Indian Institute of Technology Madras
Present

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 20
Discrete time dynamics (part 1V)

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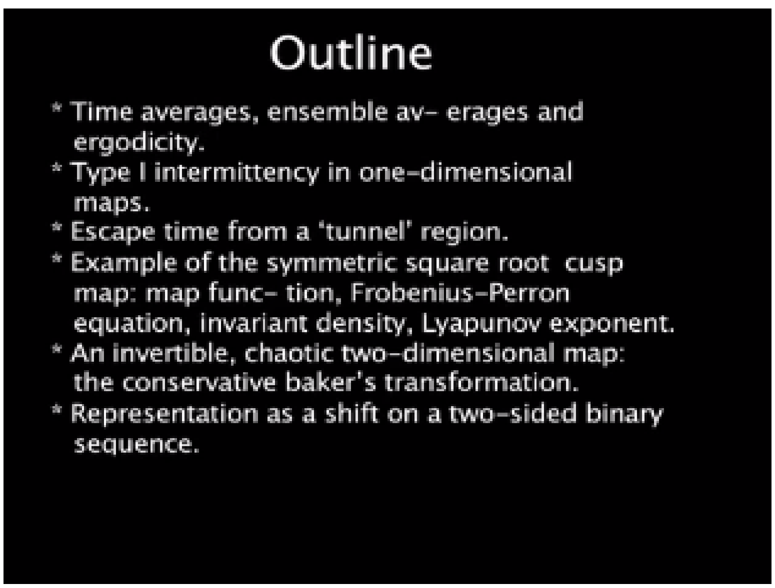

Yeah before we begin today let me answer a question that was raised which had to do with the erotic sampling and ergodicity the statement made was that if you have a time average you could in principle convert it to an ensemble average if you have a god city in the system and this is used in practice in sampling in experiments in physical practice it is used all the time and let me give an analogy for this suppose we wanted to measure we have a long roll of wire homogeneous uniform everywhere and we wanted to measure the resistance of this wire of one meter of this wire the two ways of doing this one of them would be to say .

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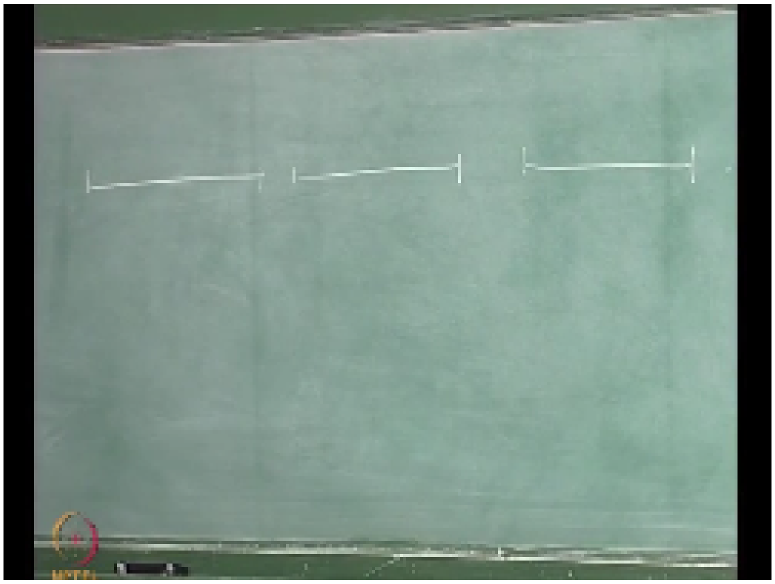

I have available to me a single piece a meter long and I measure its resistance and I do this over and over and over again and each time .I makes this measurement under identical conditions. I get a slightly different answer various fluctuations and I take the arithmetic average of all these measurements and that would be the time average of this resistance and it gives me a certain number we are as ng here that this wire does not in any way change its characteristic due to these measurements it does not become old it does not get heated up it does not get aged and so on and so forth the other way to do.

This to measure get an accurate value or average value for one meter long wire would be to take this long spool of wire and make a measurement for each of those pieces so the next piece is like this and the next piece is like that and so on and I measure each one of these guys and ask what the average value is this would be like an ensemble average and ensemble just means a collection a whole set of these identical copies and it is again assume that these pieces are all identical to each other and that gives me one average value and that may not always be practical to do you may have available just one meter and no more.

In which case to get an accurate value you would repeatedly measure the same thing under hopefully identical conditions and then assume that the .I am average that you get by measuring the resistance of one meter of wire a single specimen over and over again is the same as the ensemble average so this would be an illustration of ergodicity in the sense that you have replaced a time average by an ensemble average now what is implied in this whole business in some sense what is implied is that there are certain sources of fluctuations which lead to an answer which is slightly different each time.

When you make this measurement over and over again some random processes are operating internal external we do not care such that there are fluctuations in the answer when you make this measurement here the assumption is that the system as time goes along runs through all the realizations of that random process is leading to the fluctuations which would otherwise be reflected by the small differences in the different samples in other words whatever fluctuations are going on in time here that is already captured at the same instant of time in the different parts in the different copies or members of the ensemble which is why you equate the two and in dynamical systems as you can see when a system is ergodic.

It says instead of replacing instead of computing time averages long time average over a trajectory at a given instant of time. I compute a phase space averaged over different portions of phase space with some given measure and that is again the same ergodicity so it is precisely the same thing that is being done in both places and the hope is that under suitable conditions the random process or whatever else is determining the fluctuations here is such that a time average could be replaced by an ensemble average and this is at the root of all our goddess City yeah yes yeah this is a good question so deep question how do we know a given system is regarding first of all you have to specify a system much more accurately to do this given a dynamical system given the rules of evolution in our context.

We can certainly test if a system is a God it or not we can run a typical trajectory through or we can use other criteria for a goddess it if you know something about the dynamics if you know something in detail about whether it is expanding in one direction contracting and another and so on we may be able to prove a goddess City prove a goddess city in a rigorous sense of the word but in practice coming to this experimental question of no I am as ng that we have a dynamical system for which the equations are given to you the dynamical system is ah without an explicit solution yes oh yes you can still do that, yeah you do not need to be able to solve this system completely.

You can still prove that it is ironic you can still show that it is psychotic you do not necessarily need to solve the equations explicitly in fact in most cases. I cannot solve things explicitly so that is not the problem proving a guard city for a given the complicated dynamical system is not a trivial matter but it can be done in principle in most cases on the other hand what are you okay, the next question how do. I know how do I do this is going to involve something called coarse graining in phase space and then looking at what happens as the system visits different parts of phase space.

I will talk a little more about it a little later so it involves other criteria other quantifiers for ergodic behavior which we have not yet considered such as I break up the phase space into cells sufficiently small cells and keep track of where a representative point is I can do that numerically without actually solving the equations of motion and then depending on the statistics of how different parts of the phase space are visited and filled up I can decide whether the system is ergodic or not I can see what the dependence of various visits the statistics of recurrences to various cells and how they change how this thing changes as a function of the cell size that will give me another.

