let me work in the context of continues processes and after that I will talk about jump processes. We need to do both. Of a several ways of doing this, but let me do this in terms of something which I will motivate and so on and then two to sort of way.

So I would like to write an equation for p of x, t and I drop the x 0 for the moment, because that is incorporated initial condition. Saying the p of x, 0 is going to be some Δ of x-x0 wherever I start. Then I ask if x is continuous from the whole set of values then I asked what is Δ p is forward Δ t equal to on this sight. The two ways in which I can have contributions to the increase of the probability maths at the value x one of them would be if the system let me motivate this go back a little bit and write.

So it go back P of x, t x0 at 0 could be written in the following way, because of this mark of property you could say that you start at x0 at 0 and you evolve some value x ′ at some time t ′. So on the time access you have zero here, you have key prime here and you have t here. T ′ this < p but >0. You propagate from here to there and then you propagate from x ′ to x in the remaining time. And this is true for all t prime between zero and t, and does not integration over the intermediate steps.

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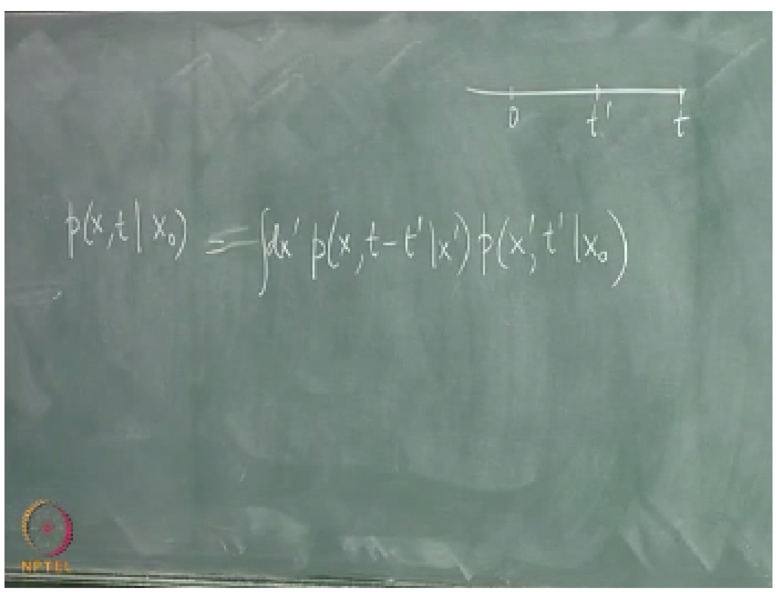

This is called a long name it is sometimes called the chapmen- Kolmogorov equations of the chain condition, because it is like a chain which properly go from step one step two, step next three and so on.

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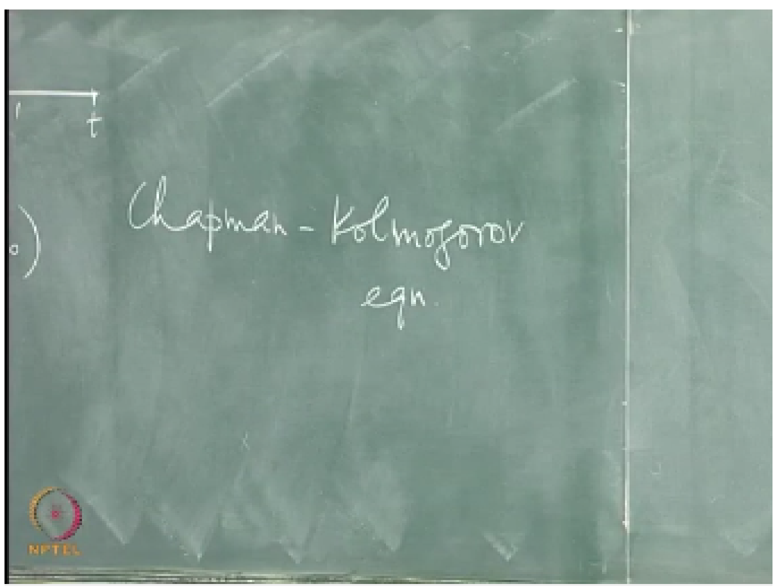

It should really be called the bachelors molecules chapmen kolmogorov equation because several people wrote it done the same time around the same time, but you would going to the history of this but this equations helps us to write something down for this. I should immediately hasten to add that one should be under the misconception that this equations this unique to mark of process.

