Indian Institute of Technology Madras
Present

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TOPICS IN NONLINEAR DYNAMICS
Lecture 2
Critical points of a dynamical system.
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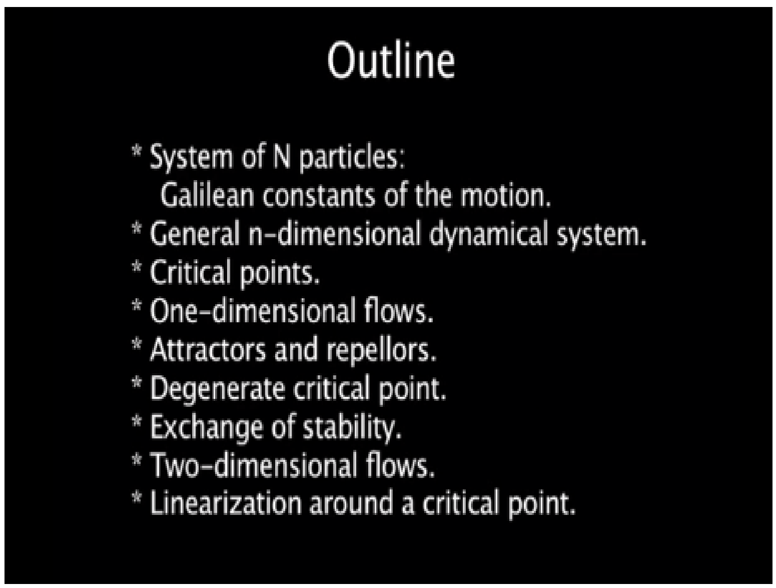

So before we start our study of differential dynamics continue with it I would like to recall to you what we ended with the last time which was the Hamiltonian for a set of particles interacting with each other by some pair wise interaction between them the force between any 2 particles being directed along the line joining these 2 particles and I mentioned certain properties of the corresponding Hamiltonian.

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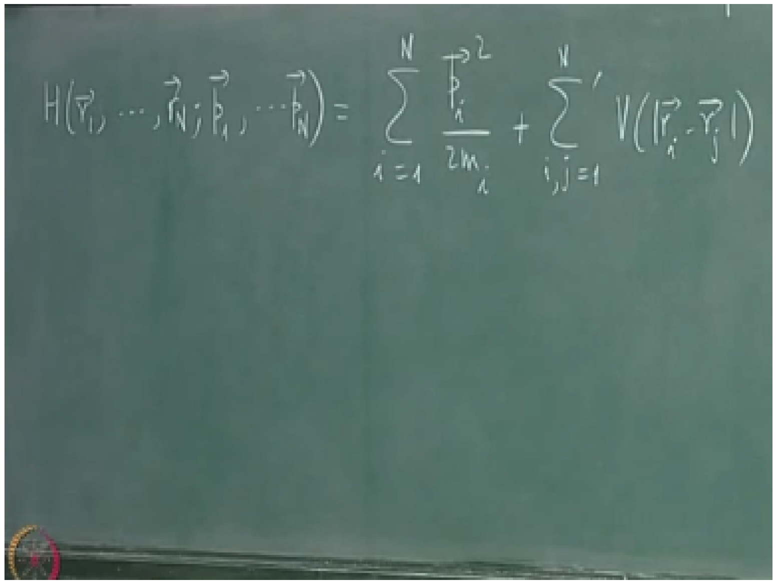

And this Hamiltonian was a function of all the coordinates of the particles n of them and of the corresponding conjugate moment P1 to PN and it was in the form of the sum of the kinetic energy of the particles and the potential energy of these particles so it was a Σ i=1 to N Pi^2 / 2 m of the particles + Σ/ ij= 1 to N no self interaction times the pair wise potential energy which was a function of ri- rj modules in this fashion.

This Hamiltonian as certain properties the first property is that it is in variant under a rotation of the entire coordinates system whatever coordinate system you choose if you rotate with that coordinate system by an arbitrary angle through an arbitrary around an arbitrary access the Hamiltonian would not change in form at all this means that rotational symmetry is a symmetry of the system dynamical symmetry of the system.

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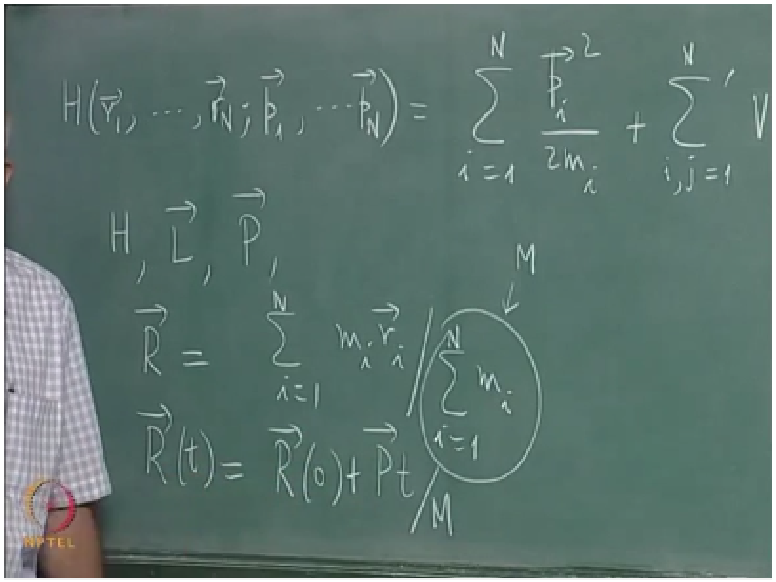

The Hamiltonian itself is constant of the motion so you have H the angular momentum about the organic of coordinates is a constant of the motion because it is rotationally in variant if I call that L total angular momentum of this system is a constant of motion there are no external forces on the system and therefore the generalization of Newton's 3^{rd} law applies the total linear movement of the system is constant and there are 3 other constants of the motion which I will write down very shortly.

And they go as follows if you consider this coordinate R which is the coordinate of the centre of mass of the set of particles this of course is defined in the standard way as a Σ from i = 1 to N of m_ir_i / 1 to N mi the total mass this coordinate is s a coordinate instantaneous coordinate of the center of mass of the centre higher system and then because the total linear movement of the system is constant it is possible to find what this R is at any instant of time.

And it would just be given by a simple formula R(t) = R(0) + since the total linear momentum of the subsystem is constant that would be the way in which this thing would move Pt/ M where M is the total mass of the system let us call this M we can proved this very regressively but physically it is very clear that this coordinate the center of mass would just drift along at uniform speed given by P/ M where this is the total momentum because this is a constant.