Indication or whether the system is a gothic or not so there are quantifiers for a God city on the other hand for a purely experimental question such as this the resistance of a piece of wire so I am talking about in practice it is clear you do not have a large copy you do not have a very large sample you certainly do not have two kilometers of wire to play with if you did then the ideal thing to do would be to cut it into one meter pieces and measure for each one of them what the resistances and take the arithmetic average since you do not have that you take it as an article of faith that whatever fluctuations are currently remaining portions are all present.

In a given sample over a period of time given enough time and therefore you use just one sample but you repeatedly make measurements in order to find an average value and then the hope is that these two averages are exactly the same thing if the system is ergodic if it's got it so this is the point of what it in fact when you do statistical mechanics or thermodynamics you are using a coda City essentially because what is going on is that you put a macroscopic system under given experimental conditions such as for instance in contact with the heat path in thermal fluctuations a thermal equilibrium.

In contact with the heat bar and then you assume that the system is ergodic in other words a statistical average over a given ensemble with a certain probability distribution is enough to give you long time averages for the system because you cannot follow the trajectories of a complicated set of interacting particles in time but you assume that whatever information you would have got by doing the time average is already there when you do the ensemble average and then it only remains to write down the correct measure the correct probability distribution and that is the task of equilibrium statistical mechanics so once again you do precisely that now what is the reason why you are able to do this again.

You do not have an infinite number of copies of an ideal gas in a container although when you derive statistical mechanics or the rules of statistical mechanics you assume you pretend you do and you have a infinite number of copies to take averages over but a given system a single system the gas in this room for example runs through all the realizations of the random processes involved given enough time so this is the whole point for instance in different samples the molecules would be at different positions at the same instant of time the assumption is given enough time the molecules of a single specimen would run through and assume all those positions.

So you certainly have to do an averaging for a sufficient amount of time that you think you've got a satisfactory enough long time average which could then be compared with the ensemble average now since equilibrium shun time in systems like this is very short this is actually true in most cases in practice need not always be true if the relaxation times in the system are extremely small then this is no longer true if the system is extremely sluggish and there are time scales for equilibrium shin or the running through of all the realizations which are much larger than the time scale on which you make measurements then the system could get stuck.

In one or two preferred configurations then it is no longer a true average this happens in many systems it's called glossy dynamics it is again happens once again when you have extremely sluggish systems with very complicated what are called very complicated or rugged energy landscapes so you do not have clear free energy minima but you have minima on all local minima on all scales and then the system could get locally trapped in some place and take a long time to get out of it.

And then of course it is not easy to take averages after that in such situations so they do these situations do occur in physics everywhere but for us in the study of dynamical systems the equations themselves specify everything so in principle that is all the information you have and everything has to be derived from there. So let us go back now and consider what one dimensional maps had to tell us we looked at the logistic map.

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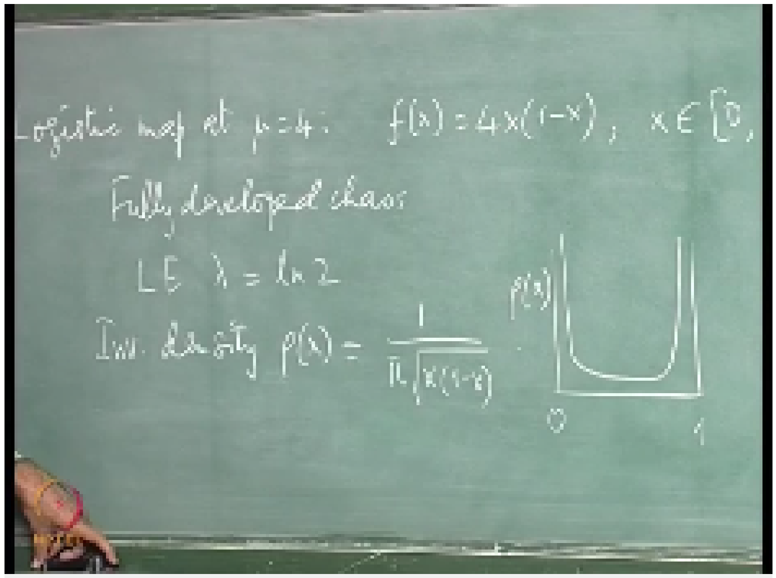

We looked at it at fully developed chaos and we discovered that it had an invariant measure and invariant density which was non-trivial and had a kind of square inverse √ shape so for the logistic map at $\mu = 4$ this was a map f of X = 4x times 1 = X we had fully developed chaos the Lyapunov exponent λ was = log to the same as for the Bernoulli map and the invariant density this was =o 1 √ x times 1 -X k so this is where we have got once you have this then it is not hard to compute various physical quantities because the average value of any function of X is simply its weighted average with this invariant density and that is guaranteed to be the long time average and of course you know that apart from a set of points of zero.

Measure all points in the unit interval lie on chaotic orbits and these orbits wander back and forth without settling down anywhere and they fill up the interval according to this density which looks like this ok now there are certain universality is about this map which are common to all one hump maps of the unit interval we will talk about some of these but there is another phenomenon. I would like to talk about today and that is the phenomenon of intermittency comes in many varieties and very roughly speaking it is the phenomenon by which a chaotic system displays periodic behavior.

In between or apparently periodic behavior instead of being fully chaotic for long intervals of time and then it is followed by bursts of chaos followed by bursts of lamanrity which is regular behavior of some kind or approximately periodic behavior so if you look at the time series of any variable such as X it would not show the truly chaotic up and down motion in an intermittent situation it would actually show long bursts of periodic behavior and then all of a sudden once again you have chaotic behavior now how does this phenomena arise several routes to intermittency as I pointed out.

But the simplest one of these is the following they can draw in pictures and show you what happens.

