Indian Institute of Technology Madras
Present

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 19
Discrete time dynamics (part iii)

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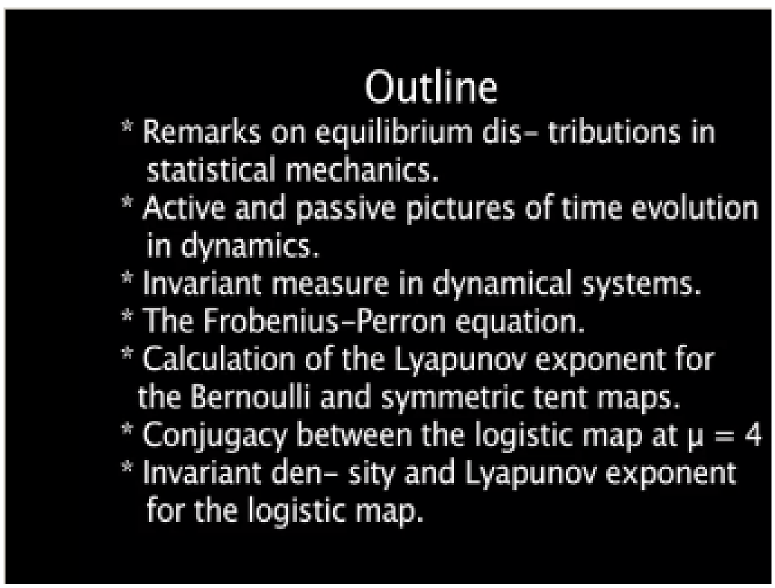

We saw some properties of the logistic map last time and we also saw that there was an intricate sequence of bifurcations followed by the onset of chaos interrupted I mentioned by periodic windows and finally fully developed chaos now one of the interesting questions that arises immediately is when you have a chaotic attractor how frequently two different parts of the attractor get visited by atypical initial condition by a trajectory and is there some kind of stationary distribution or invariant probability distribution which tells you how these attractors are occupied by atypical trajectory.

This is a question of great interest because as you know once you have the property of ergodicity then time averages which are not computable in general can be replaced by averages over some invariant distributions probability distribution so we need to compute these distributions now this the history of fur distributions is very, very long goes back a long way and just to recall to you even in statistical physics in ordinary equilibrium statistical physics the primary task of the theory is to give you the invariant distribution.

Once you have this distribution then you can compute averages physical averages using the property of ergodicity let me recall to you what happens in thermodynamics and equilibrium statistical physics for a minute if you have a system a macroscopic system isolated from the surroundings in thermodynamic equilibrium then we know that the energy of the system is constant it does not change at all.

And you could ask what is the probability with which different states of the phase-space different portions of the phase space of this system get occupied and this is called the micro canonical ensemble in equilibrium statistical mechanics and you could ask what is the probability distribution on the micro canonical ensemble and that is easy to see because it says if the system is described by a Hamiltonian q , p then for an isolated system in thermodynamic equilibrium the energy is constant.

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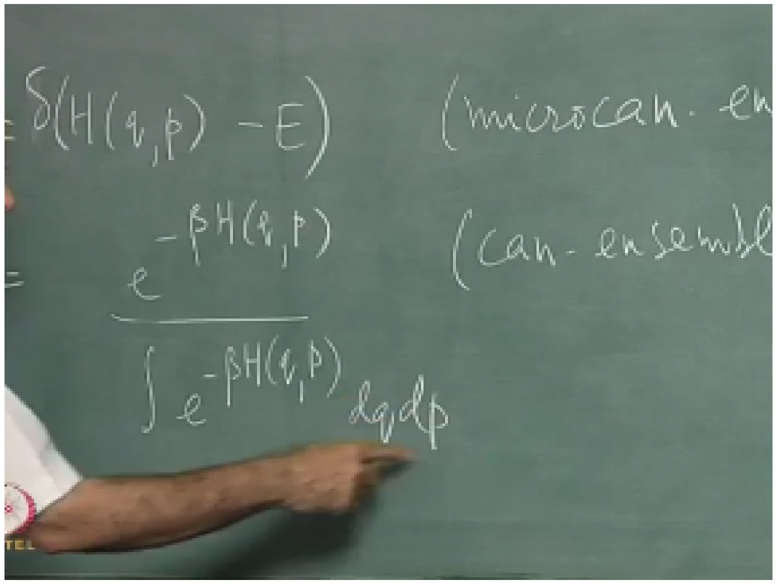

So this minus the constant value of the energy is equal to 0 and if I insist on describing my system by a probability density in phase space in the space of the p's and q's that density ρ as a function of all the cues and PS is equal to something which ensures that this Hamiltonian always have that numerical value so it is essentially a Dirac Δ function at that point on the energy surface so it is this thing that is the equilibrium or stationary distribution in a micro canonical ensemble so this is for micro the same system.

