Indian Institute of technology madras
Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 21
Coarse-grained dynamics in
phase space (Part I)

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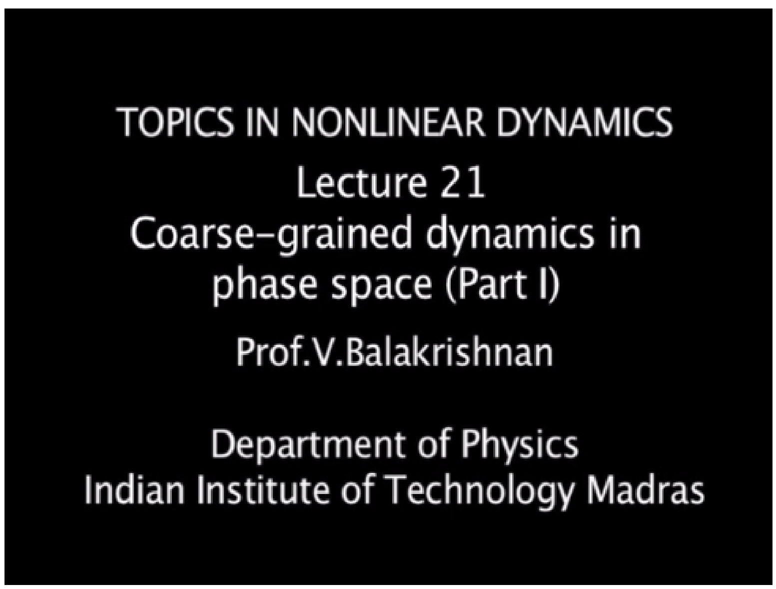

We saw that chaotic systems in some sense are like random systems the dynamics is deterministic you specify the dynamics completely and yet the outcome looks fairly irregular a periodic and has many of the characteristics of completely random motion some of the models we looked at like the Bernoulli shift I even mentioned casually could be as random as a coin toss itself and let us try to substantiate this in some sense and make some statements regarding how random deterministic dynamics can get recall the Bernoulli shift which is our prototypical model for random for care for deterministic chaos.

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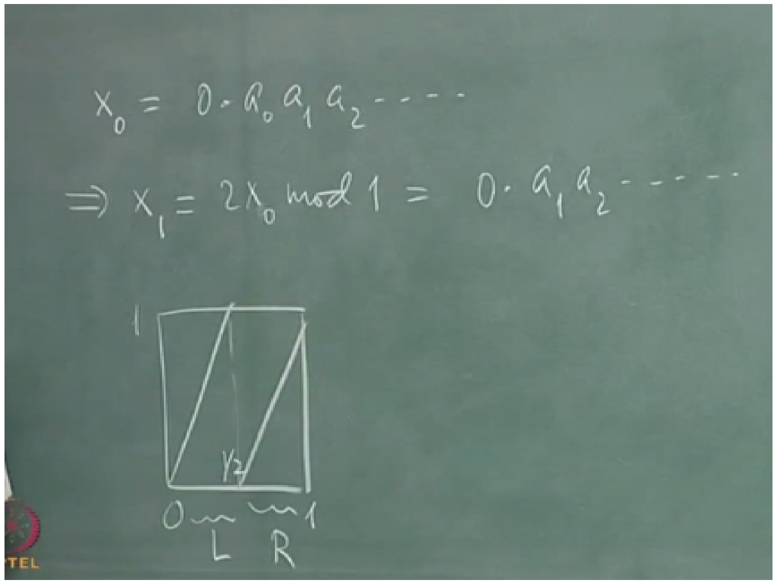

Which implied which involved taking an initial condition writing it in binary in the form a2 etcetera which would imply that x1 is $2X_0$ mod modulo 1 becomes 0 . a1 a2 and so on and this keeps getting iterated and each time you shift this decimal point one place to the right and throw away whatever is to the left of it and this was the Bernoulli shift which was chaotic with a Lyapunov exponent equal to the log of 2 the natural log of 2 and that was positive when we said the system was chaotic.

We could now ask the following question is there a way of partitioning the phase space in the problem in this case the phase space is just the unit interval between 0 and 1 in such a way that I keep track of not the precise point at any stage but merely where it is in one of the cells in phase space and given that information namely where the successive iterates are can i reconstruct what the original map is and the answer in this case is yes and we do it as follows.

So I start by saying that the unit interval and here is a map function slope 2 here I break up the unit interval into two sub intervals 0 to ½ and ½ to 1 and if the representative point is in the left I call it L if it is in the right I call it R so I will call this portion this is ½ I call this portion L and this portion R and it is immediately evident from here but if the leading digit is a 0 then you are in L because the number is less than ½ and if it is a 1 then you are in R on this side.

Now if I merely keep track of whether the representative point the iterate at any stage is in the left bin or the right bin these two cells and I keep track of this and write this sequence down next to each other what would I get.

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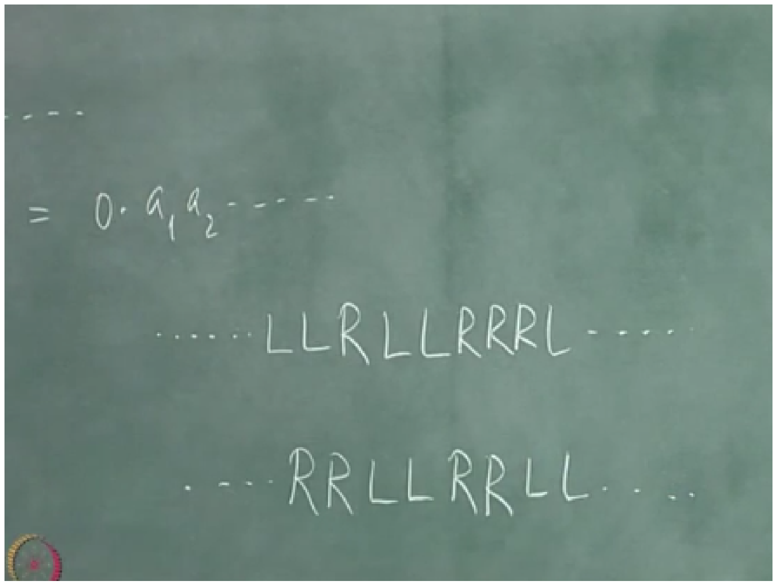

I would get after a long time a string of L's and R’s yeah you are absolutely you are not getting you are not seeing anything I mean you are simply saying that it is in the left or in the right and I am not worried about the rest of the digits then if I keep track of that this is called symbolic dynamics because I associate with each cell a letter of some alphabet and keep track of where the representative point is in which cell it is at each iteration and I get the long string of letters of the alphabet in this case the alphabet has just two letters.

Now if I took a typical point which is not part of a periodic cycle and iterated it over a long time this is what the string would look like completely and the sense in which deterministic chaos is like a random number generator is the following in this case given any string of this kind which arises by iterating the Bernoulli map for a specific initial condition given any such string the statement is there exists a sequence of coin tosses where I write L every time I get tails and are every time I get heads which is indistinguishable from this sequence.

So the statement I repeat is that given any string which has been generated by iterating a typical initial condition in generic initial condition and I get a sequence of L's and R’s and now we do not tell you where this sequence came from the point is there exists a sequence of coin tosses just supposed to be completely uncorrelated and random which would produce exactly the same string and in that sense the outcome you cannot tell whether this outcome came from a purely random machine like coin or from a purely deterministic rule of evolution for a certain initial condition yes.