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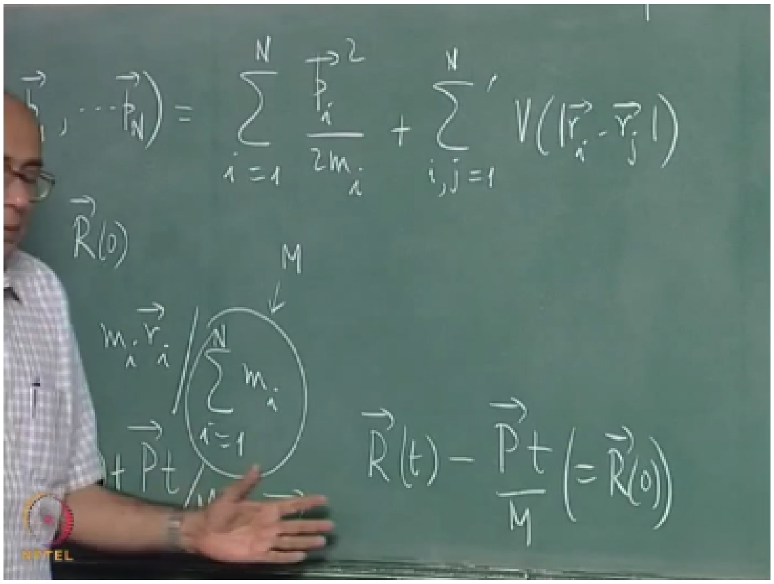

This immediately ⇒ that the combination R as a function of t – P times t / M which happens to be = R (0) is in fact a constant of the motion so this set of 3 coordinates – this combination P times t / M they are constants of the motion another vector constant of the motion so let me write down here as R(0) these are also constants of then motion but they are really dynamical variables and that combination is constant in time.

Absolutely no I do not but this quantity is a dynamical variable and so is this which happens to be constant in this simple example and the difference of these 2 is a combination of dynamical variables which is constant so it is a constant of the motion this is a different dynamical variable from that so a certain set of combination of dynamical variables happens to be constant in time.

There is however something strange about this constant of the motion it is a function of time it is explicitly a function of time so that gives us other lesion constants of the motion may be explicitly time dependent in this case that is true it is just that the time dependence of this quantity this quantity etc are such that they are just themselves in this combination is timely different.

So what do we have we have 1+ 3, 4+ 3 7 + 3 10, constant of the motion in a phase space that is 6N + 1 because times also now we are talking about extended phase space since we have got time dependent constants of the motion in this 6N+ 1 dimensional extended phase space we have exactly 10 Galion constants of the motion these are called Galion constants and 10 constants of the motions in a 6N+ 1 dimensional phase space is small.

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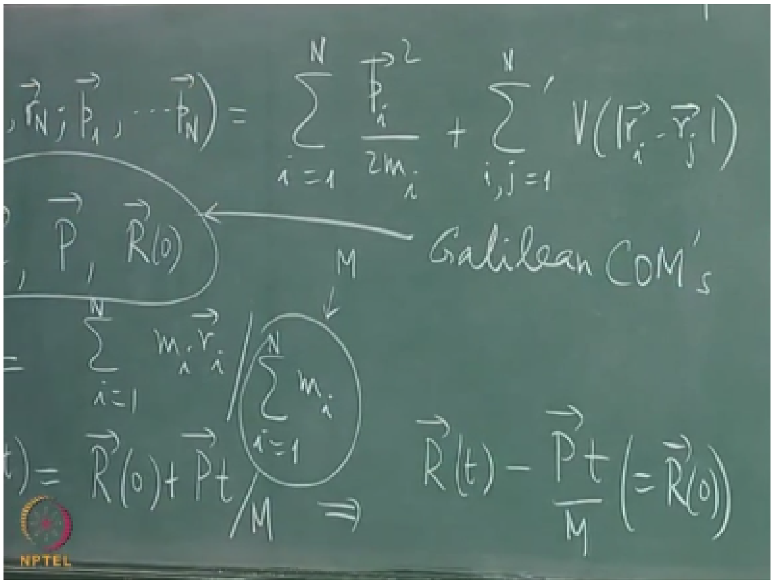

Simply too small for you make any significant progress based on just these constants of the motion say too much about the motion and this is as it should be we have after all said nothing about the we have said nothing about any of these quantities about the behavior of the system the kind of forces etc…

Other than the fact that these are forces which are rationally invariant so it is not surprising do not get too much information out but the fact is that you have 10 Galion invariance constants of the motion in this large dimensional phases space so in general the point I was making was that a given dynamical system generically typically as far fewer constant of the motion which you can find explicitly that the actual dimensionality of the phase space.

This is the typical case there are exceptions there are intolerable systems where you can find all the constants of the motion explicitly but typically what will happen is that in a given dynamical system even if it is not a Hamiltonian system could be disc petal systems and so on the generic situation is that you have far fewer constants of the motion accessible to be you that the dimensionality of the phase space.

This as to be born in mind keep this in mind okay with this now let us go back retrace our path a long way and go back and look at general dynamical systems described by a set of N coupled first order differential equations and then look at it case by case and see what the importance of various concept such as critical points equilibrium points and so on are so I go back now all the way and look at differential dynamics which if you recall was of the form.

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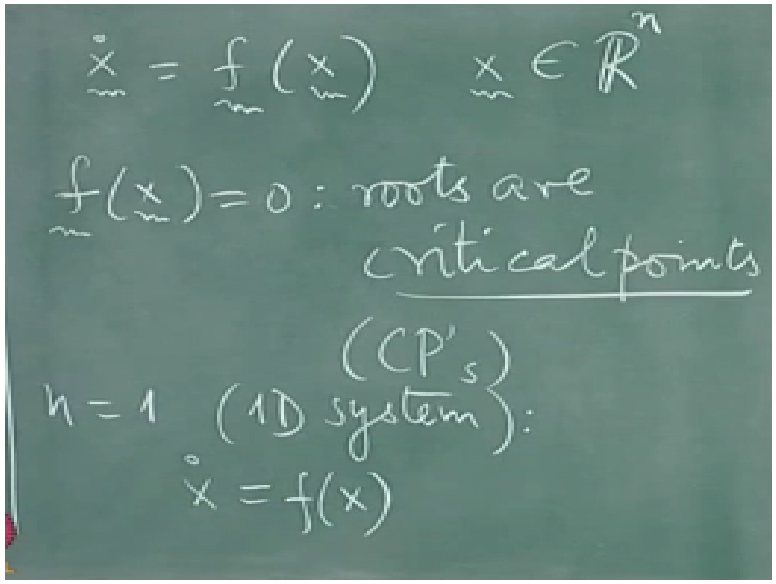

X dot = a given vector valued function f (x) and x existed x live in some space which is say n dimensional Euclidian space I write it in this form just as order set of N of numbers x1, x2, x3 up to xn and this the kind of dynamical problem we have to look at the very basic concept in the differential dynamics is that of critical points because recall the dimension that in the neighborhood of a typical point x0 you could always approximate this right hand side to leading order by f(x0) itself.