If it is put in thermal equilibrium with a heat path at some temperature T absolute temperature T then we know that the ensemble be used is the canonical ensemble it is called the canonical ensemble and that is the one in which we do normal usual thermodynamics because normal physical systems cannot do not generally remain isolated they are an interaction with their surroundings and they are in general closed systems where the system can exchange energy with its surroundings with a heat bar.

But not matter in such a case we have the canonical ensemble and the corresponding density operator row of cucumber p is something which ensures that the value e of the energy if e is one of the possible values of the energy of the system the relative probability with which the energy e is taken is equal to the exponential of -ei over KT where k is Boltzmann's constant this is the standard rule you have in the canonical ensemble.

So the relative probability is e to the minus the energy over kt -then the absolute probability is the same number divided by a normalization constant which is the sum over all values of e so it would look something like e $^{-\beta}$ H of q p whatever value this takes / a sum over all possible values or an integral in this case uses the -β H of Q P DQ DP this is an integration over all the degrees of freedom and the corresponding generalized momentum and it serves to normalize this probability density this is the thing which you normally call the Boltzmann factor because you might have come across.

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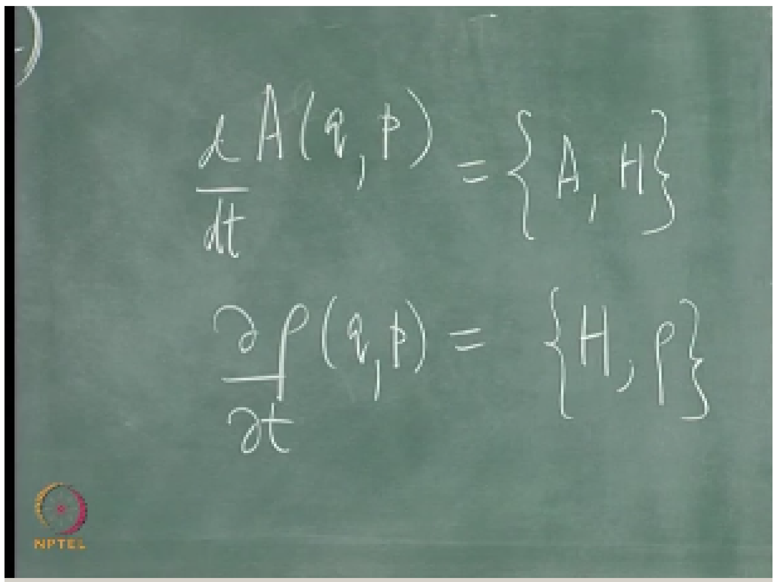

This in the form e - e over KBT I have used the symbol β for 1 over K Boltzmann times T if this is the relative probability that the energy is e if you sum over all possible values of in the denominator that gives you the normalization constant and it is called the partition function in equilibrium thermodynamics a statistical mechanics.

If you have other physical conditions such as an open system which can exchange both energy and matter with its surroundings then you use not the canonical ensemble but something called the grand canonical ensemble and the grand canonical ensemble again is of this form the density the density the probability density or the phase space density the equilibrium density is of the form e -β times h minus the chemical potential times the number of particles in the system.

And that is the grand canonical ensemble depending on what experimental conditions you are interested in you have different ensembles which specify for each of them you specify what the invariant measure is what the probability distribution is now of course we also know that in classical mechanics this probability distribution in general for a classical dynamical system obeys an equation called the liouville equation which is not quite the equation of motion for a dynamical observable call that.

If a of cucumber p is a dynamical observable we found out that the rate of change of this was equal to the Poisson bracket of this of a , H we found that this was the case long ago on the other hand the density itself row of cucumber p is equal to the Poisson bracket of H with roe it has an extra minus sign now if the system is stationary in other words the density is completely stationary steady state then this cannot change with time so the equation was this, this was the more equation of motion for the density.

And if it is completely stationary this quantity has to be zero if this is zero it means that the Poisson bracket of H with rope is zero identically but we the additional minus sign ensures that first of all it says that this row is a probability density and is not an observable this these are observables this minus sign ensures that the average values of physical quantities are independent of which picture you use to describe.