Absolutely there is a one-to-one now the point is you would think that when you have the result of deterministic evolution you would think that you would see some pattern in it you would immediately mean it cannot be caught and code truly random there is some kind of pattern in this whole business but the point I am making is that the evolution the result of the evolution of a completely deterministic rule just by looking at the end product you cannot tell whether this originated from a purely random machine like a coin tossing a set of a coin repeatedly or whether it came from some applying some deterministic rule.

So the probability of each of these sequences is exactly the same every sequence for a coin is equally probable and point is that while one would think that deterministic evolution must necessarily lead to some order some kind of pattern which you would see in the final result the answer is no that is not necessarily true if this were periodic if this were part of a periodic orbit and so on there will be a pattern which would emerge immediately you would see this and say aha this could not have been pure chance but the fact is even that is not true because after all even this sequence could well have been the result of a set of coin tosses.

So determinism and randomness as we understand it naively I kind of intricately bound with each other and just by looking at the product of just by looking at the outcome of evolution one cannot tell whether it came from a random sequence quote-unquote some random dynamics or whether it came from something which was purposeful and periodic and regular so this thing gets blurred this distinction gets blurred and in that sense the Bernoulli shift is as random as a coin toss because the typical sequences it produces are as random as that of any coin toss yeah.

Now we are getting deep questions what is the definition of randomness this is tricky it is not possible to define a truly random sequence to full satisfaction it is not possible to do so we can only give measures of randomness and find out if our strings meet those requirements or not for instance exactly he does not understand the meaning of the word random and I am not going to ignite in him because we cannot define this word too precisely we really cannot define it too precisely.

So let us give examples and see what we mean by random strings and so on one way would be to say a random string consisting of two letters L's and R's you would say is a string where there is absolutely no pattern this is one way you might want to define it by that I mean the probability with which or you take a long part of this string and you ask how many L's are there and how many R's are there if the answer is exactly ½ the fraction is near 50% each you would say okay at that level this string is random.

But then I could say suppose all the, are first and all the R's that is not true this is no longer random it is completely periodic this thing here it really does not look like a random sequence at all then you say okay the a priori probability of having an L or an R after you say have a million digits the million and first digit 50/50 then you would say it is random in other words if there is no correlation which you can detect the number of times L'S appears is on the average the same as the number of times R'S appears.

The number of times the sequence LL appears is the same as the number of times RR appears or LR or RL ¼ each of these the number of times the sequence of three L's appears together is the same as the number of times any other sequence of three letters appears etc so if all these correlations are missing if you say these are then you would say okay now we are approaching something like a random sequence but it is again tricky because you could have within randomness you could have something very regular like the digits of π.

The digits of π if I write it down or any transcendental number of that kind if I write this number down then it looks like the digits successive digits from 0 to 9 appeared completely randomly so if you take a billion digits of π and ask how many times the zero appear how many times there is one appear and so on these are all roughly 1/10 if you ask how many times does the sequence 27 27 appear it is the same as the number of times 72 appears or 63 appears it is of the order of 1 over 100.

So it looks like the whole thing is completely random on the other hand if you give me a million digits of π the million and first digit is completely determined because I have an algorithm for computing π so what do you mean by randomness here therefore we have to be much more careful and start asking what kind of randomness are we talking about are we talking about lack of correlation are we talking about a priori probabilities and so on.

And if you think about it a little bit harder you discover that there is no truly satisfactory way of defining randomness at all and the lesson I want to without getting into that kind of meta mathematics the lesson I would not draw here is that chaotic time series this is after all a time series here could look quote-unquote just as random as a truly random time series we would not go further into this for the moment we will keep this at the back of our minds that no pattern need be discernible all.

Could be completely quote-unquote random again I put this word random in quotations yeah any random yes that is the statement I take any random set of any in the result of any set of any coin toss experiment then the statement is there exists an initial condition for which the successive iterates would give you exactly that sequence so in this in that sense you can put these into correspondence with each other.

Now let us let us get on with this let us let us see okay now that we know that chaotic dynamics in some sense could be very random looking in this restricted sense let us see what we can do with it we would like to find some measures for these chaotic in this kind of chaotic dynamics so what can one do one should ask alright if I have more complicated alphabets or symbolic dynamics I take my phase space I break it up into cells then can I reconstruct by looking at the sequence of letters can I reconstruct the dynamics and in this case we could.

So this example that I gave with just two L's and R’s just two cells for the Bernoulli map is called a generating partition in other words I broke up the phase space into two cells I partitioned it and kept track simply of whether the point was in the left cell or the right cell and I generated a string but that string has in it the full information of what the initial condition was I can reconstruct from that string where I started you can reconstruct the dynamics and that kind of thing is called a generating partition.

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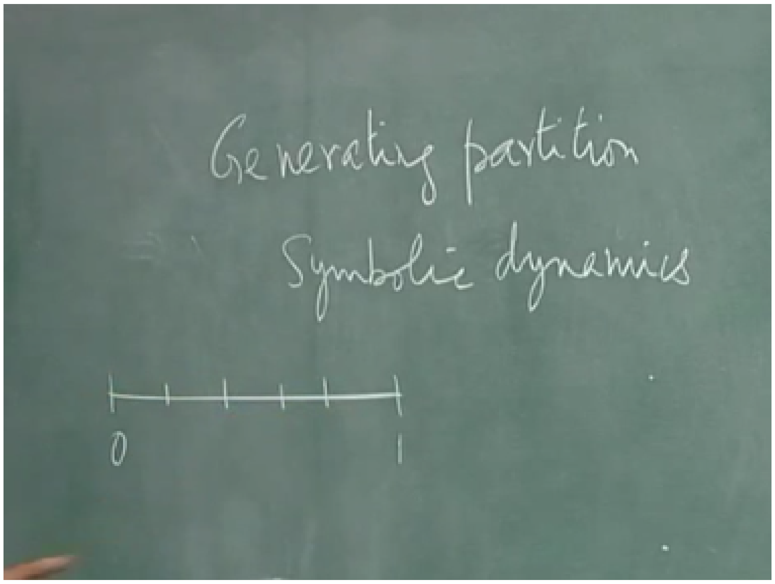

And one of the aims of chaotic dynamics would be to try to get a generating partition after which you do not have to look at the full the actual dynamics but you need only look at the dynamics of these symbols of these letters and that goes under the name of symbolic dynamics suppose we have such a partition and we are going to talk about several kinds of partitions now very shortly suppose we have such a partition and I am going to take very specific examples so let us go now and look at a phase space which is perhaps in a one dimensional map the unit interval for example 0 to 1 or it could be more general.

And I break it up into little cells and keep track of where the system is that is all I do at each iteration so in some sense I coarse-grain my phase space my resolution is not at the level of a point but rather of a small interval and I simply ask is the particle in this bin or that bin.