Other processes which are not mark of also base such a change condition, because it is some sense of renewal process, all renewal process could obey similar equations, but not process certainly do. Now what one does is the following one takes this t-t' to be infinite some Δ t, for example very close to t and then writes instead of this a transaction probability for unit time and that changes this chain equations from this kind of integral equations which is not linear, it is an integral equations and it is not linear in this p to a linear equations for the rate change of p.

And that equations looks like this. So from this it follows in the suitable conditions δ / δ t = p(xt $x_{0)}$ = on this side an integral over intermediate states the explain the probability that you are x' at time t having started x_0 then you make a transition from x' to x per unit time if the transition probability is w x given x' in other word it trans for the transition probability per unit time of making a transition from the value x' to x per unit time, because we writing a rate equal.

But you also have a lost term it says you might will have reached x itself at time t, you could jump out of x into any other state. So there is a w of x' x this looks exactly like the rate equations you write down chemical reactions or any other systems which is govern by rate equations, this is like a game term. You go from x_0 to an intermediate point x prime and then you jump from x prime to x. This is a rate of jumps. Or you go from x0 to x and you jump out of x to some other value, how you write the same thing if you had the mark of chain in describe time again.

So suppose the values that this variable could take being continues set of values was descript set of values, x_1, x_2, x_3 etc, x_j let us suppose this whole thing to place in time. Then how would you write this set of equations. What would you write? What sort of equations would you write here? If it is continues, if the process is descript but the prime is continues then it is very straight forward. You still have rate equations of this kind here. But you would have a submission period.

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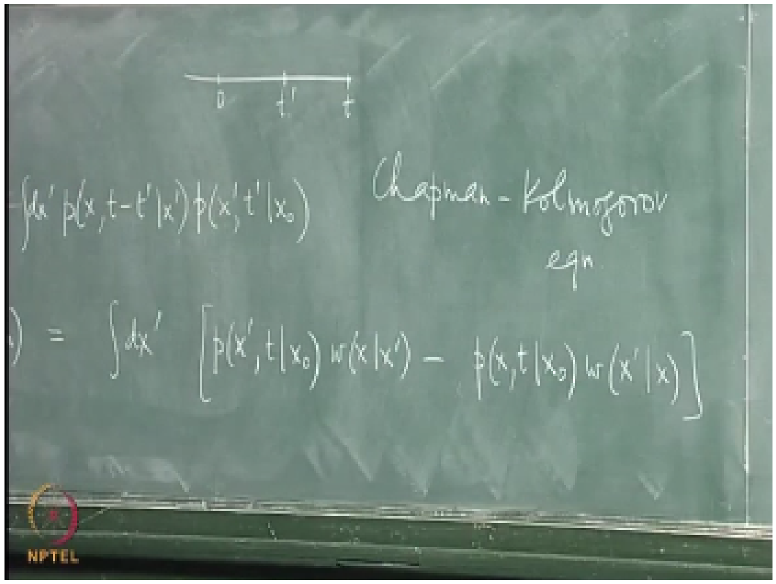

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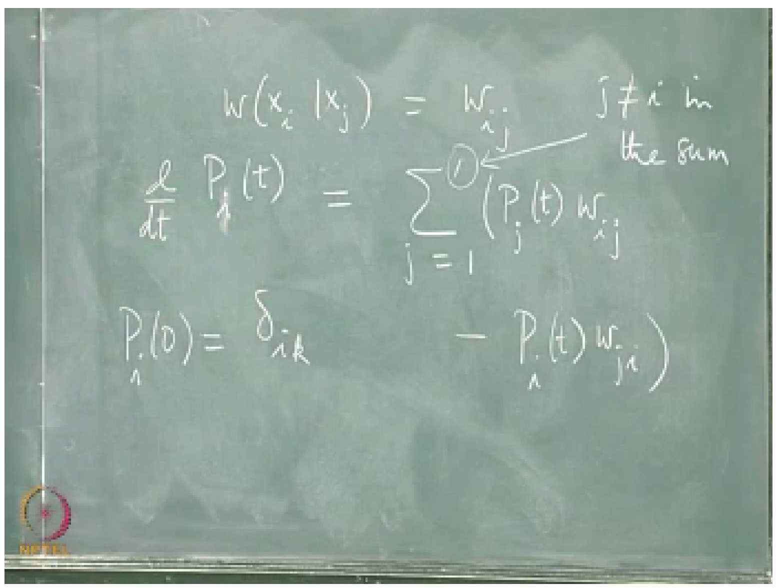