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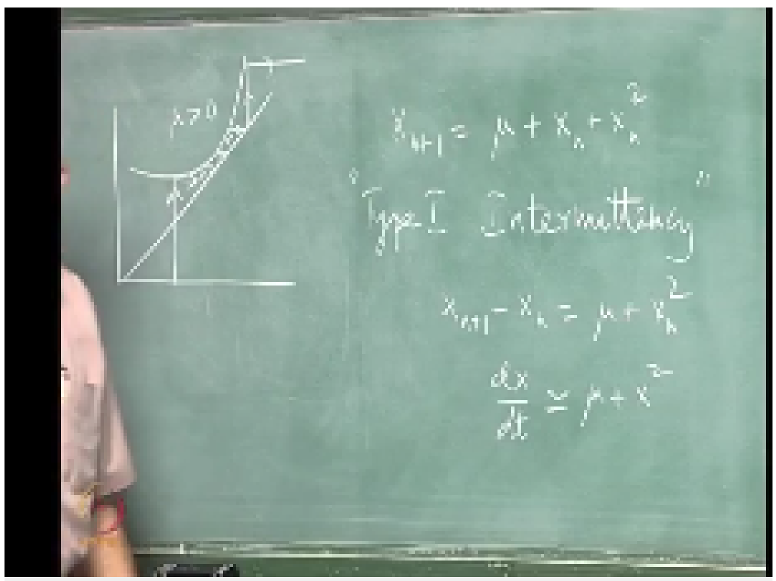

Suppose you had a chaotic one dimensional map here is the bisector which looked like this and this was the map function it is easy to see that in this case you have an stable fixed point here and an unstable fixed point here where the slope is >1 so if you start with points in between they would end up at this point you have started points here it would end up in this stable fixed point now suppose you varied a parameter such that this map function as the parameter very moves up and there comes a stage when it is got a tangency at that place so the slope is 1 at the point of tangency and then as you vary the parameter further the thing moves out and goes off like that there may be other portions.

To this map function but locally in the neighborhood of this point where of tangency suppose it looks like this as you tune a parameter and an example shortly what would happen and here is an example right away suppose you consider xn + 1 = some constant μ + xn +s X and ^{2} for instance near the origin so I have shifted this point to the origin and the map looks like that what would happen well clearly if you had the map near the origin if mu is 0 then 0 then it is xn +X N ^{2} and it looks exactly like this there is a point of tangency with slow plus 1 and then it takes off like an X^2 so this would correspond to μ = 0 and this would correspond to positive values of μ.

But it just moved off from there completely and this would correspond to negative values of μ so as you cross μ =0 from left to right the picture would go like this from here to there now at this place at this in this situation there is no problem this thing here is a stable fixed point things get attracted to it at this point you have something that is marginally indifferent marginally indifferent marginally slope = 1 so the fixed point is marginal here as you come in here things would flow into this point but if you started off on this side things would flow out so you would have a behavior like this in here could go in but if you started here things would get out in this situation if I start here I go to this function.

I come here, I go here, I go take the staircase route and then I am off as you can see if this is infinitesimally close to this point the time the bisector there is a tunnel region where the system takes a long time to get out of this tunnel region and then eventually it does and does chaotic motion elsewhere and then once it gets trapped in this region again for a long amount of time there is again approximately period regular behavior it is not really going anywhere it is stuck in this tunnel and the moment.

It clears the tunnel it goes off gets out and comes back we can even estimate how long it would take to cross this tunnel behavior this tunnel region and this is type 1 min intermittency the simplest type of intermittency we can easily estimate how long it would take to get out of this tunnel region as a function of this parameter mu which is supposed to be infinitesimal here so mu just a little bigger than 0 μ = 0 and this was new <0 so let us look at this map here and say is xn + 1 is μ + xn + X^2 so it is clear that xn +1 - xn μ+ X^2 and in this region the dynamics is essentially differential dynamics because it is making ever.

So small steps and I can replace the difference equation by a differential equation in time and it is clear this thing here is just the first derivative so it looks like DX over DT is approximately μ +X ^{2} itself in this region which have drawn very exaggerated way but the solution that is obvious.

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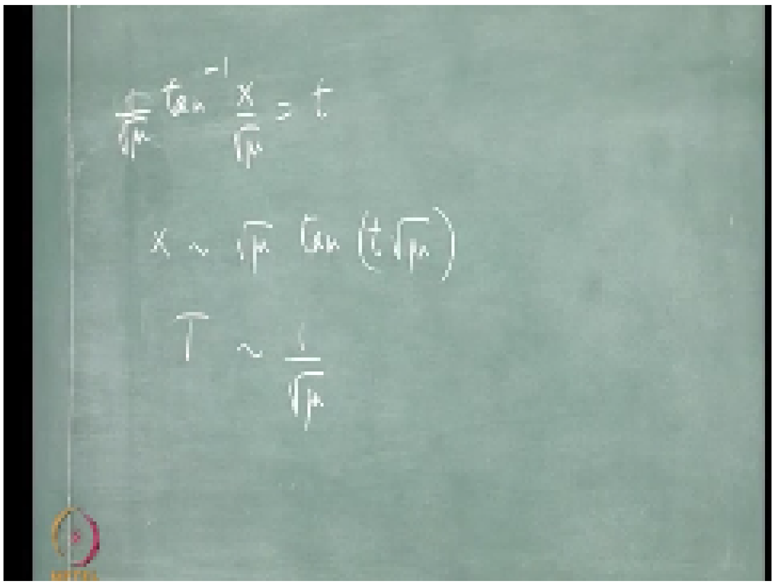

Because it says √of 1 /√ μ can inverse x over √ of μ =t as ng that i start near 0 at the odd at T =0 and I move out it is of this form so that immediately says that X is like √ μ and T roomier in other words to reach a point X to the right of the origin you need a time which is related to the space X by this relation here and what happens to this when T hits T root μ hits π/2 becomes infinite.

So essentially it says that to cross this tunnel region the time of crossing T is of the order of 1 / √of μ that is the time it takes to get out of this tunnel region in an order of magnitude way that is the reason why if μ is infinitesimal and you get closer and closer it is going to take longer and longer in other words the laminar interruptions the laminar regions in this chaotic time series are going to become longer and longer and it looks like the system is not chaotic at all but in reality it is except that for long bursts of intervals of time you would SC see essentially periodic behavior or regular behavior now this kind of thing is seen in experiments in a variety of situations in liquids for instance very well known.

That there are models of liquids dynamics of liquids fluid dynamics this appears all the time there are many other areas in semiconductor physics chemical reactions and so on where intermittency has been seen different types of intermittency have been seen the reason I said this is called type 1 let me mention this very briefly and perhaps we will come back to this little later is the slope at this point becomes 1 this is in marginal fixed point but in higher dimensions if you have maps in more than one dimension then the marginality appears not when the slope hits one alone.

But in the eigenvalue plane of the local Y acoubian matrix every time I glen values cross the unit circle you have this kind of behavior you have marginality the three ways in which I glen values can cross this unit circle one of them is to cross the value one which is what happens here in a one dimensional map they could also cross - 1 this direction and then you have what is called type 3 intermittency which perhaps. I will come back to later and then you could have a pair of Eigen values crossing at complex conjugate points and this will only happen in 2 or higher dimensional maps and then you have what is called type 2 intermittency we will try to come back to this.

When we do higher dimensional maps but right now in one-dimensional maps the slope crosses the value 1 and this is type 1 intermittency and what you need to know is that this phenomenon is very common and it is part of chaotic dynamics and the travel time through the tunnel region can increase scales like this parameter μ like a 1 over square root of μ now let us try to study this in a little more detail and see what happens in a map which we perhaps could solve and see the effect of this marginal fixed point.