Them either you use an active picture in which the dynamical variables vary with time and the density does not or you use a passive picture in which the dynamical variables do not change with time but the density probability density changes with time and these are called the Heisenberg and Schrodinger pictures respectively they are related to each other the transform transformations of each other but yeah exact this is exactly the point by insisting that averages of physical quantities remain independent or be the same.

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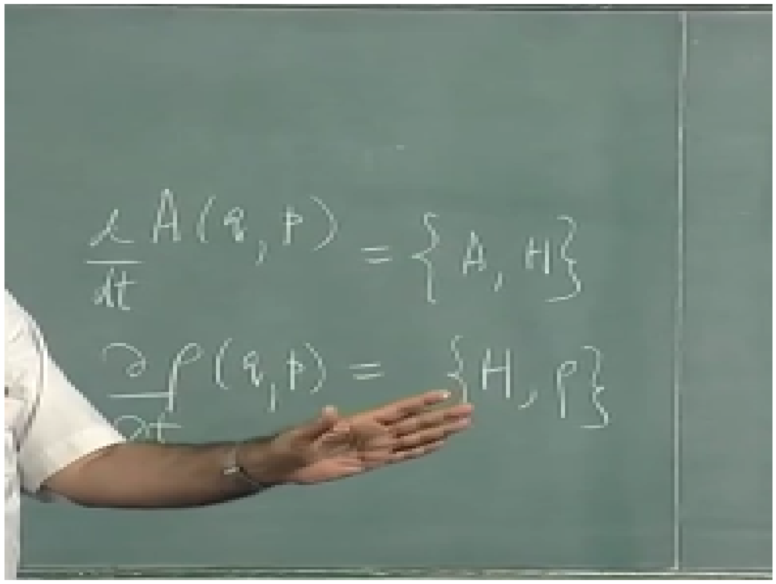

Whatever picture I choose to describe this evolution in so it is a little bit like to give an analogy it is like saying that if you have a particle moving in space then two ways of doing this either you say the position of the article changes in space and the axis remain fixed or I could equivalently say the particles really stays where it is but its axis that are changing in space and both these things should give exactly the same results they just 22transformed ways of looking at the same thing one of them is an active picture.

And the other is a passive picture in quantum mechanics we would say one of them is the Heisenberg picture and the other is the Schrodinger picture and physical answer should be independent of which picture you choose that imposes this equation on the density matrix on the density of operator itself no yeah by insisting on this namely the time average of any physical quantity now I take the average value of this quantity.

And when I do that then it turns out that if I write the formula down for the average value and look at its time derivative then depending on whether I take the average value of a and let us suppress the Q and P for a moment if I take an active picture and say this is the average value of this physical quantity which changes with time with respect to a stationary distribution row this must be exactly the same as a picture in which the a does not have explicit time dependents does not have time dependence at all.

But it is with respect to a row which depends on time and when I insist that this be the case then automatically it says consistency demands that row itself obey an equation with an extra minus sign here okay and this follows from the cyclic invariance of the trace I will demonstrate this after a little while okay so eventually the point I was trying to get across here is that since Δρ ΔT equal to this if, if the distribution is to be a stationary distribution.

Then this should be equal to zero and that implies that the Poisson bracket of row with the stationary distribution or the equilibrium distribution let me call it stationary be equal to zero identically what does that imply what does this imply for this distribution this is a function of all the phase space variables and this is the Hamiltonian of the system and you are saying the Poisson bracket of this should be identically equal to zero.

So what is the most general solution you can think of what condition does it put on row stationary it is a constant of the motion but it is not a physical observable so what condition does this put on row if I say that the Poisson bracket of row with H ρ stationary with H must be identically zero in general it implies that this row stationary should be some function of H that is the most general thing you can think of under all conditions if this is to be true this must be a function of that H.

And that is exactly what has happened as you can see the stationary or equilibrium distributions row equilibrium of functions of H but what functions is not specified by this equation here that is specified by your experimental conditions whether the system is isolated or the system is closed system or not and that is why invariably the in equilibrium statistical mechanics the probability density in phase space is a functional of the Hamiltonian of the system and not of anything else and that follows from here.