So you would have a summation which is let me say w, x, I from the value x j to the xi this transition probability per unit time, let me write this as some w ij. Then what would I write down instead of this. I would write d / dt and if p_j is the probability that you are in the value xj at time t then this rate changed is equal to summation over it is called ti. Summation from j =1 to whatever the number of set of value you can have could be infinite, could be finite times what, what is the integral, what is the summation inside this remember you have to some more are values so what would you write here? P j t w you have to go from I to j from j to I.

So you have a I, j-p, ji. With the prime here to show that j is not equal to I, in an integration that does not matter because the value x’ = x is the set of measure zero here. But in the summit matters. So it could a prime here this denotes this thing here and imply j ≠ I in the sum. So that is very much like a rate equations write down and once you specified the matrix w you can write this transition probability matrix. As a matrix once you write this matrix down the job is essentially done; now what kind of solutions you have for these matrix equations. So let us see if you have convenient way of writing this thing down.

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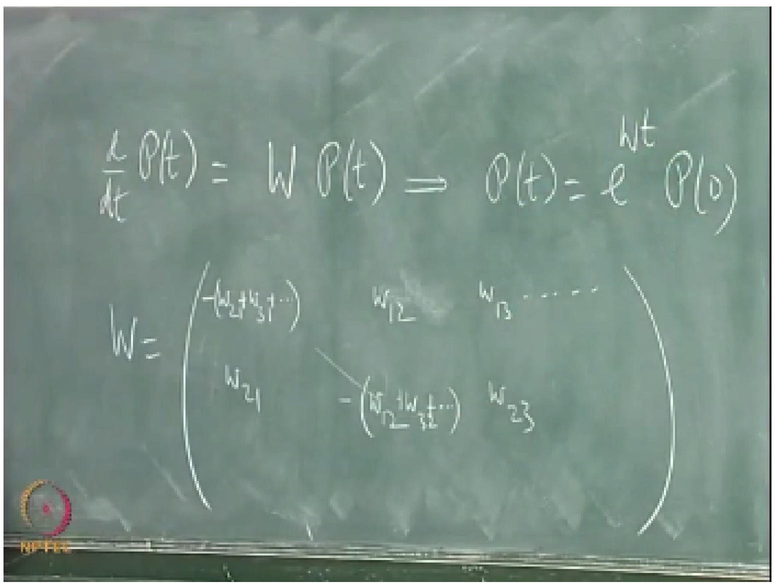

Let us suppose the recall p of t by the way what is the initial conditions of this, what is the initial condition. It is anything at all depends on what you are x_0 I start with state k x sub k then of course it says that the I of zero = δ IK. If it is at the state K the probability is one otherwise it is zero. So I am starting with some specific value the T=0 and with that initial condition you have to solve this equations. But it is the very convenient way of doing this.

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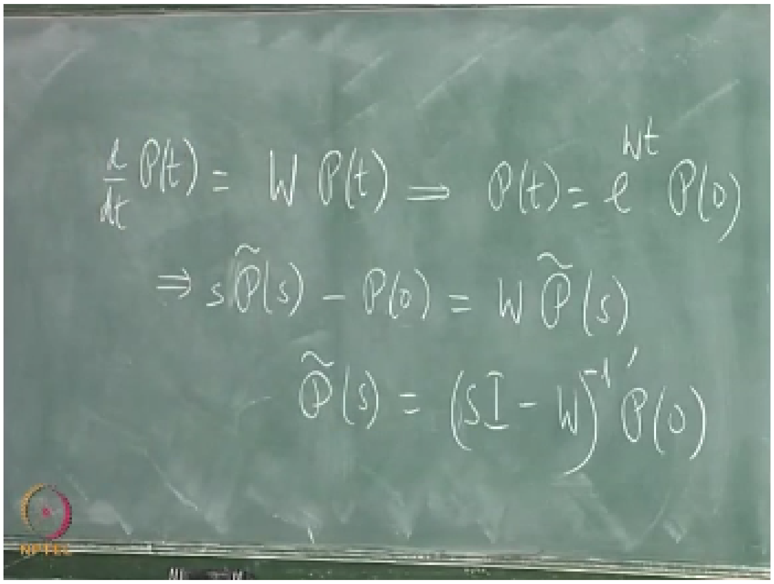