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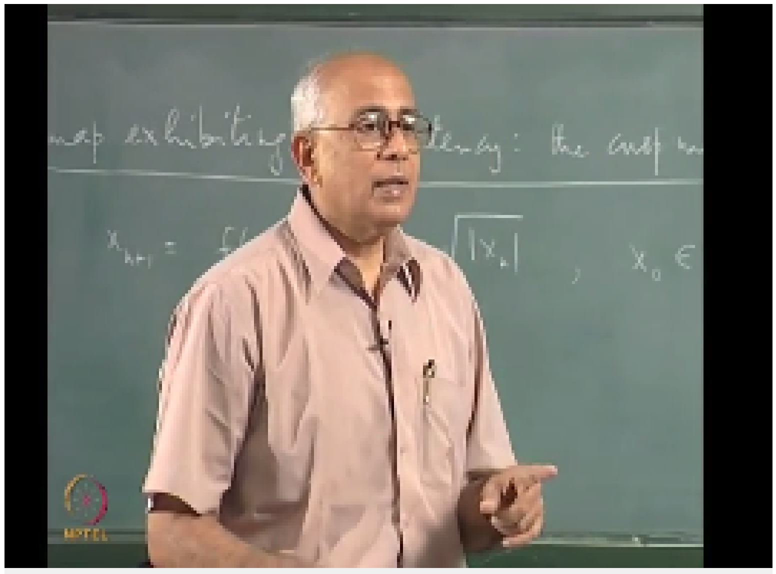

So let me do a map a map exhibit yeah there is no chaos everything gets attracted there yeah ,I said that the time that I evaluated was for $\mu > 0$ the situation > C when it was < 0 it just got stuck there that is the end of it what good will it to when eventually things are going to fall into the fixed point right but as I am interested in finding out what the time scale or the way this the intermittent region scale as a function of the parameter in a chaotic situation in a chaotic night when you have a fixed point it just falls in so it is not upgrade interested in the stable region it will fall in yeah so that oh but that integral is not true anymore right and this is not true I mean DX / $\mu + x^2$ is = 1 over $\sqrt{\mu}$ tan inverse x $\sqrt{\mu}$ if μ is negative right then it becomes logarithms and so on so it is of course.

The tan inverse function and the log function are essentially the same by analytic continuation right but the interpretations are very different altogether so it is not just the same tan inverse function okay ,let us look at a map exhibited intermittency let us call it the cusp map again this is a map of an interval this time say for convenience I will take it from - 1 to 1 and it looks like this xn + 1 = f of X sub n = 1 -2 $\sqrt{}$ modulus x n and x 0 is an element of yeah between mu equal to 32 for the logistic map has regions of values of μ where the intermittency is displayed once again but the intermittency is not in the map function itself that does not have a slope but you will easily recognize that iterates of this could have this behavior.

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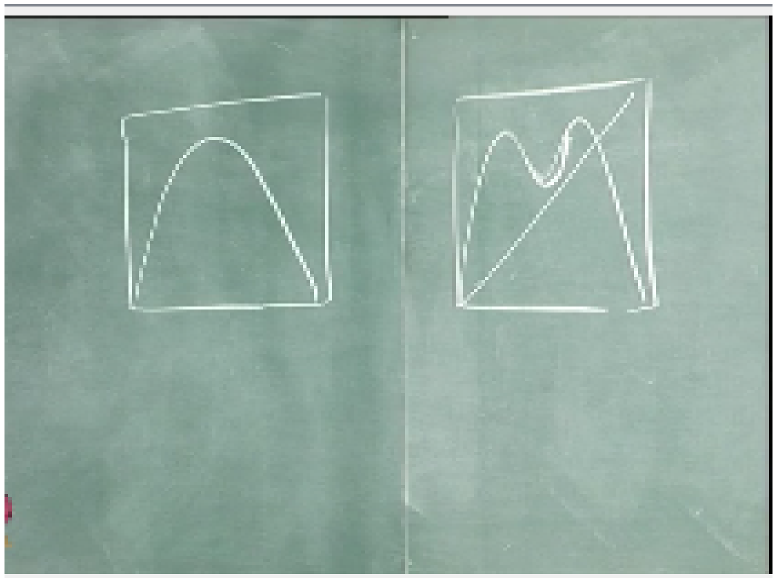

So again a digression the logistic map itself perhaps look like this at a value $\mu < 4$ for instance on the other hand the iterates of the map would start looking like this so the plenty of opportunity for reasons like this to be set up there is plenty of opportunity for regions of that kind to be set up which could then lead to intermittency because it is not just the map function that determines the dynamics but all it is a traits as well so that is why the map does exhibit intermittency in between not at fourth at four it is fully developed chaos it is not intermittent between $1 + \sqrt{8}$ not quite up to 4 1 + rotate to another value numerically determinable the map has stable period three cycles has a stable period 3 cycle.

So this is actually periodic it is not intermittent it is actually periodic it is not chaotic in that region at all so the chaos disappears and then it comes back so now you could ask how does this happen we will do a little more. I will bring we will talk about it numerically let me show you the exact bifurcation diagram for the logistic map when you this has been well studied what happens is at certain parameter values they could be collisions between the chaotic attractor and unstable fixed points and these could lead to things..

to four because it is not an on to map unless mu is for it does not fill up it does not map the called crises so that with different kinds of crises there are boundary crisis there are inside interior crises and so on and they could lead to sudden changes in the nature of that tractor and that is what happens in the logistic map.

So to sum K yeah < four yeah yes there is when I say fully developed chaos what I mean is the following the entire unit interval is the attractor that is not true unless μ is = 4 because it is not an on to map unless μ is for it does not fill up it does not map the unit interval to the full unit interval but rather to a point which is < that. Something like $\mu/4$ it maps it from zero to μ over four and unless μ is one you do not hit the full interval.

So let us look at this map and see what happens yeah is another question pardon me yeah yes yeah the system gets stuck so the stickiness if you like many dynamical systems of this kind including some Hamiltonian systems, where you do not quite have this kind of phenomenon but you have stickiness of some kind and we will talk a little bit about that too and here you see the mechanism by which it gets stuck as you can see.

And so very when μ is small you could see it is really could get stuck for very long periods of time but there is no doubt that it will escape eventually and then the system becomes chaotic again. So it is laminar bus occur in chaotic bursts occur in the middle of laminar regions and vice versa, so that is really what intermittency is. So let us look at this map this will fix many ideas of intermittency clearly it is a map which is solvable where you can actually write down solutions and so on explicitly in the following sense.