And that was like applying the rectification theorem and you could find the flow locally in an infinities time interval 12, 30 the only exception I mention was when this vector field vanished at some point if it became 0 then you could not do that you have to go to higher orders you have compute teams you have to compute the problem once again now what happens then and why is it so important.

The reason is that when this right hand side vanishes x dot vanishes at that point and since it is a set of first order equations if you give me the initial point I know the feature but the derivative at that point if like is 0 for all the x is in therefore x remains at that point therefore you could define all points with satisfy f (x) = 0 to be the equilibrium points of the system since equilibrium as this coronation of being a of a mechanical system and we are not talking about mechanical systems alone.

We are talking about all kinds of dynamical systems I would like to call it by general name and I would say f(x) = 0 the roots critical points and I will abbreviate critical points by CP's if you like they are the generalization of equilibrium points nothing has been said as yet whether that is stable equilibrium or an unstable equilibrium or a marginal kind of equilibrium a nutria equilibrium or anything like that.

They are just the critical points of the system the other statement which is work bearing in mind which I will make as statement now and then try to substantiate it as we go along is that by enlarge the nature of the flow the nature of the phase is controlled to a great extent by the behavior of the system near it is critical points so it is important to find where there critical points are after which we have handle on what the general dynamical system does.

Now let me illustrate this by the simplest of flows a flow in which N = 1 a 1 dimensional dynamical system and what happens then so let us look at it n =1 a 1D system or a 1D flow given by an equation of the form x dot is f(x) where f(x) is some scalar function of the single scalar variable x.

The equilibrium points of this would be given by the roots of the equation f(x) = 0 and if f(x) is a reasonable kind of function it 0's would form an isolated set of points it would not have in general a continuous set of points put just isolated roots of the equation f(x) = 0 let us take an even simpler example of this specific case of this to see what happens.

So let us look at f(x) = x itself so linear function of x in this case jut x itself what does the flow look like this ⇒ that x dot is x or solving it is very trivial x(t) = what would you say is the solution to this simple equitation it just an exponential so this is et so I will put in the initial condition which determines the constant of integration that is determine by the initial condition x of 0 and you can check very trivially where t0 at some t reduces to x^3 this portion what does the graph of x of t as a function of t loop like so here is t here is x of t it is immediately everyone that if we start with the positive value for x of 0.

The positive initial condition then it starts at this point and takes of exponentially passed as a function of value diverges like e^t, if on the other hand x of 0 over negative it would diverge this fashion, so this corresponds to x of 0 less than 0 and this is nx of 0 > 0 effects of 0 should happen to be 0 you started with 0 as the initial value then of course you remain write here, this graph itself so this will correspond to x of 0 = 0 so it is clear that the critical point at x = 0 that is only the critical point.

Is actually separating two different kinds of trajectories those which take you to + ∞ and t goes to ∞ and those which take you to - ∞ yes t goes to so what should we do we draw the phase plane in this case of the phase space and this is one dimensional so it is just a phase line, so here is a phase line this is phase space in this case, is not much of a space it is just a line here is the point 0 and there are only two different kinds of trajectories possible on this those which would start anyway here and move on to ∞ which I denote by an arrow outwards.

And those which we would start here and move off to the ∞ so it denote by an arrow pointing left wards this point 0 is a critical point so this is the only critical point in this problem is this is a stable critical point or an unstable one based on our elementary notions of stability I make a little perturbation away from it and the question is to I go back to it or do it move away from it, it is every done we will make this more regress that this is in fact an unstable pointing notice that it is a trajectory by itself.

Because if I start a text of 0 = 0 I remain on it at this point so there are three classes of phase trajectories here, those which started 0 and remain at 0 that it is just a single point the trajectory which represents the positive x axis and another one which represents which is represented by the negative x axis would you call this an attracting critical point or a repel in critical point it repels on both the sides, so it is in fact an unstable critical point and this case it happens to be what is called a repellor.

It is immediately clear that yes well we must make precise the notion of unstable which we will do very shortly and repellor at this level is simply because trajectories on either side of better repelled away initial points on either side of it repelled away, so an unstable critical point can be have many types and this instance it happens to be a repellor which are use in the louse sense as something which pushes away trajectories on either side of it, we will define the notion of unstable more care.

Can I make that coat and coat a stable critical point and attractive something which attracts thick points to it base points to it, can I change this flow to do that 1/x will have a critical point at ∞ and like to have a critical point in the finite part of the plate of the line – x would do that so if I took x.= - x.

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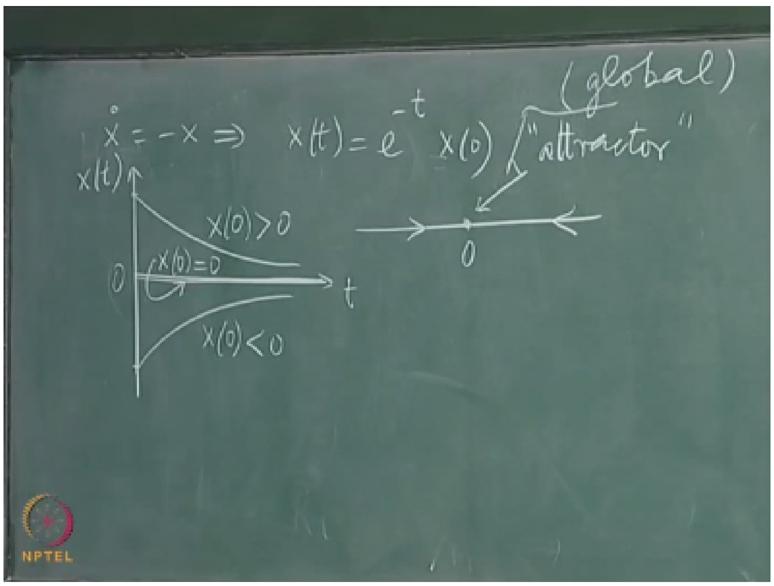

This would implies x of p where e – t and the phase the graph of x of t versus t again we by started 0 I remain here, but wherever I will such start by started this point or this point etc, I would exponentially dk towards 0 as time at all here is okay so this case if I plot the similar phase diagram or a phase portrait as they call it this case is 0, it is an attractor because a trajectory is flow inwards and you began to see that the phase portrait gives you much more information then specific.

Solutions of the differential equations the point being that this is not much difference between a phase the behavior of the system for an initial condition which starts here or here they all form all in the same class and the same is true on this other side, so it is not so important to worry about the specific initial conditions but we are now dealing with whole classes of initial conditions in one short and that is it, there are only three phase trajectories in that problem there are only three here and this one is in.

No matter where you start you are going to as you we though it will flow into the point 0 eventually perception weights of essentially long you come arbitral close to 0, and this is true for any finite initial condition throughout this space every point flows into this origin the critical point at the origin, so this thing here is actually a global attractor reason I call this what I call this attractive global is because you can envy short situations where you have more than one attractor and part of the initial conditions are flow into one.