Eventually now what can we learn from this for our problem well in our problem unlike equilibrium statistical mechanics where this is all you can say you can say that this must be some functional of this but it does not say what functional at all that has to be put in from outside and the fundamental postulate of equilibrium statistical mechanics or equal to you says that all accessible microstates of a system and thermodynamic equilibrium an isolated system are equally probable.

And that is this measure so it says on this energy hyper surface you have a uniform measure it is completely erotic and you have a uniform measure because this immediately tells you that this measure this Δ function says there is only one constraint that h = e it does not preferentially say this portion of the energy hyper surface is better than that portion or anything like that on the other hand when you have a canonical ensemble when the system is in equilibrium with a heat part then it says this thing it is called the Gibbs measure.

This factor the Boltzmann factor appears here and this is the particular functional of H so in the case of statistical mechanics you need entrain puts it is not yet fully derive able from dynamics at all and that tells you what the form of this invariant or equilibrium distribution is but in our case we have some dynamical system specified by set of equations and everything must follow from those equations there is no external input possible anymore so where does that leave us well let us see how to find invariant measures and this is our next task.

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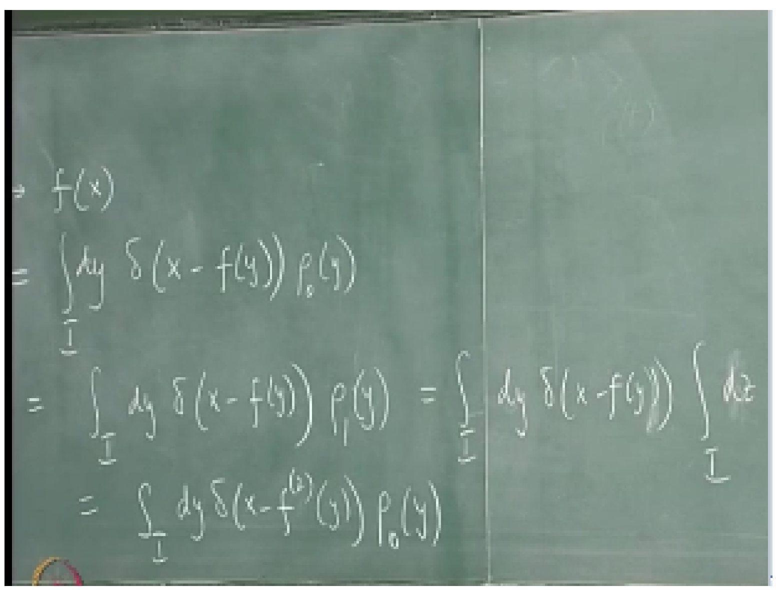

So let me call it the invariant measure or the invariant density again to make things specific we work with a one dimensional map which is chaotic say map of some unit interval or something like that and then the equation the map is the following xn + 1 is equal to some non linear function of xn and xn is an element of some interval that is the sort of situation we have in mind and.

Now let us assume that you are in a region of the parameters where you have chaotic motion completely and this is the interval this is the attractor and you would like to find is there an invariant measure or not okay how do we do this well if I start with a point X in this interval and I iterate ones then it goes to a point which is f of X therefore if I start with some initial point why and I ask what is the density ρ after onetime step as a function of X what is this equal to well clearly.

If I start with the point why it goes to a point f of y and therefore the support of ρ 1of x is a Δ function of X- F of Y if Y is my starting point so this is clearly true what happens if I start with a distribution in why what happens if I start with a distribution which is Rosie row of why a given Y goes to an X which is given by X - F of Y therefore a distribution or a probability density in y r 0 of why each Y goes to its corresponding X according to this rule.

And if I sum overall possible wise I end up with Row 1 of X so different wise would go to different axis but each Y goes to an X which is given by this Δ function condition because it is completely deterministic you give me the why and I know exactly which exit goes to and now the probability density in X is just the integral of the probability density in Y weighted with this colonel so it is an integral equation for Row 1 of x given after one iteration a density ρ 1 of X what is ρ 2 of X what is ρ 2 of X going to be equal to well.

Once again you take the density after one iteration and then you again weight it with the corresponding Δ function kernel and you get ρ2 of X so this is again equal to I d Y Δ of X -F of Y but here you put Row one of why because everything that I want here is a function of X and I keep integrating this well if you like you can write this out in several ways you could write this out as I d Y Δ of X -F of Y times on integral over I Δ some D Z Δ of Y - f of z Ro not offside you could do this.