Let me write column vector T1 of T up to P N of T if you have for example fine number of state. But if you do not it is keeping going for ever, you really do not care, if you one of T, P_2 of T, etc. Let me call this column vector. The script P of T. In that looks very much like matrix equations, what it says is that D / DT of P (T) is equal to some matrix W x p (t).

And what is this W? What would you say as W? What are the diagonal elements of W, for example, suppose I = 1 and then it says DP1/DT is a summation over J ≠ 1, so it start off in this fashion and it says w1 to x p_2 + w_{13} x p_3 and so on. So it is clear that this W has matrix elements which are like W_{12}, W_{13} all the way in this fashion and then you have a diagonal element here, what is that diagonal element equal to. So you put I = 1, DP_1/DT, there is term here which is minus p1 but there is summation over WJI, so this is W_{21}, W_{31}, W_{41} and so on. So you really have structure which looks like W = $-W_{21} + W_{31}$ + etc, that is the first term.

The second term is W_{21} and the term here is W_{23} but in the diagonal here, you have minus instead W – w what will you have here? 12 just you add 21 31 etc you have 12 + w 32 etc and so on. So you have matrix equation of this kind which is not very difficult to solve if W is a constant matrix which is what it is and our stationary case in principal and then you have a matrix W which determines the transition probabilities in which the off diagonal elements give you the transition probabilities from 1 state to another.

This is a rate at which from state 2 you jump to state 1 and so on. But the diagonal elements are the negatives of the sum of all of the rest of the column. So the sum over the each column in this matrix W is 0, so you have matrix with property and the elements are all real and the diagonal elements are minus in the rest of the elements in the same column and formal solution to this is of course p(t) = p wt p(0) and if p(0) starts at a particular state then it is column back there with 0s everywhere the one at that particular state and e wt acting on that initial column vector gives you the probability vector for all the states, so this is how you will handle discrete valued process in continuous time.

Now if you go to discreet time as well, you have what is called mark of chain. That too is determined by precisely this matrix of probability something that analogues to W. Once again instead of an exponent, ewt if you discreet time n what you think this will be replaced by indubitably.

I have discreet time, what you think this guy is going to be replaced by. Well, each time to go from time step 0 to time step 1, you are going to apply this matrix W, to go from time stamp you are going to apply the matrix W once again, it will just W^n just as the discreet version of the Laplace transform is a z transform it is the power it is exactly the same principal. So you need to know W raise to arbitrary power which you incidentally need to know if you want to expatiate it as well.

But it is an easier way to handle the matter and that is to look at what happens to the Laplace transform of this equation. This immediately implies that the Laplace transform of P(t) we call this p (s) it says s $\tilde{p}$ - p of 0 = w $\tilde{p}$ s which implies that $\tilde{p}(s)$ = the unit matrix minus W inverse on P (0). So it implies that if you have the resolvent of this matrix W as a function of S you will approximate and then you can invert Laplace transform to w of t.

This is all went to an exist as long as you do not hit an Eigen value as long as the S is not one of the Eigen values of W. You can already see from here, this is very similar to what we had for dynamical systems, so the entire behavior of this probability, the matrix will covered by the high end values of this transition matrix W just as it was covered by high end value of the Jacobian matrix per our dynamical system. So in that sense this mark of process is really have reduced to set of first order equations that is very similar to a dynamical system, very similar mathematics is used.

Now what I am going to do next is to take a map like that, do a partition of it and show that this partition can behave like a mark of chain, the discreet time dynamic will behave exactly like a mark of change and therefore I can actually tell what happen to n time arbitrary time probabilities and we will see how that is done next time.

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