And another part would flow into another in which case those will be local attractors but this is a global attractors everything falls into it, this classification of critical points into attractors and repellors is very specific to the kind of one dimensional system if you talked about things will get much more complicated as we go long, but already here we can illustrate the fact that more complicated things could happen for instance let us look at the case let us modify slightly this is good point to actually introduced the idea of a bifurcation.

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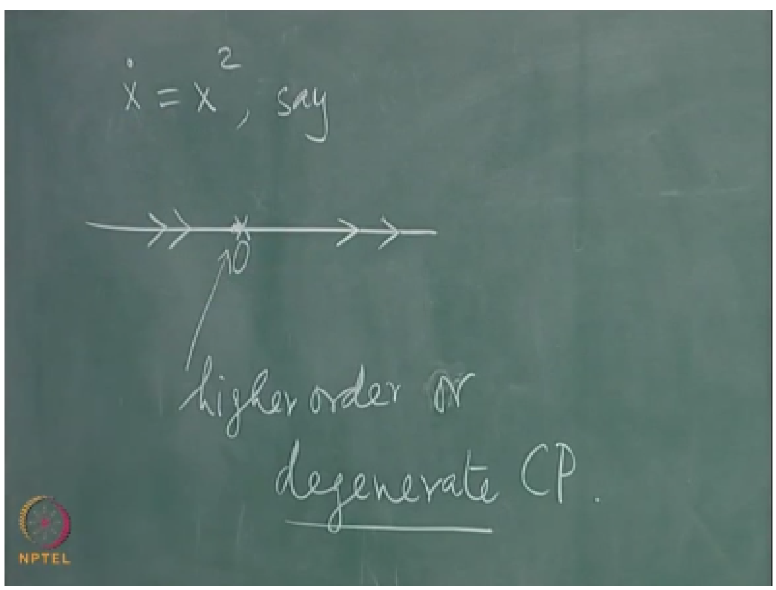

So let us modify this slightly and consider x. = x^2 say what kind of solutions so you have what kind of phase partita would you have where is a critical point it is again at 0 that is the truth of x^2 = 0 but it is a double root now what kind of arrow should I draw here for trajectories here well one way would be solve the equation explicit there is a function of t and c what it does for thus t increases as the function of t but we can do that by inspection by looking at just this differential equation itself.

If x is any positive number x. is positive and therefore x increases and since x is increases it is clear that the trajectories must move outwards if x is negative x x^2 is still positive and therefore x. makes x increase in algebraic value and therefore it again in this way is this an attractive or repellor well it looks like as far as point so the right are concerns it is repellor but point so the left or concern it is an attractor, so it is neither and attractor or a repellor it is in fact the higher order critical point because this is not a linear function of x.

So let me call this higher order or degenerate degenerates critical points the reason we ended up with the degenerate critical point could be interested will be important it is because we start at with an x^2 here if I ask you to write down the polynomial function of the right hand side you would typically start with the constant plus a first order term plus a second order term and so on, the constant can be got read off by shifting the origin to that value the constant after which you have a linear term proportional to x typically.

But here you do not have that you start with x^2 so this is not a typical or generic polynomial of the right hand side and typical one would start with the linear term and then unless this is an accident will always be present and then go on to a quadratic and cubic terms and so on, and therefore since this is not generic we would like to make it generic by saying this came about this double root at x = 0 came about by 2 roots coinciding with each other so let us split one of those roots and consider.

Instead x, = x times let say x – ε is where e is some small positive number to start where are the critical points of this flow. 0 and e now of course we do not have to solve the differential equation that is the whole point of phase plane analysis explicitly we try to get as much as we cannot from the differential equation, so here is 0 and here is ε at movement we have no way of knowing which one is the stable one or which one is unstable one or both and whatever and no way of knowing what their stability is but it is hard to diffuse let us look at the neighborhood of x = 0 that it is a critical point, so I look at in a small neighborhood of this point.

So small neighborhood that if any given ε I could neglect the x^2 terms compared to the linear term in x and then near this in this neighborhood we would write x. it is approximately equal to this is $x^2 - \varepsilon x$ and I leave out the quadratic term and write this as $-\varepsilon x$ does that suggest an attractor or a repellor it is an attractor because the solutions nearby would go like $e^{-\varepsilon t} x$ x of 0 therefore the arrows would points inverts here, and inverts here.

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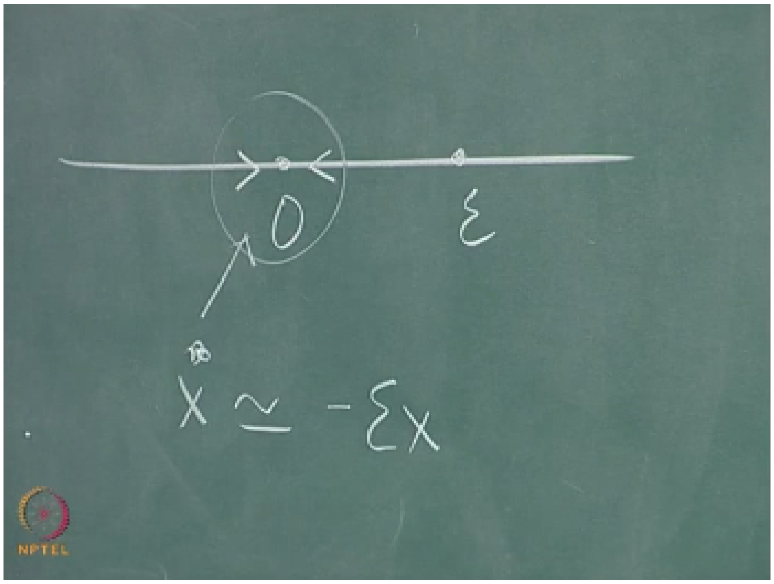

By continuity since nothing happens between these two there are no other critical points it is not hard to show that even further beyond it would still have to do this, because there is nothing in between to demark it two different kinds of qualitatively different kinds of behavior, in exactly the same way no matter how far you are on the left it would still flow in to 0 from this side, simply by continuity.

What happens in a neighborhood of $x = \varepsilon$ one would have to liniearized the problem near $x = \varepsilon$ and that is going to be a big tool in our analysis of critical points we would liniearized the flow near this point by simply shifting to that point.