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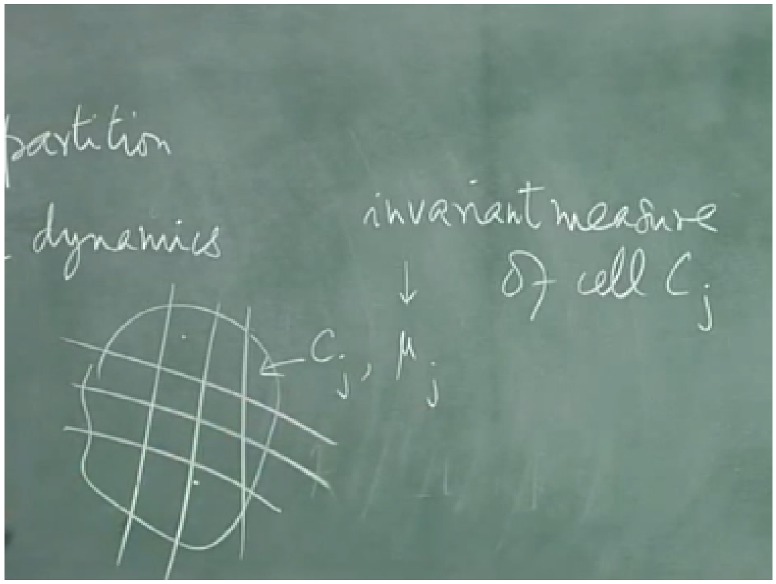

And more general phase spaces if I have a multi dimensional phase space I imagine doing something like this I break it up into various cells and suppose this is cell C sub j they label it with a j and I also assume that there is an invariant measure for itself.

Namely if there is an invariant density for this process the chaotic dynamics then the integral of that density over the volume of this thing I will call that the invariant to measure and let me denote it like μ subject this quantity here is the invariant measure of cell Cj then is there some way in which I can understand from the symbolic dynamics is there some way in which I can quantify.

That the actual dynamics in other words there could be some cells which are visited very often there could be some cells which are visited very infrequently is there some quantified by which I can probe this there could be some cells which are weighted very heavily which have a high μ j and there are others which I was low μj.

And I assume that each of these Mj are non negative number between 0 and 1 and the total measure is 1 if I normalize everything the total measure is 1 then we define the following set of numbers which helps us do this.

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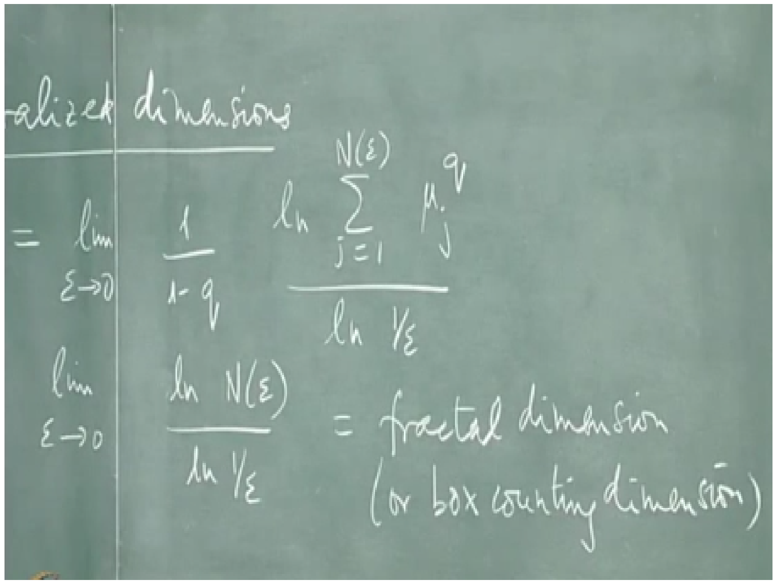

And this set of numbers is called set of generalized dimensions and it is defined as follows I introduced a parameter q and write D sub q this is the generalized dimension the cute dimension q is a number a real number and it plays the following Road this is defined as the limit in which the size of each cell goes to 0 and let us for simplicity assume that each cell is of some size ε they do not have to be of the same size but it is completely arbitrary however partition it but I would like to look at the limiting case because finally you would like to go to the dynamics of the point itself and therefore the cell size should go to 0 eventually.

But formally let us write limit ε goes to 0 that is the linear the dimension of a cell the size of each cell of the following 1 / 1 -q the reason for this will become clear the log of a Σ j =1 to the total number of cells and that is a function of the size of each cell so let us call N (ε) the total number of cells into which my phase space has been partitioned the log of the sum of μ_j^{q} /log 1/ ε.

So let us consider this quantity but q is a real number we could start with q and integer and later continue to all q's all real numbers from - ∞ to ∞ but let us study this let us define this quantity and see what it is trying to tell us if I set u= 0 D 0 is the following it is limit ε goes to 0 this becomes q = 0 log of if q is 0 this thing becomes 1 here and it is just the number of cells into which you partition things so log N(ε) / log 1 / ε.

And what is this quantity what is it telling us this quantity is called the box counting dimension of this set and it is a fractal dimensionality of the set on of this attractor or whatever phase space you are looking at now I should have mentioned earlier that I have in mind a situation where the system has fallen into some chaotic attractor some region where it is completely chaotic dynamics.

So all the transients have been removed and I talk now about the stationary distribution that is why I use the invariant measure so all transients have died out and now the system is continuously going on this strange attractor perhaps or the unit interval in the case of these maps which are chaotic and this is the fractal dimension or box counting dimension of this attractor so let me explain what I mean by right many factor dimensions here but this is the first of the all.

Now I should spend some time and take a little digression and talk about fractals since we Have sort of mentioned this word fractal several times without actually defining it I ought to do that now we Have time has come when we need to do this but first I want you to notice that the following fact which I will come back to this quantity here if q is positive then clearly this number Dq is going to be dominated by those cells for which the μj is large relative to those which are where it is small in other words when q is positive Dq will be heavily dominated by those cells which are visited very often.

So the total fraction of time spent there will be larger in other words μj is larger for such cells on the other hand if q is negative it is just the other way about those cells which are infrequently visited would contribute to this number Dq and those which are more frequently visited would not because this would then become a negative number here q becomes a negative exponent therefore the smaller μj would dominate over the larger μj we keep that in mind for a moment and but q = 1 we seem to have a problem we really seem to have a problem because this blows up on the other hand if q is 1 you end up with Σ j =1 to N (μj) but that is the total measure which we have taken to be unity we have normalized it to be unity.

So you get a log 1 there which vanishes and you get this which vanishes and the question is there a limit or not and we will see there is a limit definite limit which will be surprising limit and so on for high values of q etc but now with that let me come back let us backtrack a little bit and give a little quick tutorial on fractals and what we mean by fractals the traditional way of doing this.

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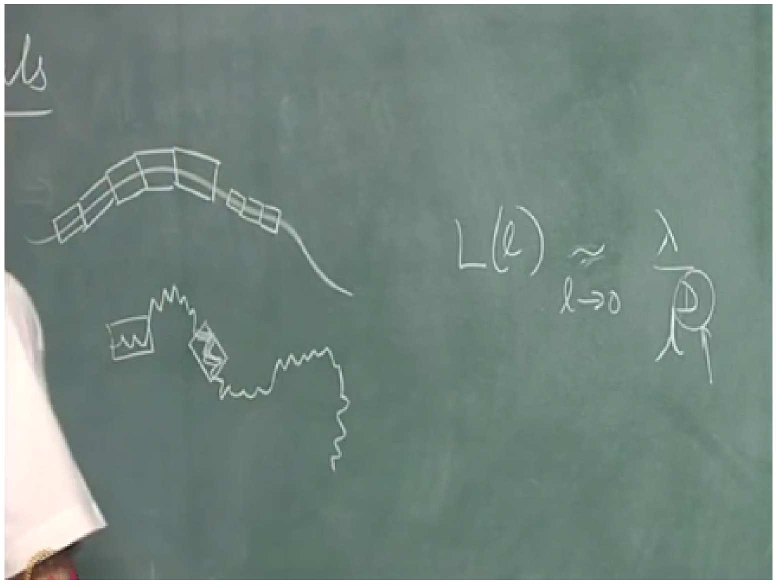

Now is to go back and look at fractal lines this has been around for a long time mathematicians have known about fractals for a very long time but only after the 1960s and 70s when Benoit Mandelbrot introduced the term fractal into the physics literature and started applying this to various physical problems this became extremely popular and almost fashionable today we have fractal art and fractal dimensionality is everywhere fractals everywhere the titles of popular books and so on and what we mean by this and the simplest way of doing this is to give you simple examples of this.