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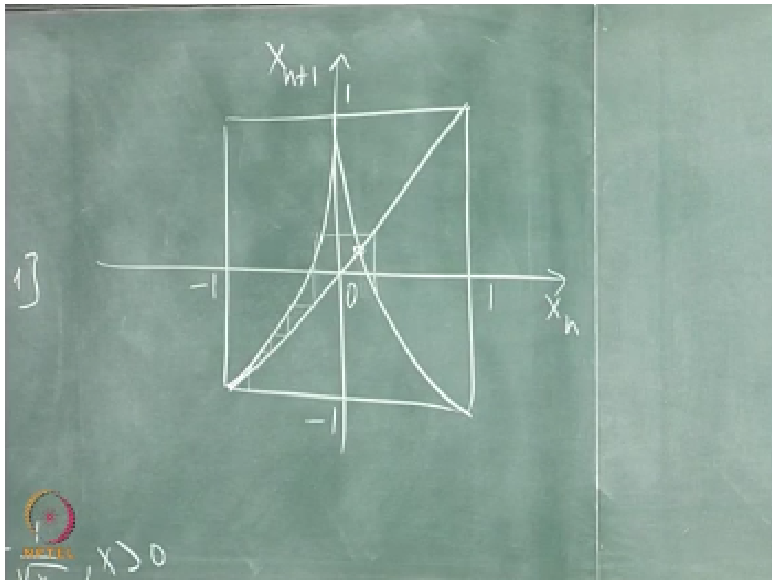

So let us first draw it and the let me draw it here and then I come back there this map is from - 1 to 1 so let us draw a little square here from - 1 - 1 1 and 1 here and this is f of X or if you like xn + 1 as a function of X of N and that is the origin. Now this is the bisector once again and it is clear from here that this is =1 - y √- xn for X and negative and it is =1 - y √x n for x and positive and the slope at any point if i differentiate this.

And compute this f ′ of x is =- 2 over twice √ - x and then a - 1 again, so this is =1 over √ - x for x <0 and for x >0 it is 1 over f ′ of x =- 1 over √ x x >0. So the slope diverges at the origin from both sides this is a sort of cusp air and the slope at x =- 1 is + 1 and x =+ 1 is – 1, so it is immediately evident that this graph goes like this is infinite at that point at that point and then it is symmetric and falls back there in this fashion and this slope here is exactly one and for any point X >- 1 it increases.

So this is the region of the marginal fixed point where you could get stuck for a long period of time and in fact what would happen is that you do this staircase thing here and eventually go up go there and then you get re injected and so on and so forth. So this map oh by the way the fixed point here is unstable the slope is >1 it is easy to check now what would iterates of this map look like.

Yeah they would even be even steeper than this for example the first iterate would look like this go up like this and come back the slope here would always remain one for those maps. So they would always be this marginal fixed point but then you have unstable periodic points and all the periodic points are unstable and the map is actually chaotic but you have the effect of this marginal fix marginally unstable fixed point here and therefore you have the phenomenon of intermittency.

In this case we could write the Fresenius Peron equation down for this, so let us do that we need to be able to solve this guy right down the two √s and then compute what the invariant measure is would you like me to do that or do I take it that you will do it yeah you can also try it.

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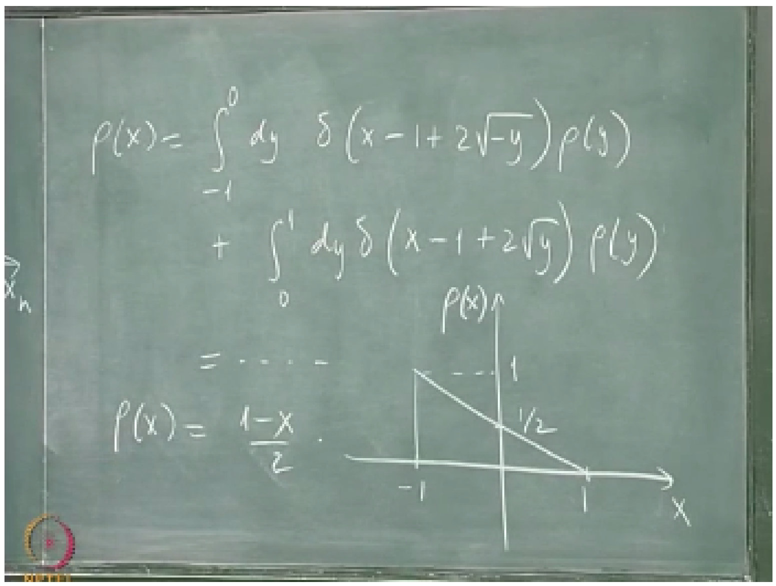

All you have to do is to write the provenance para equation - 1 to 1 dy, so let us write this as - 1 to 0 Δ of X - F of Y but f of y 4 y negative is x - 1 + twice - y in this fashion ρ of y + = 0 to 1 dy Δ of X - 1 + 2 √ y. Now you have to convert this to a δ functions in y find the slope of the functions at that point divided by the magnitude of the slope and you get a functional equation for ρ of X so this is =some functional equation which is fairly complicated on this side.

And the question is does this equation have a solution or not it is non-trivial again the functional equation but this solution has been found and the exact answer in this case and the normalized distribution is 1 - x over 2 which is quite remarkable because it is an explicit function of X the linear function of X looks quite simple and if you sketch this function, since ρ of X must be non-negative, so let us plot ρ of X here versus X it runs from - 1 to 1 and at - 1 it is =1 and at one it is =0.

So it is simply a linear function of this kind so this is a half and this is one that is what the invariant measure looks like. Now in your terms in crude terms why do you think the map is symmetric about the origin, but why do you think the invariant density is piled up on the left rather than on the right rather than being spread out uniformly. Yeah there is sort of this thing here is doing it this marginal fixed point it is unstable it is marginally unstable but because of this intermittency because of the fact that the system spends long periods of time here.

Remember the measure in any region is proportional to the fraction of the time a typical trajectory spends in that region and it is clear that the typical trajectory chaotic trajectory would spend a lot of time here. So that is reflected in this fact here but unlike the logistic map where you actually had unbounded invariant density and very little at the middle here it is not like that it is actually quite bounded its linear goes down in this fashion and the area under the curve is one.

I might add that this stickiness is actually sufficient to prevent you from having an invariant density of this kind in fact you would have just a δ function here things will get stuck here the only normalize able solution would be a δ function here unless you had this ∞ here for reasons I will not go into here you need to have some point in the map of other than the fixed point where the slope actually becomes ∞ infinitely shock, then under suitable conditions you can have an invariant measure of this kind.

So what I would like you to appreciate although I am not proving this is that the behavior here and the behavior here are related to each other you need to have yeah, suppose you do suppose or you instead of cutting it off suppose, you did something like this suppose you came along like this and did this or something like that so it is an then it is not an on to map I would like to have a non to map I would like to have - 1 to 1 map down to - 1 to 1.