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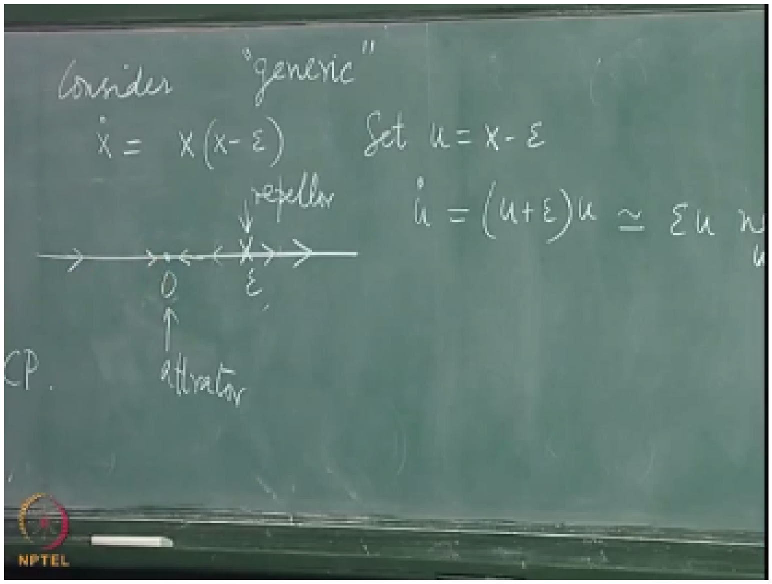

So let us set $u = x - \varepsilon$ so $x = \varepsilon$ has been shifted to $u = 0$ and this plane the equation become u. = x is $(u + \varepsilon)$ u itself is approximately equal to ε u near $u = 0$ so once again I neglect the quadratic term and the linear term is ε u near $u = 0$ and is that a repellor on the attractor, since ε is positive that is a repelled. So the picture would look like this and if it is a repellor it is unstable and I use the convention where I mark constable critical points by cross and stable critical points by dots in which case this is what the global flow looks like.

Again by continuity this is all it can be so I delete this and that is the phase portrait of the system you have an attractor, and you have a repellor, notice the arrows cannot properly change direction without passing without going to critical point in between. And I do not need to solve the problem explicitly do anything else, I could write down the solution explicitly and then you would discover if you eliminate a t that the phase flow does looks like this in this passion.

Now what happens when you ended up with $\varepsilon = 0$ was that this point and this point quails so this region became narrow and narrow over and disappear and you ended up with this flow with the higher order critical point here. So this procedure by which I took at degenerate critical point and I found it genesis by the quiescence of two separate single simple critical points it is called unfolding the singularity.

This is the most elementary example of this unfolding of the singularity. So I went to what I call a generic or typical case this is generic that is not generic it is degenerate and all this was true for $\varepsilon > 0$ that was crucial I could now ask the question what if I start with ε as a tunable parameter I make it smaller and smaller so this critical point gets closer and closer hits the origin gives of this picture and moves of the left of it I make ε negative. What would then happen? What would be the picture for ε negative?

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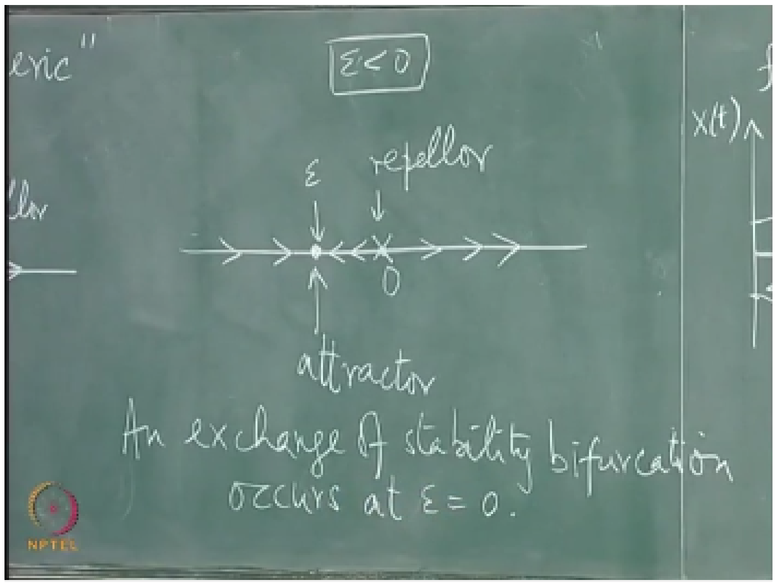

You will still have to critical points this would be 0 this would be 0 and this critical point would at ε and remember that ε would be negative now, what is thing would happen? Well near the origin the behavior is minus ε x and if ε is negative – ε is a positive number so 0 becomes repellor, and critical point at ε becomes an attractor. So you end up with the picture where this is an attractor and that is a repellor and thing flow in this passion.

So this 0 becomes repellor and that becomes an attractor, so while this was the picture for ε > 0 this was the picture for ε = 0 and that is the picture for ε < 0. So it is evident from this simple example that act ε = 0 the systems behavior changes qualitatively it changes from a situation where ε was the repellor and 0 was the attractor to the opposite situation where 0 is the repellor and ε is the attractor there is a qualitative change in behavior across this point ε = 0.

In other words a bifurcation of the system happens at parameter value ε = 0, what kind of bifurcation would call this? so there fancy names for it but the most illustrative one is name actually given to this notice that this was an unstable critical point and this was a stable critical point and after the bifurcation the roles have an exchanged what was unstable become stable and what was stable becomes unstable therefore this is called an exchange of stability bifurcation.

Might as well define a bifurcation, a bifurcation happens at some critical values of parameters in the problem when the qualitative behavior of the dynamical system changes from one kind of behavior to another. So small change in ε from an infinite decimal positive value to an infinite decimal negative value changes the system behavior completely quite drastically that is what a bifurcation is.

So one studies bifurcations in parameters bifurcations happen in the parameters so the dynamical variables behave in a very different wave from one side bifurcation to the other kind of other side of the bifurcation we will study several kinds of elementary bifurcations, but for the moment this already serves to illustrate and extremely simple phenomenon, this is just one last point with regard to this trivial example almost trivial example and that has to do with the following.

If I solve this differential equation for x is function of t you end up with the traditional exponential behavior in time either exponentially divergent or exponentially converging the same situation is valid here too. But here at x. = x2 it is not hard to write down the solution of this equation let us solve it for specific values of initial conditions, so at this special value 0 of the parameter ε if I solve this equation this implies like 1/ x of t.