So let us look at for example a line of this kind of smooth line does not have to be really smooth it could have corners and so on and ask what Is the length of this curve that is a traditional way to introduce fractals now the length of this curve is measured in practice by putting little foot rules everywhere little meter scales everywhere and adding up all those pieces now if you have for instance we know this thing has some finite length in some sense and I take a meter rule and start measuring its actual length then you are well aware intuitively that this length has some absolute value independent of the size of your ruler certainly should.

It does not matter whether you measure it in centimeters or meters there is an absolute value if it is length is 1 meter you discover it is equal to 1 if you expressed it in meters it is equal to 100 if you express it in centimeters a 1000 if you express it in millimeters but the fact is there is an absolute length to this curve and that is because the assumption is that if you measured it with a foot rule of resolution this much then the number of such things will you put in order to measure it would give you in this unit would give you a certain number but now if you use a smaller ruler of this kind you need more of these guys.

So the actual number would become larger but then the conversion between this and that would give you the same you same actual absolute length for did the curve is smooth on the other hand imagine what happens if the curve has kinky corners of this kind everywhere then if I have a curve which is very jagged of this kind is roughly in this fashion and I use a resolution of this much then this entire curve from this point to this point is just put in subsumed under one unit.

Because in some sense this ruler is too coarse to see these little bends on the other hand if I start using a much finer scale then indeed I would be able to put this here and then this here and this here and so on so I really call it as three of these smaller units as opposed to a single one of the bigger I would miss those little bends in the curve this suggests immediately that this curves length the actual length depends on the resolution that you have.

And if the resolution is L then the actual length L could be a function of l the length that you measure and this is not choice of units that I am talking about the actual length the absolute length that you measure would be equal to some constant which depends on the units that you choose and it could well diverge as this l goes to 0 like l^D it could well diverge could actually tend to ∞ if you had little kinks and corners on all length scales right down to 0.

So it is evident immediately that if the absolute length of a certain curve is to divide it means it must have a large number of bends which are missed if you have a resolution which is too gross or coarse so if indeed you can construct a curve which has kinks and corners everywhere on the curve at all points then if such a thing can be constructed its absolute length would actually become infinite as your resolution goes to 0.

So this would look like this as l tends to 0 it would diverge like some power of L some negative power of L that power with which it diverges this thing here is called the box counting dimension box because the same thing can be generalized to higher dimensions like 2, 3, 4 and so on we are talking about a one-dimensional curve here.

So this D here is called the fractal dimension or the box counting dimension of this curve and we will see by examples that this curve this D need not be an integer the topological dimension of this curve is 1 for any curve just as the topological dimension of the surface is 2 of a volume is 3 and so on a point or set of points is 0 the topological dimension of this continuous curve is actually 1 however this dimensionality could be a number between 1 and 2.

It could even be two it could even be as big as the dimensionality of the Euclidean space in which you have embedded this curve in this case the blackboard namely 2 it cannot be bigger than 2 so you could have a curve on this which is space filling namely it comes out literally close to every point in this plane and may even intersect itself and such a curve would have a dimensionality of 2 a fractal or box counting dimensionality of 2 in any case it could be some number between 1 and 2 and then we call it a fractal curve and let me give an example right away or how you compute fractal and the fractal dimensionalities.

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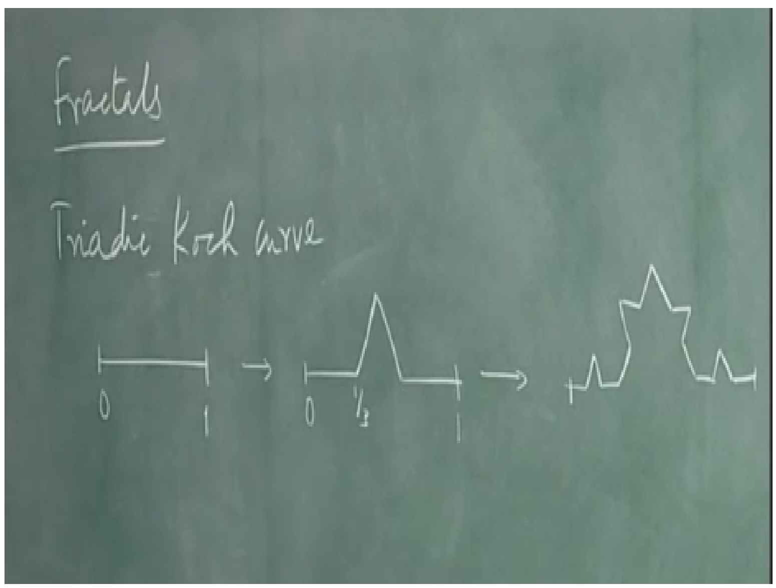

In this example is called the Triadic Koch curve and it is specified by a geometrical recursive construction in stages so you start with a unit length 0 to 1 at the first stage at the next stage the construction is break it up into three equal parts and tilt the middle part upwards so this goes into something which looks like this so this is still 0 to 1 but this is 1/3 that is another length 1/3 that is another length 1/3 and that is another length 1/3.

So you are still between 0 and 1 except you moved up in this fashion so the total length of this curve in some units has become 4/3 from 1 each of these is 1/3 the original link at the next stage you do it all over again for each of them now how many pieces do you have you have four here four here four here and four here so at the neck first stage you had one piece then you had four pieces and now you have sixteen pieces and each of these is one-third of this so what is happened to the length.

Yeah it is become $(4/3)^2$ and so on and it is evident if you keep doing this forever that this whole curve will still occupy a finite portion of this plane it will start at this point and at that point but its length would actually become infinite and you would have these corners everywhere so it would become differentiable non differentiable almost everywhere it is actual length would become infinite it would have a corner everywhere and you could.

Now ask what is the fractal or box counting dimensionality of this namely how does it diverge how does this length diverge and a moment's thought will show you and I want you to work this out that the length at any stage is going to increase geometrically increasing like four thirds to whatever power of the generation or the number of iterations and it actually becomes infinite and you can now ask how does it become infinite as the size of each of these pieces goes to zero.

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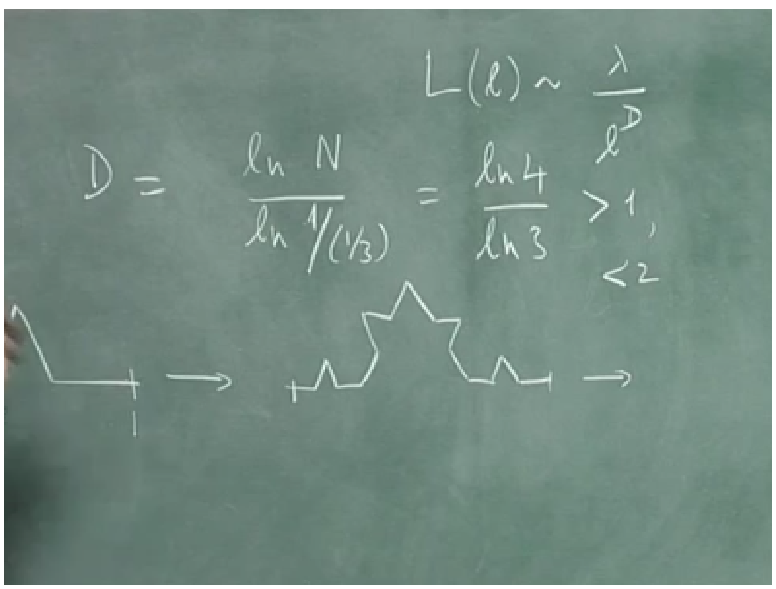

And that power is the fractal dimensionality and a moment's thought shows you that this D here which after all if you take the resolution to be l then it is going like L^D and therefore what you should do is to take the log of that and divide by the log of L and that will give you D if L of L goes like constant divided by L^D in order to find this D all you have to do is to take logs on both sides and then let the limit L go to zero.