No not at all you do not need to have it yeah oh yes yeah okay you need not have an invariant measure of this kind you need to have an invariant measure, for example if you took a map like this then there is no guarantee that you have a map an invariant measure of this kind you could just end up with a δ function here and nothing more yeah, the behavior will change the invariant measure will change.

There is no normalize able solution which is non negative of this kind at all you could just have a Dirac measure here you could just have a δ function here and the system gets stuck because what would happen in that case is that instead of chaos, the maps Lyapunov exponent would drop to zero because the effect of this stickiness is so strong it prevents the chaos from happening it actually makes the Lyapunov exponent zero.

And you need to have some place here with infinite slope unbounded slope in order for it to actually be chaotic yes they may not young. So the statement is there is no normalize able invariant measure which would do this, there is no normalize able invariant measure you need this thing to be normalize able this density to be normalize able it should not get singular suppose for argument is sake the density went like 1 over X + 1 what would happen then?

You cannot normalize that you cannot integrate from - 1 upwards because it is not it is it not as a integral singularity at all, so this can happen yeah because you cannot decouple the two because anything that comes here is bound to also go there sooner or later does not matter. Now we are talking about what happens about reinjection we are talking about the entire dynamics not a single passage talking about it has to once it gets re injected here then that differential approximation I made would be a reasonably good approximation.

Provided it gets re injected in a finite amount of time right, so it is the map is actually exploring the entire phrase space it is not that trajectories are just exploring the neighbored of this in which case the behavior is trivially determinable but you need to know what is happening everywhere else what sort of collection region do you have, what sort of reinjection do you have and so on all of them play a role.

And the statement I am making is that in order to have a normalize able density like this you need to have and I am not proving this statement you need to have something which has an infinite slope the slope has to become unbounded at some other point. I urge you to verify that this is indeed a solution to this equation you have to first convert it to a functional equation and then verify that this is a solution.

And as I said if you have a non-negative normalize able solution you are guaranteed by certain theorems that the solution is unique. Now where does that get us we need to know what kind of we have the invariant measure we need to know whether it is chaotic or not so the first thing we do is to find out what the Lyapunov exponent is let us do that for this map.

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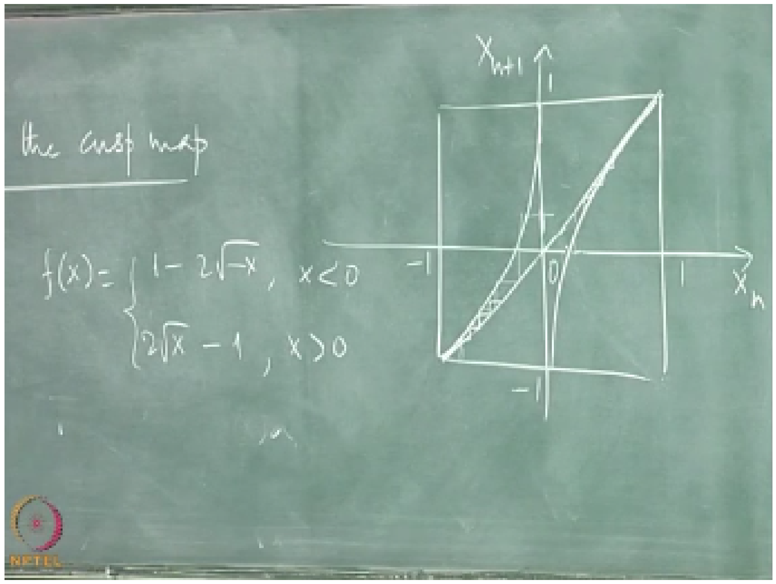

So the λ for this map is integral - 1 to 1 DX 1 - x over 2 times log of the slope of F ′ of X the modulus of that and that is =log of 1 over the √ modulus x because the slope was 1 over √ - X for ex- and - 1 over √ x for x positive neither case by took the modulus it is just this number here, so it is mod x to the - half modulus x in this fashion of course it is an even function and that is an odd function.

So that portion goes away and you are left with just an integral half log mod X it is easy to do and I usually do this and the answer will turn out to be a half notice this is negative in the region of integration so that cancels the - sign and you end up with a half, so this is certainly >0 implies chaos but it is intermittent chaos. In fact finding this finding this invariant measure invariant density numerically is non-trivial.

You do this by finding a taking a long time series and drawing a histogram and this thing takes a long time to build up here so you really have to run for a very long time you have to run a trajectory and you have to leave out the initial transient switch would be specific to the initial current given initial conditions and then eventually you end up with this but this is an analytic solution. You can easily check that this is an exact solution to this okay.

So we have a chaotic map with a chaotic map you still have intermittent behavior and this is exactly solvable so it is like a paradigm it is like a model like the logistic map, now of course you could make this uniform you could make this density uniform without changing any other properties of this map by taking this portion of it and doing exactly what we did for the Bernoulli shift instead of the 10th map in other words do this it is in this fashion.

So this map function would be f of x =1 - twice √ - x for x negative but it will be twice √ x - 1 for x positive and that is sufficient to make all the difference because, now you have a marginal fixed point here and you have a marginal fixed point here too and they compete with each other. So you do not have any right to expect an invariant density of that kind you can write down the functional equation once again I you to do this for this map and verify that in fact the invariant density is a constant.

So for this new map the invariant density is simply this, so the area under the curve is again one and it's just a constant. So ρ of x is just = 1/2 for this anti symmetric map the earlier map was symmetric but the invariant density was not symmetric on the other hand here the map is anti-symmetric but the invariant density is symmetric. In the logistic map and the 10th map at parameter value to with slope to the map function was symmetric about the midpoint and the invariant density was also symmetric.

This does not pardon me at well it will have intermittency, why not why not necessarily true not necessarily true, because what has happened in the crude sense is that these regions have kind of overlapped in this case. So once again it is easier to handle than the other map simply because the invariant density, so I do not have to take weight it with any function of X everything is uniform here yeah but the way it jumps from point to point is not necessarily periodic.

Whereas here once you are here I am drawing this in an exaggerated way but once you are here there is a long staircase behavior that behavior is lost in this middle region and again out here there is a long staircase behavior which is lost in the intermediate region. So definitely it would look very periodic but not with the constant amplitude slowly increasing maybe something like that when you are in these regions, so that is what intermittency is it is not as if it's strictly periodic any a function which is monochromatic or periodic would just go on forever - ∞ to ∞.

So the periodicity stops definitely and if you look at it with infinite accuracy certainly it is not periodic there is no single pleaded discernible, yeah the same oh yes I agree I agree because of this because of this once you have a density then the time the fraction of the time T over a long trajectory the fraction of the time that it spends in any interval is in fact proportional to a to be pro of x of x for a normalized invariant density.