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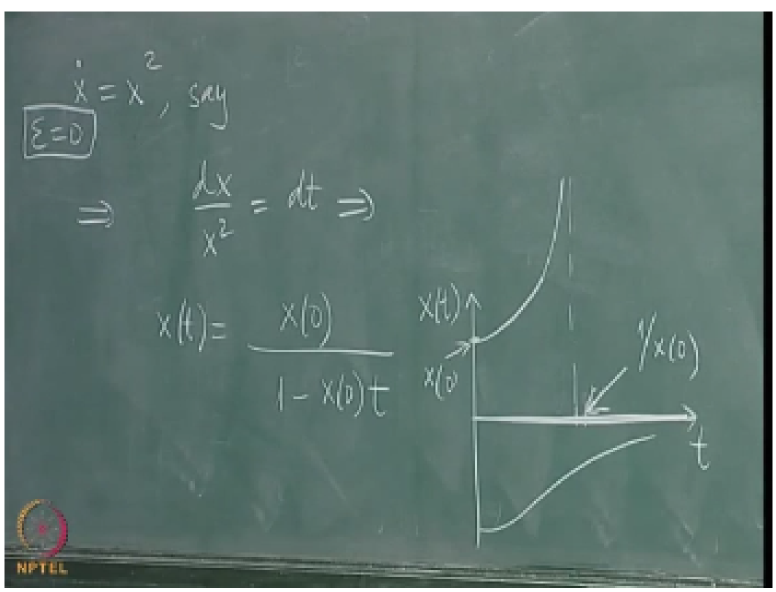

We differentiate both sides it says if I separate variables let us say this is equal to dt which implies 1/ x of 0 – 1/ x of t = t itself, therefore x of t = x of 0 we move in to that side divided by 1 – x of 0 x t, this is right, now what is the solution look like? as before 0 is a critical point so if I started at 0 I remain on 0 for all time, what happens if I start with the positive value of x0 say here, what happens to that solution?

It is clear that it would diverge at some t, it would diverge at t = 1 / x 0 so at some finite t which is 1/ x of 0 if I start with x of 0 here this solution would in fact diverge to infinity, this is at this point to diverge, so this space point this appears to infinity ion a finite time. That never happens in first order dynamics it happens in this kind of higher order dynamic x2 it is non linear to start with interracially non linear.

On the other hand for negative values of x0 this quantity in never vanish this since t runs from 0 to infinity so to start at some value here and slowly approach like a 1/ t it would approach the point 0 as t goes to infinity this vanishes like a1/t. if this is negative, so this is again two qualitatively different kinds of time behavior simply because of the non linear nature of this problem. So although it is a very simple example it is serves to illustrate some general issues such as what a bifurcation is and the fact that at a finite time the system actually go off to infinity the phase point may go off to infinity, this is very characteristic of higher order degenerate critical points. Having seen what happens in first order dynamics we could now generalize it little bit go on to high dimensions but before that let me make one more point which is the following what would you say.

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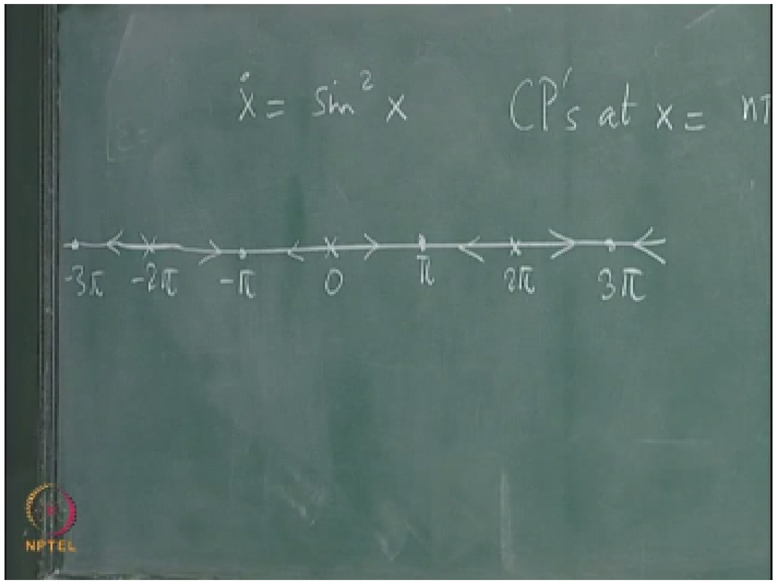

Is the dynamical behavior of a system given x, say sin x, what kind of behavior you would expect here? Where are the critical points? Any multiple integer multiple of π etc, what kind of critical point it would be, by now a lesson with a earlier example tells us because of the continuity that is involved in drawing these arrows showing the direction of the flow it is quite clear that stable and unstable points must alternate.

There is no way of having two stable points next to each other because in the system in between will not know where to go to. What happens near 0? You need to linearize sin x near x = 0 what is the leading term in sin x = 0, x itself and it has the positive coefficient, so the x = 0 replier on attractor, it is a replier. So we know that the 0 is the replier in which case we immediately know that the π is an attractor and therefore 2π is a replier and likewise – π, -2 π and so on.

And therefore the arrows should go off like this etc. you could solve this equation the x over sin x = dt and you integrate both sides and so on so forth you get explicit solutions but you will have already all the information you need about the simple dynamical system by just writing phase down. So you begin to see the power of the phase plane analysis you need not necessarily have to solve the equations completely.

You can deduce a great deal by figuring out where the critical points are and the nature of the stability of this critical points. As an exercise I urge you to try next following sin ^{2} x and what happens in that example. What kind of critical points these are? Would they be simple critical points no near x = 0 this behaves like x^2 so right way this is the degenerated critical point and therefore the behavior would be very interesting we will check out what happens?

Now that we have a little bit of an idea of what one dimensional systems will do let us go over to the 2 dimensional systems and instead of calling the variables for x 1 and x2 let me call them x and y for short.

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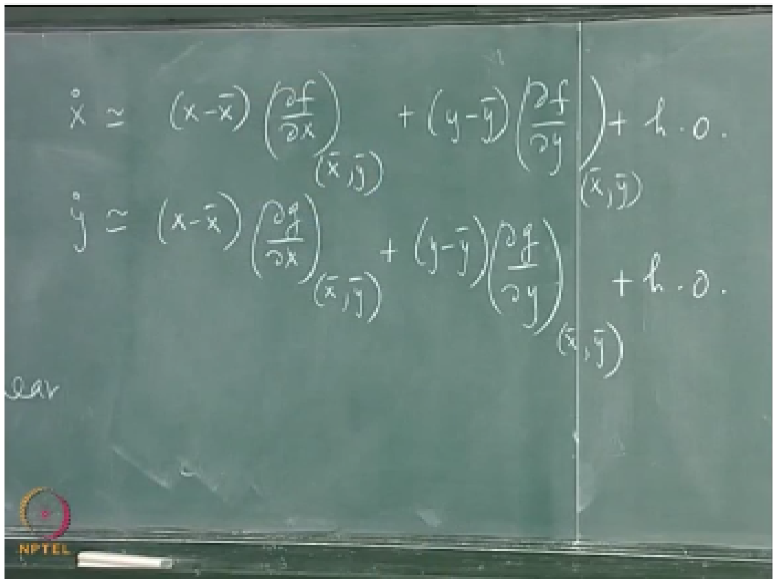

So now let us talk about flows in the phase plane, 2 dimensional systems and this would be a special case on the n dimensional system I talked about earlier, instead of calling the variable x1 and x2 as I said let us just call them x and y and they specify by the equation of form x. some (xy) and y. (xy). What would be the critical points will be given by? They would be the roots they will be the simultaneous roots common roots.