So it is not hard to see by this regular construction that this is also equal to log in this case I do not have to take the limit ε or l goes to zero because this procedure is generating a regular fractal in the sense that the relation this bears to this is the same as a relation this bears to this and so on at each stage so all you have to do is to ask given this construction at some stage.

What is the next stage do how much does it increase the length by and that will give you an indication of what B is so a moment's thought will show you that this is the same as the log of the number of pieces into which you break the system at any stage divided by the log of the ratio of the size of each unit at one stage to the unit at the previous stage and in this case this was one third of this and therefore it is log of one over one third and that is log three.

So if the D magnification factor is on 1/3 then it is log of one over that factor and in the numerator you have the log of the amplification factor namely each unit here gets broken up into how many pieces here and what is that equal to in this case it is four in this case it is four so this is equal to log four over log three and this is greater than one but less than two that is the fractal dimensionality or box counting dimensionality of this fractal curve.

This construction for all regular fractals you can decide now what the fractal dimensionality is namely how does the measure the length of the volume or the surface or whatever it is diverge as the size of each box unit goes to zero by simply writing down the ratio of the logs of the number of pieces into which the previous steps unit is broken at each stage divided by log of the reciprocal of the deme-magnification factor.

And I will leave you to play with this because you can have all sorts of fractals you can construct all kinds of fractals if you look at text books and books on practice they give you a huge number of geometrical constructions where you would see the calculation of these fractal dimensions done.

But that is exactly what we have here this is exactly what we have for Dq and q is zero because you are breaking up a system into cells you are letting the cell size go to zero each time and then the ratio of the log of the number of cells of dimension ε of size ε / log 1 over ε because ε is the D magnification factor is in fact the fractal dimension of this set now depending on what this the original phase space is what the attractor looks like.

And so on this number could be a fractional number it need not be an integer at all so it is some number which tells you what the fractal dimensionality of the attractor is could be a set of points with a fractal dimensionality between 0 and 1 this is possible I will give a construction very shortly tell you what the fractal dimensionality of a set of points is we talked about the fractal dimensionality of a curve here.

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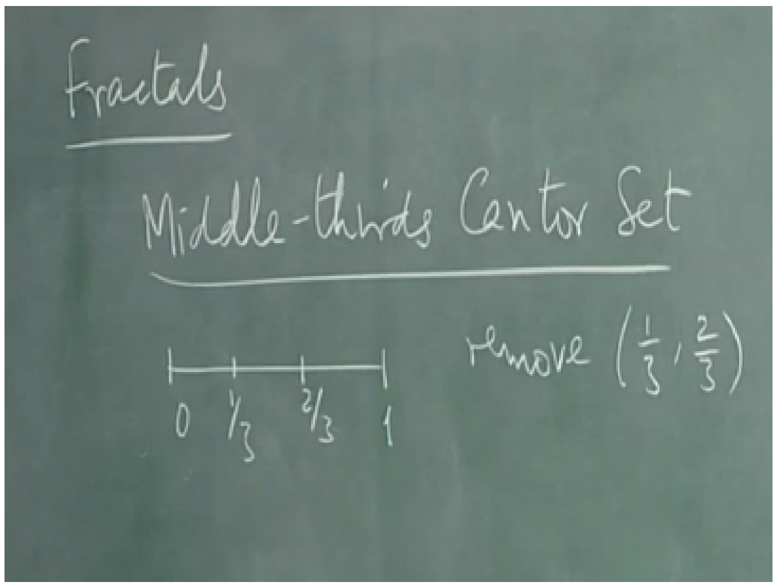

A one dimensional topologically one dimensional object but now let us look at the following construction and this is called the triadic cantor set so that is another very common factor this is called the middle thirds Cantor set and the construction goes as follows says take the unit interval 0 to 1 and remove from it open into the middle third open interval so here is 1/3 here is 2/3 and at each step in the construction remove the middle third thing so remove 1/3, 2/3 remove this open interval leave the point one third here and 2/3 here they belong to the set but remove the point points in between so where does it get us that gets us to this.

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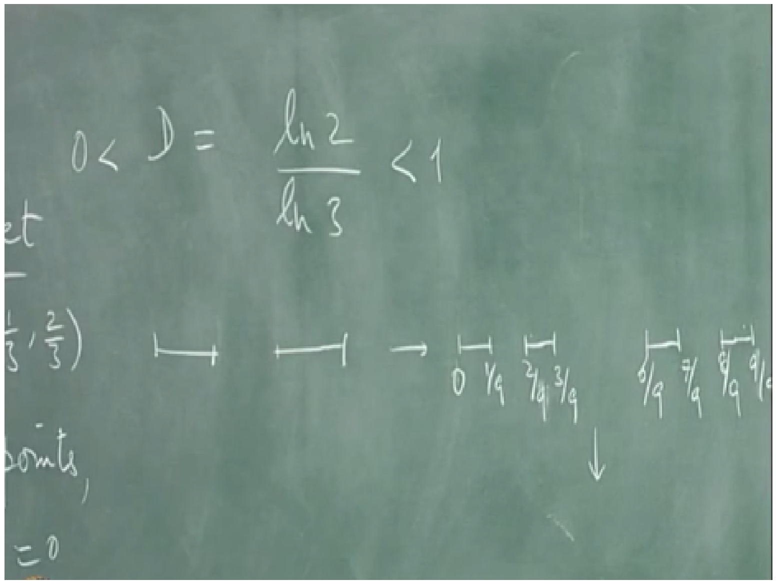

So I have this and I remove the middle thirds in the next step I remove the middle one third for each of these and where does that take us so it looks like this and there is a big gap so this was zero and that was one but just I had a walk as I had a 1/3 here this is a 1/9 that is a 2 9 that is the 3 / 9 that is 6 / 9 which is 2/3 at 7 / 9 that is 8 / 9 and that is 9 / 9 then I continue this forever and I keep doing this each time I remove the middle third open set what am I left with finally I am left with a set of points I am just left with a set of points.

Now what is the total length of this set of points that I am left with the length is 0 how do I assert that how do I assert that yes how am I guaranteed to the length finally is 0 I mean how do I know that yes absolutely so you can see that you are reducing the length by a factor one third each time one third of it is gone and as you go to ∞ the total length is 0 I urge you to work this out I urge you to work out what the length is after n such iterations yeah it is 2/3 to the power n and as n goes to 0 that goes to 0.

So it is clear that the actual topological dimension the measure the will take measure or whatever is left is 0 so it is a set of points now is it a countable set of points or uncountable it is an uncountable set of points it is just a set of disjoint points but it is uncountable so it is an uncountable total equal to 0 in the limit what is the fractal dimensionality of this set now that is almost by inspection because instead of adding something you are simply removing something.
So what is the fractal dimensionality of this set?