So this is certainly true this is a fraction, so if you took a long orbit and you asked how much of the time does spend in given region that is measured directly by the integral of the invariant density the measure of that region measured it is given directly by that that is certainly true. So this map has interesting properties so does the other one but the other one has this extra feature that the map is symmetric whereas this one does not.

Now you could ask why did I choose why did I choose something which went like X + perhaps x squared here if we expand the map near X = - 1 the map function would have a leading term like X + 1 and a term which goes like X + 1 whole ^{2} and so on and so forth so let us see if we can generalize that a little bit and let me shift it to the origin.

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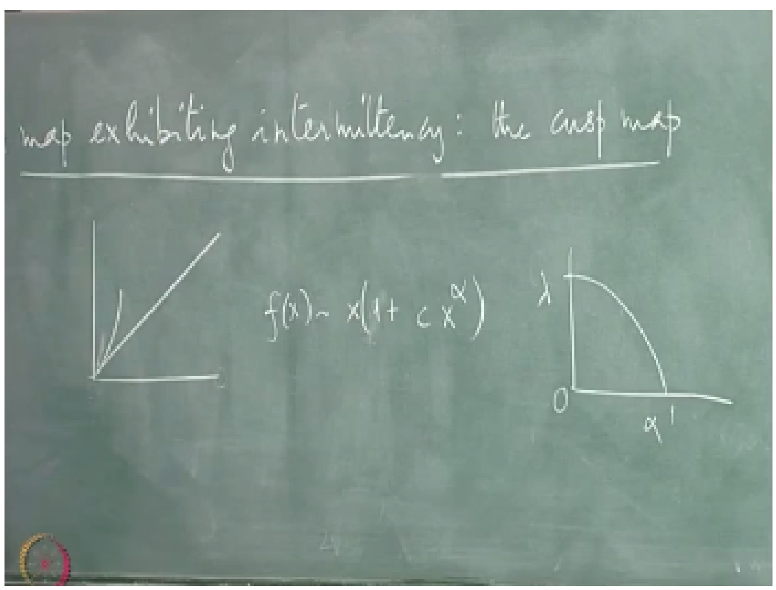

So let us suppose your income it and maps are like this here and this map function f of X near the origin is like X that is the leading term + something or the other so + some constant may be multiplied by X $^{\alpha}$ let us take out an X 1 + this thank you this is typically what would happen near the origin α if it is a constant, if α is 0 first of all the nothing interesting happens it just is a map which is going to look like this straight away linear with the slope >1 because it is slope is 1 + C.

But if α is not 0 but a positive number generically if you have a function which is expandable smoothly around this point α would start with one and then higher powers go on like in the case of these maps this map can be written in the neighborhood of this point as X + 1 + a term which is X + 1 whole^2 and so on in this neighborhood but in principle you could have an α which is positive does not have to be 1.

Could be >one could be even <1, so the degree of stickiness of here as you can see is measured directly by α and for such maps you can actually plot the Lyapunov exponent as a function of the parameter α and it turns out that if you plot it λ versus α. Then if you are if you start at some point like that which is certainly positive because if α 0 this is this map with slope 1 + C and this is unstable and the assumption is the map is chaotic by its behavior elsewhere.

So it start with some Lyapunov exponent and as α increases it is getting more and more sticky here and therefore the Lyapunov exponent actually drops in this fashion and it turns out at α =1 it drops to zero and the map is no longer chaotic some kind of phase transition takes place. On the other hand we also know that in the cast map for example there is a Lyapunov exponent which is non zero which is positive and that happens because of the infinite slope elsewhere.

So you actually get rid of this stickiness of this point by having a sharp spike somewhere else in the matter, so they do not fall in this class these maps do not fall within the purview of this general statement here this is a technical aside I want to get too deep into this but let me go back. You could also ask can I construct maps of this kind can I construct a map where for example I have not a square √ cusp up there and the square √ behavior in the map function which had √ x.

And so on what about x $^{1/3rd}$ or x to any other power cube √s and so on the answer is yes you can construct whole families of these maps and whatever exponent α you have here you need to have a 1 over α type behavior on the slope up there roughly you need to have something which becomes unbounded. And therefore the nature of this stickiness here can be related to the nature of the cast spells where to get a finite point in fact you could construct an infinite family of such maps.

For which you have invariant densities which are all sorts of prescribed functions of X not necessarily linear functions this is the inverse Fresenius Peron problem if you give me a smooth function. As an invariant density can I construct a map of which this is the invariant density that problem can be solved modulo certain concern qualifications and it turns out a huge variety of such functions ρ of X you can actually find you can tailor make a map whose invariant density the given function would be.

So that is actually done turns out we are non not too difficult problem yeah yes because of where yeah pardon me I can find the map function yeah I can find the map function, it is a non-trivial problem but it is doable after all the Fresenius Peron equation is an eigenvalue problem. So in some sense you are saying you are given me the eigenvector and now you go back and construct the colonel this is yes the whole thing is to only for certain class of chaotic systems absolutely and in one-dimensional max.

So things are restricted to one dimension very much, so I do not know how these things I do not offhand have direct statement about how this generalizes to higher dimensions I am not too sure so for one-dimensional maps a great deal is known much is known about these chaotic maps. Yeah the whole thing is we are dealing only with chaotic systems. Now let me go on to we will come back a little bit later to understanding what coarse graining in phase space is and so on but let me go on to a two dimensional map such maps exist too.

And let me give you in particular an example of a very simple two-dimensional map which is invertible and yet exhibits chaos and this is in fact a model which is used as a model for Hamiltonian systems, where as you know things are conservative. So we will look at an area preserving map where you have chaotic behavior unlike the case of the Bernoulli map or the logistic map and so on they do not model conservative systems but now we are going to talk about a map which models a conservative system and yet exhibits chaos.

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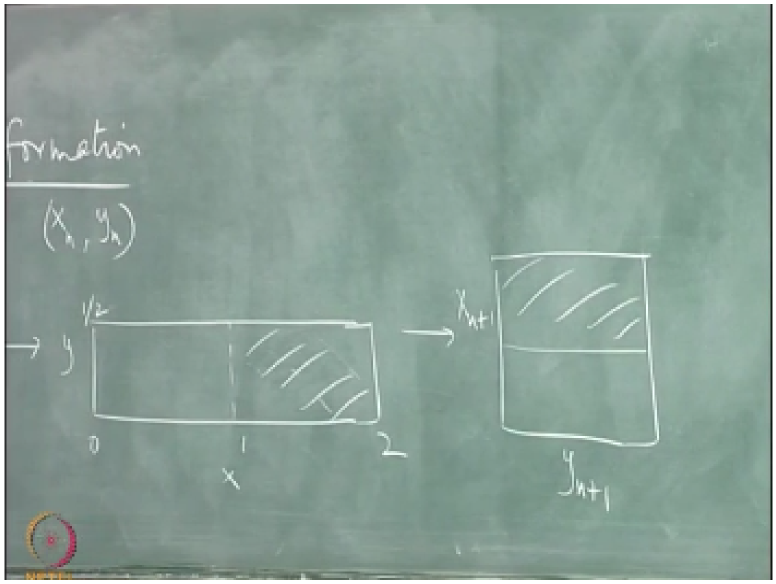

And the map is the following it is called the Baker's transformation or the Baker's map because it is supposed to mimic the way in which bakers make bread from dope, Baker is not a proper name it is common noun and it is supposed to model the wave bakers make dough which is take the dough to stretch it and fold it back and they keep folding it back and this mixing is what is producing chaotic behavior eventually we saw on the Bernoulli shift or the 10th map you stretched and you folded you stretched and you fold it in one dimension.