Since f(xy) = 0 it is some kind of curve in the x y plane and so is g this is the intersection of the two curves given by f is 0 and g is 0. Again the statement is the critical points in the phase plane control the behavior of flow to a large extent, so we would like to analysis what kind of critical points you would possible have in the space plane. And what could be the way to do this, typically wherever we are near any point of interest I would do a tailor expansion of x and g near that point.

And study the flow in the neighborhood of the point with the help of the expansion, if it is the critical point and those are point of interest then these functions are 0 at the critical point. And therefore the terms that would start with would be linear terms and what would it do? So let x and y be a critical point, Taylor expand f and g near critical point.

What would you then get? You will get x. is approximately = the leading term f (x, y) is 0 by definition because you are at a critical point and what would the 1^{st} term be, it will be linear in the difference x – x bar this would be the 1^{st} term so it is x – x bar times co efficient. Now what would be the co efficient of the linear term?

It is a partial derivative $\partial f/ \partial x$ evaluated at x bar y bar + similar expansion in y$\partial f/ \partial y$ evaluated at x bar y bar + higher order terms these would be the terms which are the form of x – x bar ^{2} y – y bar ^{2} or x – x bar times y – y bar cross derivative and so on so forth. Exactly similarly y. is x – x bar $\partial g/\partial x$ at the critical point + y – y bar $\partial g/\partial y$ at x bar y bar + higher order terms. I could therefore make life simpler by shifting my origin to x bar y bar.

It would just correspondent to taking the original phase plane the x y plane and if this is x bar y bar I could shift origins and work in new co ordinate say u and v such that u is x – x bar and v is y – y bar, instead of notation of kind let me just call x – x bar x1 again and y – y bar shift origins and use the same symbols. In which case the flow in the vicinity of the origin.

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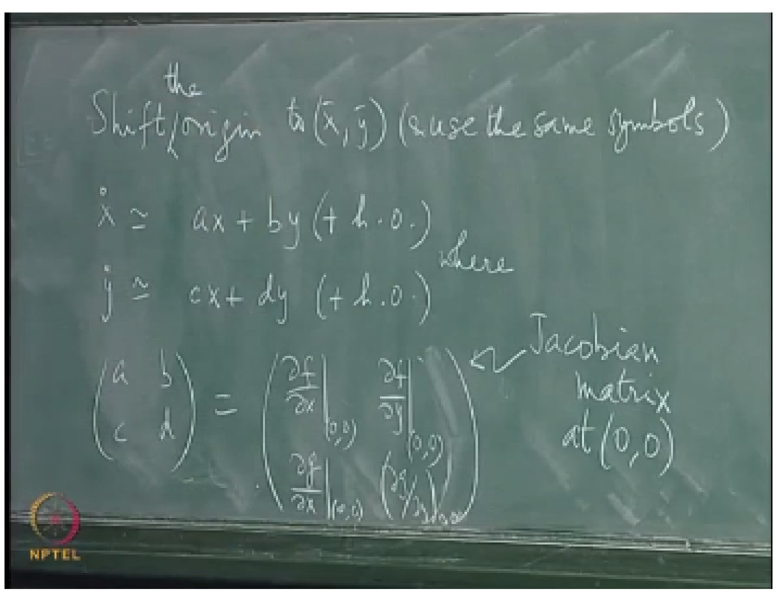

So let us shift the origin to x bar y bar then the flow looks like x. use the same symbols x and y instead of x – x bar I just call it x and similarly for y. then x. is to perform ax + by + higher orders and y. cx + dy + higher orders, where a, b, c, d it is just the matrix of partial derivation of f and g evaluated at the origin which is where the critical pint is, so this is simply ∂f/∂x at 0 0 ∂f/∂y at 00, ∂g/∂x at 00 and ∂g/∂y at the origin.

This matrix of partial derivation at any point is called the Jacobean matrix at that point and the system looks like this, provided these terms are present provided these partial derivatives are not 0 this is what the flow looks like near this point. Now the question we would like to understand what the flow does in the vicinity of this point by looking at the approximation the linear approximation in which you get in these terms and therefore sufficiently close to this point. You could neglect these terms and just look at the linear system and therefore linearize.

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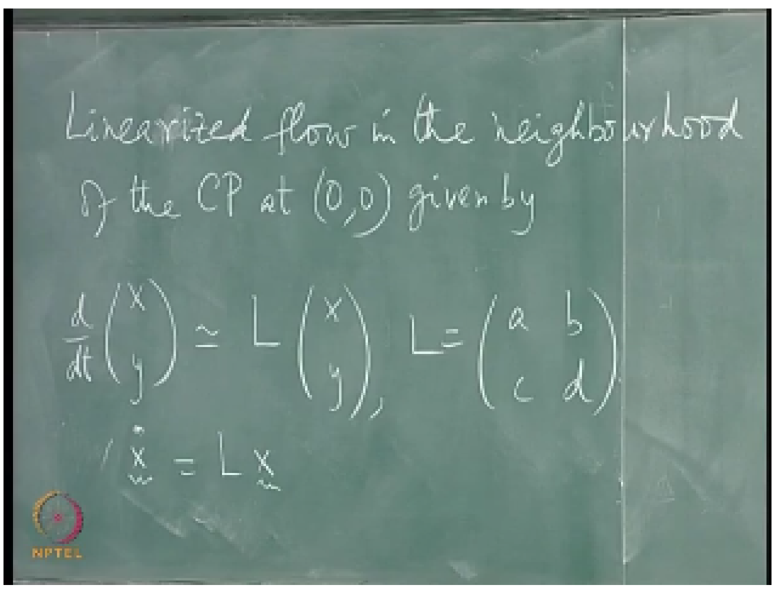

Right this is a vertex d over dt of this where l is coefficient matrix off course you would ask immediately how could an approximation is this statement is that we can estimate that we should need to know how big these terms are there therefore to any prescribed term of accuracy if this matrix is well behaved when the partial derivative is known vanish then sufficient close to the origin it looks like a good approximation will come back and see when it is justified and when it is not but if this where true.

Then this equation is extremely simple to solve because again you have an equation of an form x. where x is a column vector with components x and y this is equal to l times x where l is the matrix of constant coefficient time independent just numbers and what is the solution to this equation the formal solution to this is exactly as a x word just an ordinary scalar function and this would imply immediately that $xt=e^{lt}$ so this square matrix e^{lt} acts on the column vector of the initial condition and gives you the value of x and y at any time.

This is what the formal solution looks like you could off course do the following and I leave this you as an exercise you could differentiate this is the second time and eliminate y from this equation by getting x.. This ax.+by. For y. you substitute from this equation but that still involves y what could you do to get read of y how would you eliminate either x or y from the set of coupled equations I differentiate this so I get x.. is ax.+by. For y. I put in cx+by how would I write y how do I get rid of the y.