So what is D exactly so you are breaking up each unit into two units so n of ε is in fact two at every stage so this is equal to log 2 but then each unit new unit is one-third the original unit and therefore you have log of 1 / 1/3 and that is log three so this is log three here and this number zero less than this less than unity it is a fraction the low limit could be young exactly the number of I start with a curve yes that is true if I start with a plane and start depends on the construction that I do to get rid of things yes it is a fractal dimensionality which lies between the topological dimension of this object right and the Euclidean dimension in which you buried this is of the space in which you have embedded it.

So in this case it is some number between zero and one and it actually gives you some indication of what the nature of the set is what now this was easy to compute because we actually looked at the reason we did not we did not go all the way to ∞ we did not take this recursion indefinitely far but we realized that this whole curve is self-similar exactly self-similar in the sense that at any stage the relation the curve bears to its immediate preceding stage is identical as you go along everywhere.

So it is self similar but then real fractals which occur in nature could only be statistically self-similar in the sense they would be very irregular objects they need not be this kind of regular construction that takes you from one stage to another which does not change at all.

And it could be its statistical properties could be exactly the same at all stages it does not have to be actually physically self-similar not a geometrically regular fractal at all could have very easily lured motion such as diffusive motion or Brownian motion and so on these things are statistical fractals but the formula for the box counting dimension is precisely what I wrote down which is limit ε goes to 0 log n of ε / log 1 / ε okay.

So much for quick tutorial on front fractals we will come back to this a little bit later now going back to our yeah but yeah it diverges the length of such objects will diverge or if it said but then I have not talking about a fractal curve here it is a fractal set of points okay his point is I defined my fractal by saying I started with the example of a curve and I said this curve is going to have an infinite length eventually.

And now in the set of points there is no length from the curve I could go to higher objects like areas volumes and so on and so forth the scale it is question of how it scales so now I generalize that and say that I define my D as the scaling exponent nothing more than that in the limit in which the size goes to zero.

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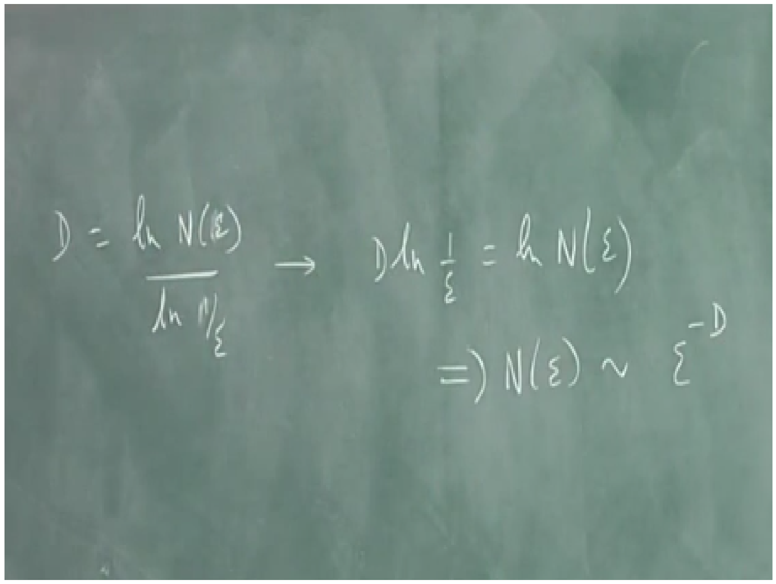

And that is also it has no connotation of being a length diverging or anything like that it simply says what is the scaling exponent namely how does see if D equal to log n of ε / log 1 / ε this implies q you imagine I forget about the limit for a minute then it says d log 1 / ε = log of ε which implies that n of ε scales like ε to the - D that is all I am interested in that is how I define my fractal dimensionality so just a scaling exponent nothing has to go to ∞ or anything like that exactly what is what is that a fractal is a geometrical object which has a non-trivial fractal dimension.

It which is different from its topological dimension in this case there is a set of points the topological dimension is 0 but this dimension is some number between 0 and 1 it is sum could be less than do you think it could be less than the topological dimension not the way we have constructed these things and it cannot be greater than the Euclidean dimensionality of whatever space is embedded in so those are put bounds on it so now the plenty of examples for example if I look at this surface and I pretend to look at it at greater and greater magnification then it would become irregular on many scales it would become more and more rough and the actual total surface area could well tend to ∞.

Now physically of course that is not true because you soon come to a stage where the physics of the problem gets in and things are no longer fractal because the very structure changes I mean once I let get to the level of atoms for example then it is not even a continuous surface so there is always a lower bound and an upper bound in physics from physical consideration so when things scale in this fashion so clearly if you look at a very classic example would be coastlines of countries and this is how Mandelbrot originally introduced this whole business how long is the coast of Britain or how long is the coastline of a country.

Well if you look at it from 100 kilometers up then the country has seems to have a coastline which looks like this but you get closer then the coastline is actually larger because now you are going to start looking at structures on smaller scales eventually you would start counting you know how do you define a coastline suppose you say it is all those portions of the coast which are not wet or which are wetted the boundary between the water and the land then you would have to start looking at individual boulders and then individual rocks to see if it is part of the coastline or not.

And of course the total length is increasing then you start looking at individual grains of sand and some would be wet some would not partially and so on but of course once you come down to extremely fine scales like atoms we know that this is not even a continuous curve so it is gone right so it is obvious in all these cases that there is a lower scale lower resolution and an upper resolution within which the system is factor and beyond it is not although mathematical fractals of this kind like the coarse curve and so on.

The assumption is it is a geometrical construction so it goes right down to $\varepsilon = 0$ so we must keep that in mind but in physical applications there is always some resolution some orders of magnitude or a range of orders of magnitude inside which the system is fractal okay.

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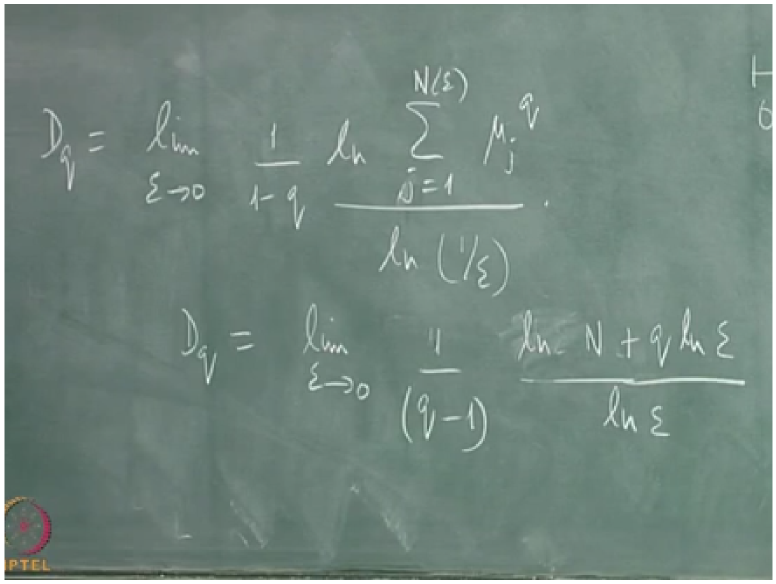

So let us go back and examine Dq we are going to come back and compute this in many cases so this is limit ε goes to 0 1 / 1 - q log Σj = 1 to n of ε μj ^{q} divided by log 1 over ε now we can start computing this in simple instances just to see what things look like what would happen if I took.