Now let us do this in two dimensions so the map looks like this you have two variables X_N and Y n at time n and each of them runs between 0 & 1 0 & 1 and the map function looks like this, so here's X n here is why n 0 to 1 and you take this square and do the following manipulation on it stretch this ^{2} by a factor of 2 in the X direction and compress it simultaneously in the Y direction by a factor of 2.

So at the next stage this becomes like this so this is X and that is why and 0 1 this is 2 and that is a half and then cut this piece exactly as you did in the Bernoulli shape and put it on top here so this goes off to this and this is xn + 1 Y n + 1. So just to see what we have done let us do the following let us take this map and shade this region the other half so I keep track of where it is gone and when I extend it that shaded region has come here.

Now I content put it back and that shaded region has gone there every point on the square has been mapped onto some other point on the square but this is the transformation, what is the actual function and it is also clear the area has been preserved we have not done anything at all you stretch this side by a factor of 2 but you also compress the other one by a factor of 2 and you put the square back onto the square but you mixed up things here.

So a point here we will go somewhere else a point there we will go somewhere else and so on in this map and what is the map function look like.

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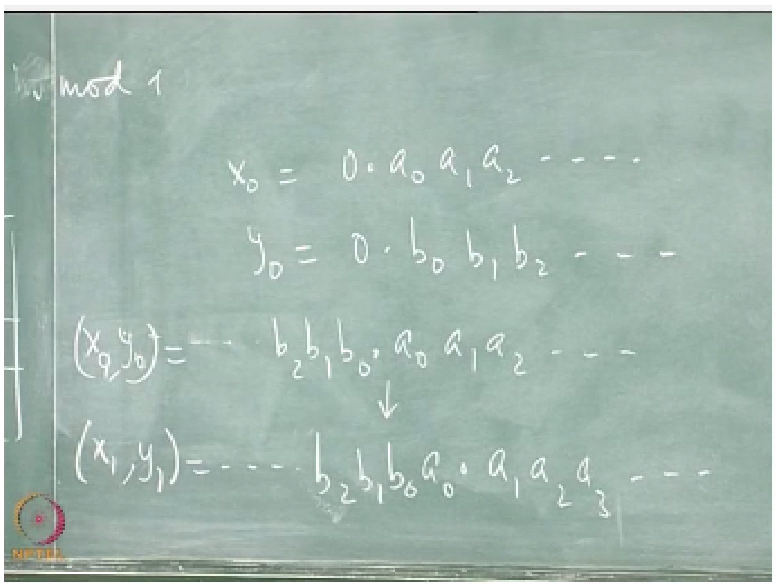

Well it says xn + 1 =2 Xn modulo 1 right because you are going to extraction then you are going to put things back YN + 1 is =1/2 Y_N brought it down provided xn was between zero and a half otherwise you added a half to it, so you compressed it but then you cut and pasted it back so you actually added a half to it right so this was =this so provided xn is < a ½ but it was =half + half x n xn was >a half and the whole thing is modulo 1 everything is modulo 1.

The map is invertible it is definitely invertible because i can tell you precisely we where each point came from there is no 221 business here every point has a unique reimage, if i started at some point here i stretched. So it went to double this and it came to half its height on this point here if i started at some point here, then when I stretched it came down somewhere here and then I cut it and put it back so it went up somewhere here so certainly you can identify the pre image of every point and the area is preserved.

So it may mix a conservative system on the other hand there are two Lyapunov exponents, one corresponding to the stretching or contracting in the X direction than the other in the Y direction so you have to lay open of exponents and let us call them λ sub X and λ sub y and these are easy to write down by inspection. What would these be what would be the stretch factor in the X direction it is the original Bernoulli shift.

So this is log 2what would λ Y be log a half absolutely right and it is clear that the area must remain the same because after all under this map what is happening is that dx dy is going to DX ′ the y ′ this is the area element if you like and this is supposed to go like any area element expands, so it goes like e to the power λ X + λ Y t dx dy that is the whole point about the Lyapunov exponent and it does not change, so this cancels that so the map is area preserving.

But it is definitely chaotic definitely losing information because what happens to the next what happens to the next iterate of this rate once more and I do the same thing then it is not hard to see but you are going to have something like this region is good get scrambled up a bit it is going to look like this you increase it a little more it is going to get even more striated and so on. So you are going to have a you take a cross-section here you are going to have a sort of fractal structure this whole thing.

So certainly mixing up the entire thing so if this is pulling and cutting and putting it back cutting and putting it back is like a baker transformation and the fact that it has one positive Lyapunov exponent this is enough to show that it has chaos. Yes but now I pointed out that this map and I will stop with this but this map is invertible so you should really be able to recover you are not losing any information in the sense that you should be able to say where you came from is that really true or not.

We saw in the Bernoulli shift the way to understand the shift was to write x not a zero point a naught a 1 a 2 etcetera in which case you ended up with X 1 is zero point a1 a 2 a 3 not because you x 2 and throw away the integer part which means you got rid of the knowledge of a zero. What would you do for the bundle for the Baker transformation? One of those goes to the y coordinate.

So there is a clever way of doing this and I will stop with this in the clever way is to say suppose X_0 is this and suppose why not the initial is zero point B_0 b1 b2 or not right the pair X_0Y_0in the following strange way, so put a dot here and write a not a 1 a 2 and write this guy backwards V naught b1 b2 in this fashion represent the pair X_0 Y_0 by this strange number and then in the next shift all it does is precisely what it did earlier.

So you have B to B 1 B_0 a_0 dot a1 a2 so if this is X_0Y_0 it is represented by this then x1 y1 is represented by that half which was sitting here and is moved over to this side so you haven't lost anything in other words you can tell precisely where you came from so this in fact establishes that the map is invertible you are not losing anything at all and yet you have chaos.

So it is important to remember that chaos does not imply always shrinkage of volume and loss of information you still have exponential sensitivity to initial conditions but yet you could have something which is invertible in the sensor completely solvable if you like and still display exponential sensitivity. So I stop here this time and then we will take up some other aspects such as course next time you.

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