I would write y as x.-ax/ and that would give me the second order differential equation for x along and do the same thing for y if the second order equation and you know you get a second order differential equation with constant coefficient and how do you write the solution second order linear differential equation with constant coefficients what is the solution look like.

It is some of the exponential what are those exponents you have to satisfy the character tic equation corresponding to this differential equation and in fact we can see without much to do that this quantity e^{lt} if I imagine being able to diagonals this matrix l then e^{lt} will involve in the diagonal form eλ1d1λ2t where λ1 and λ2 are Eigen values so the general solution to this equation we can do.

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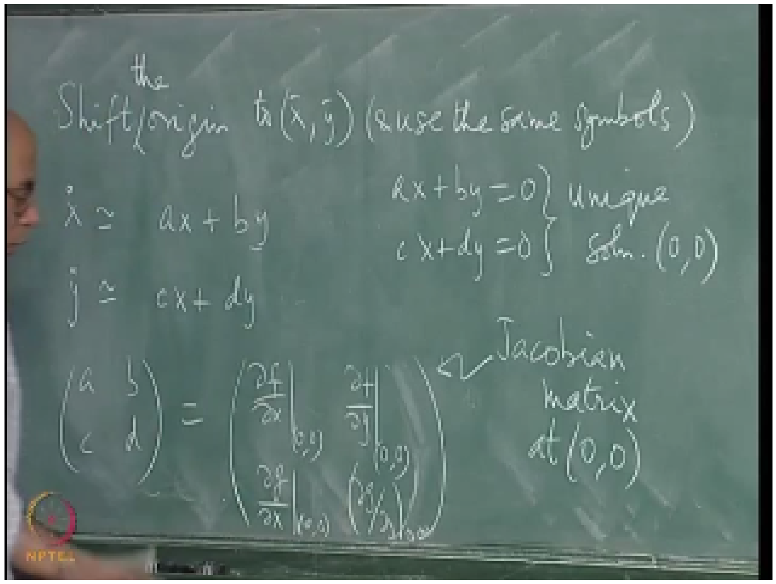

This step could involve something like x of t equal to some coefficient $c11e^{\lambda 1t}$ $+c12e^{\lambda 2t}$ and similarly y of t c21 $e^{\lambda 2t}$+c22 $e^{\lambda 2t}$ in general how would you find this constant c11,c12,c13 what determines the constant initial conditions absolutely we have initial conditions no x of 0 and y of 0 but then this is little paradox or to puzzle here I have four constants but I have only two constants I have only x of 0 and y of 0 so given x of 0 and y of 0 are given how do I find four constant yes how do I get four constant out of two absolutely.

I need to know x and y where do I get those from the differential equation themselves so in this couple form you use the differential equation themselves you set t=0 here and x0 of 0 is y0 when you use that in those equations therefore it is completely adequate in the specified as we know very well we have two couple first order differential equation and therefore two initial species of back that I have written it by eliminating by the other way.

And so on really can find therefore you can write what assumption we have made about this statements that is not singular that this matrix is not singular in another words I took this set of equations and I set 0,0 is a solution for the critical points namely the right hand side will be 0 and I assume that the only solution of ax+by=0 cx+by=0 as a unique solution 0,0 another words I set this simultaneous equation as only a trivial solution 0,0 when is that true.

When this determined is not 0 when the Jacobean matrix is non singular and when l is non singular matrix then and only then this is we assume that l was constant but exactly what happens when it is singular that is a primary assumption and when is this form of solution so in another word determine l not equal to 0 that was an assumption and what was the another assumption even if we write solutions whatever I assumed about λ1 and λ2 written two terms separately.

So what is the perception there not detailed as an Eigen values another words Eigen values are not repeated λ1 and λ2 are dissect what happens if λ1=λ2 the Eigen values will be repeated what happens to the solution of this differential equation well if I write it has second order differential equation it is clear that when you have roots repeated you have a different form of the solution the linearly independent solutions are no longer eλ1p and eλ2 p suppose if λ1=λ2=λ.

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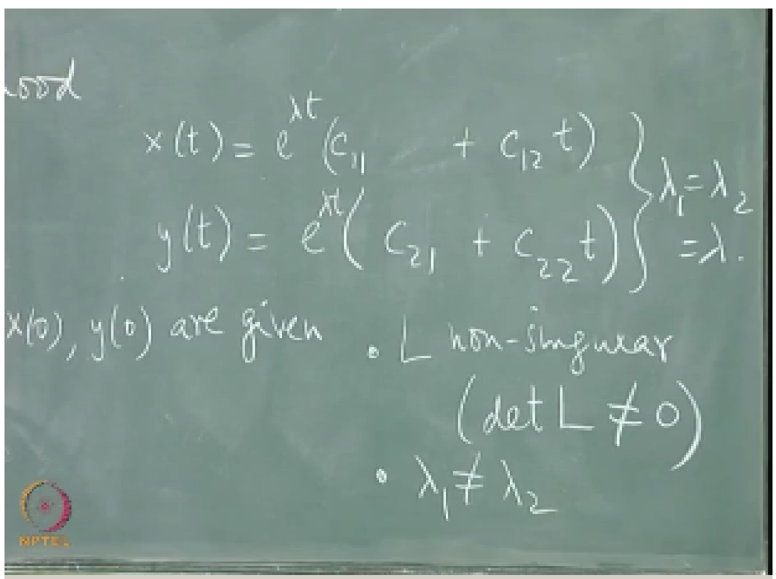

Does not see in equations then $e^{\lambda 1t}$, $e^{\lambda 2t}$ this set is replaced by there is an $e^{\lambda 1t}$ and what is the other term $e^{\lambda 1t}$ therefore these equations will be modified to be something like $e^{\lambda 1t}$xc11+c12t and similarly $e^{\lambda 1t}$ c21t+c22t stop at this case and we take the analysis for the analysis of this final remark is that everything seems to depend on the behavior of this expression so this will help us analysis what happens nature of the Eigen value tell us everything about and notice one second I am not going to use explicitly forms of the solutions.

I do not use here we Will define next time we define what is meant by the exponential matrix if it is the square matrix as this one is then square is also an n/n matrix and cube is also an matrix so on the exponential of the matrix is defined exactly as a^z is defined for the complex so it's the identity matrix plus the matrix for one factorial to the square over two factorial and this is guaranteed to be well defined convergences property.

The exponential of the operators are extremely important over and over we will see the more general matrices the domination will appear in $\lambda1$ and $\lambda2$ both are positive and $\lambda2$ is larger or $\lambda1$ is negative and $\lambda2$ is positive it is clear that the divergence behavior the explosive growth would have happened from the positive side sure we do not have to worry about which one is positive and which is negative okay interestingly things are going to happen because they could either be both positive or both negative or one positive and one negative or complex because the matrix coefficient the coefficient ABC are real and therefore the roots may appear complex conjugate this will be the role.

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