For example the Bernoulli shift and I did the following here 0 to 1 and in this case I am just going to break it up into a finite number of into a certain number of cells and let the resolution go to 0 the number go become larger and larger so let us break it up into n cells each of length ε so n is simply 1 / ε nothing more than that and what is the invariant measure for this map what is the invariant measure for the Bernoulli shift it is a constant and therefore what is μ j every μj is exactly the same this is equal to an integral over Cj dx J ρ invariant of X.

That is the natural invariant measure for each cell but ρ(x) is 1 for the Bernoulli shift or the tenth map at fully developed chaos and therefore this is simply equal to the length of the cell and nothing more than that and what is that equal to just ε each of these is absolutely nothing more than that so what happens here what happens to this definition it becomes trivial right so this is equal to limit ε goes to 0 1 / 1 = q then what you get up there this is constant right.

Yeah so this thing here and that is the total number and it is exactly the same for each of those guys so what does it give you n ε to the q so log n $ε^{q}$ / log 1 / ε nothing more than that and what is that equal to so let us write this as log n + q log ε what does that give you so if you like write this as q minus one right this is log ε so what is log in remember that n ε in this problem equal to 1 yeah but cannot depend on X.

So what does it give you Dq what yeah so what do you get for Dq1 this is such a trivial problem you had a unique URI you are the unit interval we broke it up into equal cells we let the cell size go to 0 and it simply says this is such a simple problem I mean the invariant measure does not distinguish between one part of the phase space and another that all the Dq s have turned out to be one every one of them is one it is not giving you much information and this attractor has the dimensionality same as the topological dimensionality of the interval itself it is 1.

So it is really not strange it is not a fractal dimension in that sense it is an integer it is 1 it is a very trivial problem in this case now what the same thing happened if you had for instance the logistic map at fully developed chaos it would not happen that way even if I broke it up into equal size cells in exactly the same way it is clear certain cells near 0 and near 1 would be heavily weighted because the invariant measure is very large there and much smaller elsewhere.

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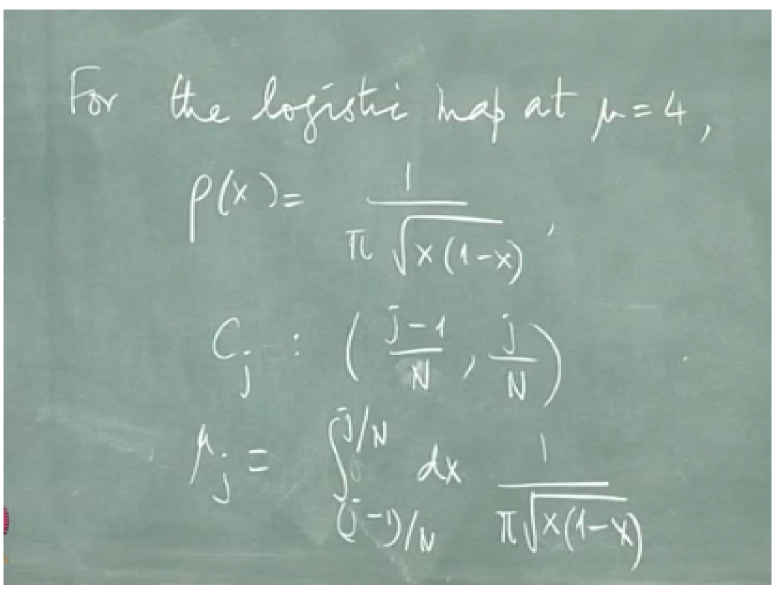

So I leave it as an exercise to you to calculate Dq in that case and it is not a trivial matter it is a function of q now but you have to be very careful in doing this because so take for the logistic map for the logistic map $\mu = 4$ we know that $\rho(x0 = 1 / \pi \sqrt{x} (-x)$ and then what would μ_j be what would μ_j be if I took the unit interval and broke it up into equal parts of size ε each.

So this number is ε this number is 2 ε and so on or if you like to break it up into n cells this is the same as 1 over n 2 over n etc so the μj cell the cell Cj this is the interval between J - 1 / N and J over n therefore Mj equal to integral J - 1 / N 2 J / N of dx 1 / π √ x times 1 - X you have to evaluate this integral and then you have to raise that to the power q and then do this construction here in this case n over ε is just 1 over ε or if you can replace this down here 1 / ε by just log n here and work with just n because everything there is written as a function of N.

And then you have to compute this so we are left with the problem of computing this quantity 1 to n μj^q / log N and μj is this quantity here now one might be tempted to do the following and here is where you have to be careful one might be tempted to say that this quantity here as some as ε goes to zero is really going to some kind of integral.

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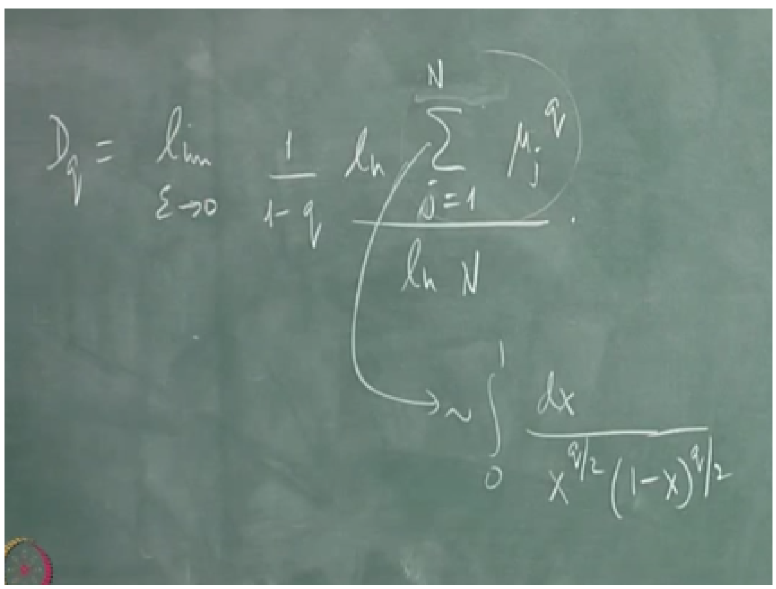

So you might be tempted to replace this summation by an integration of the form dx $\rho (x)^q$ the full interval from 0 to 1 so this quantity here is related to this integral here but you have to be very careful here because this thing here because the in T invariant measure has that structure is of the form x / q / 2 1 - x / 2 and this does not converge this does not converge if q is sufficiently large positive it is immediately clear.

That if q is bigger than because then 2 if it is equal to 2 also you have a problem because then you have dx/ x and that is logarithmically divergent so this replacement of the sum by an integral will work only if q is less than 2 and so you immediately begin to see that at q = 2 you have a abrupt change something happens and you no longer can replace.

The summation by an integration on the other hand the summation can be well-defined this quantity is well-defined absolutely and you therefore have to treat this separately you have to leave the sum as it is for Q >= 2 and for Q < 2 you could replace it by a suitable integral so that is where the non-trivial and it might it comes and I will leave you to do this analysis next time I will write out the solution for what this is and how to go about this so you can immediately see from here that DQ need not be a constant it will change by the way what do you think d0 is in this problem.

What if B 0 is given in the logistic map what do you think B 0 is it is in fact the fractal dimension allotted of the attractor which = which is 1 because it is the full line right its lebesgue measure is 1 so it is actually the fact be 0 will turn out to be 1 definitely but as soon as Q exceeds 2 you have a problem so you need a function which will be 1 at Q = 0 but something else at Q > > = 2 now let us we will come back to this but let us look at what D 1 is because that's non-trivial we need to be able to write down what D 1 is and then physically interpret anyone how do. I do that what should I do I need to find the limit of these quantities as Q tends to 1.

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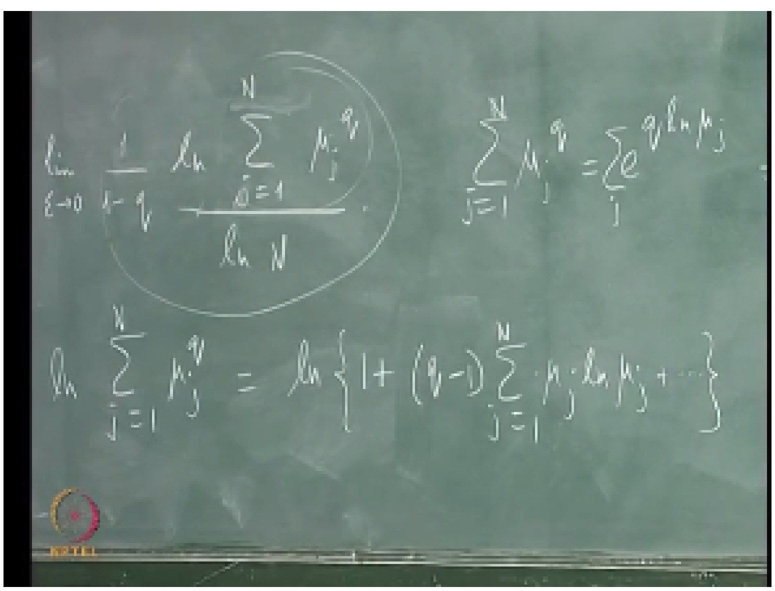

So let me write this as d 1 = the limit as epsilon goes to 0 limit as Q goes to 1 limit ε goes to 0 of this quantity here this the question is what am I going to do so let us find out what this quantity goes to after all the numerator goes to 0 the denominator goes to 0 so what should I do I thought to write μJ^Q as e to the Q log μ J and do a Taylor expansion of this function about Q = 1 because there is a Q - 1 sitting here so terms which are quadratic in Q -1 will disappear and all that will survive is the quantity that the coefficient of Q = 1 right so this thing here is equal to e^q = 1 is e to the log μ J + Q -1 the derivative of this quantity at Q = 1 which is log μ j e to the log near J + order Q - 1.

The whole squared and that gives me e to the log μ J is just μ J +Q - 1 and this is μ J log μ J + higher orders which you can ignore out there and I have to do the summation of this from 1 to N so let us do that so Σ J = 1 to n of ε is the summation here the summation over J and that gives me a summation over J of this thing here but that is = sum over J μ J is 1 + Q - 1 comes out sum over J μ J log μ J + higher orders now I have to do a log of the whole thing so what does that give me log summation J = 1 to n μ J to the Q therefore = log of this whole thing 1 + sum over J = 1 to N 1 + Q - 1 sum over J = 1 to N μ J + higher orders and I want the leading term at Q = 1 so I have log 1 + Z.

And what is the series for log 1 + Z for small Z it is Z + Z^2over 2 or - Z^2 over 2 + etc so the leading term is just Z itself which is Q - 1 times this and therefore we have a formula now for D 1.

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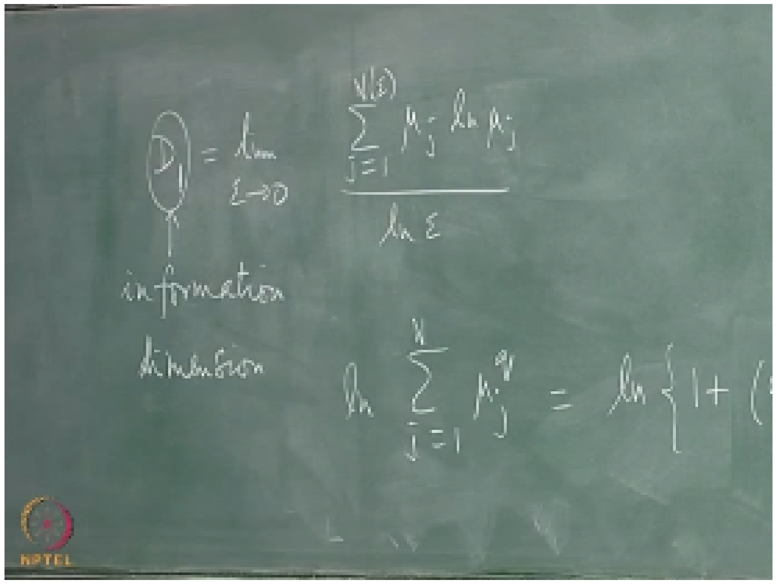

Which is = limit ε goes to 0 1 over so we have something here which goes like 1 /1 - Q and then you have a Q - 1 so they cancel each other and you end up with summation J = 1 to n μ J log and this is n of ε μ J log μJ divided by log 1 over ε with - sign because this was a Q - 1 and this was a 1 - Q therefore this could be written as log well log μ J say but you could also write this as log ε does this remind you of something.

When Jove entropy reminds you of entropy for which if you have many possibilities and if you have a set of events with probabilities P 1 P 2 P 3 etcetera you know that there is something called the information and there is something called the negative of the information which is the entropy which is something like summation over I P I log P I that is exactly what it is except the probability is being replaced by it a measure of the self again a number like between 0 & 1 so in fact new J gives you the a priori probability of being in cell J you start completely randomly and put your pencil.

There according to the dynamics then μ J tells you that the typical point will inhabit cell J with probability μ J once you are on the invariant attractor so it is very much like information or entropy and D 1 is called the information dimension just as the earlier one was called D 0 was called the box counting or the fractal dimension this is called information dimension this is the only value of cube for which you have a log sitting there and that is because of the special way in which q = 1 led to this log immediately because summation over the measure μ J gives you one summit.

So it ended up with this log it is not true for any other Q you do not get these logs for any other Q because mu to the power J would be perfectly for Q to the power cube not P not reduce to one in that case so that was the reason for putting the 1 / 1 - Q to make a finite limit here need this and of course at Q = 0 it does not matter now I leave you to find out what D 1 is for the logistic map in fact you would like to find out what D Q itself is for the logistic map a result which, I will quote which I would not prove is the following if Q ' > Q then DQ '< = DQ in other words DQ is a non increasing function of Q it generally goes down or could remain constant for a while and then go down.

So this is not hard to prove it is actually provable from the definition of B to itself again I did like you to try this out and see if you can establish this it is fairly straightforward to do this just the property of the measure is needed the fact that the new days are non-negative numbers between 0 & 1 that is all that is needed now where does this get us what does it all get us well we are going to look at some partitions run out of time today but we will look at some partitions of simple maps and we will push this idea further that this chaotic dynamics is starting to look more and more like the realizations of some random process some sort of a random process.

So what I will do next time is to show you that you can partition the phase space in such a way that the jumps of the representative point from one cell to another would not look different from a Markov process and therefore you could use the machinery of Markov chains in the theory of random processes in order to understand what the dynamics looks like at the coarse-grained level so this is bringing together deterministic dynamics with stochastic or random dynamics and it becomes very interesting to analyze the system from this point of view so we do this next time.

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