So it would start with Z that goes to Y then it goes to X so certainly this is also true you could so have written this as equal to i D Y Δ of X- f2 of y r o not of Y so I could have said well you have to have wo iterations now in time and the function that takes you from a given Y to a given X after two iterations is f2of Y so if this is your colonel then you integrate the original density with this Δ function kernel and you have the density after two iterations.

So these are more or less obvious statements so therefore we generalize this and say if you give me the density after n iterations then the density the n + 1iteration is given by this right which you could also write if you like you could write this as f n out here and then put a row 0 here but now suppose you have an invariant density in other words under iteration it does not change at all that would correspond to taking the limit n going to ∞ in this n goes to ∞in this.

So what would that lead to so let us call limit n goes to ∞ row n of X = ρ of X and let us call this the invariant density that is like your equilibrium or stationary density in the case of statistical mechanics but now we have a dynamical equation here like the Louisville equation we have a dynamical equation here and you need to take the limit n going to ∞ in this and what would be the equation obeyed by this invariant density.

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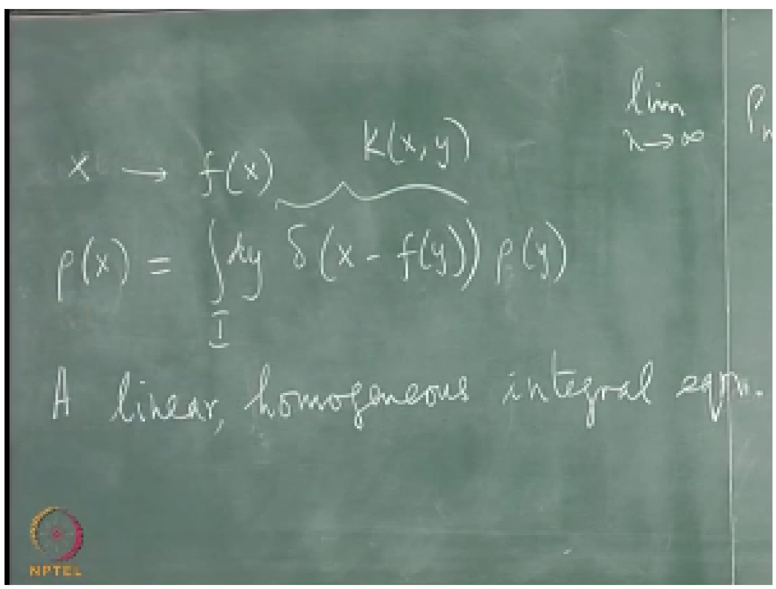

If it exists yeah so all we have to see is to see what happens to this equation when n tends to ∞so if you call that ρ of X that is it because as n tends to ∞ so does n + 1 so the invariant density of ways this equation what kind of equation is this is it a linear equation this is a horrible nonlinear function but now remember the equation we are talking about is an equation for this function so is it linear yes indeed of course because just the first power of it so it is a linear is it a homogeneous equation or in a bun me this is horrible.

This is some function this is some complicated some K of X , Y in row that is the unknown that is the unknown quantity this is unknown right the suppose will be given this you give me the map function and I know this so it is very much a linear equation is it homogeneous or inhomogeneous where is the in homogeneous term here it is an integral equation definitely it is not a differential equation it is an integral equation.

But is it homogeneous arraign homogeneous in other words if I multiply Row it by a constant what this will be true yes of course so it is a linear homogeneous is it a differential equation or an integral equation or a difference equation it is an integral equation it is a very much an integral equation is it an Eigen value equation in some function space this is some function x defined over some interval so yes indeed it is an operator it is an integral operator acting on this rope.

And that gives you this row again therefore it is an Eigen value equation this it is an Eigen value equation for an integral operator with Colonel K of X , Y is this kernel smooth or is it singular it is singular because it is a Δ function so it is a singular integral equation not an easy thing it is got a singular Colonel if it is an Eigen value equation then in some abstract notation we have an equation of the form row some vector in some function space.

The language of linear vectors is equal to some abstract operator k an integral operator acting on row that is the way this equation structure is so it is very much an Eigen value equation operator on this Eigen vector is equal to the eigenvector what is the Eigen value 1 in this case plus 1 so you want to solve a singular linear homogenous integral equation such that they the solution corresponds to I + 1 for this integral operator with a Δ function kernel is there any other condition on row based on the physical requirement.

That there should be a probability density it should not yeah exactly it should be normalizable can it be negative it cannot be negative it is a density therefore it could be unbounded does not have will between 0and 1 but certainly it must be normalizable.

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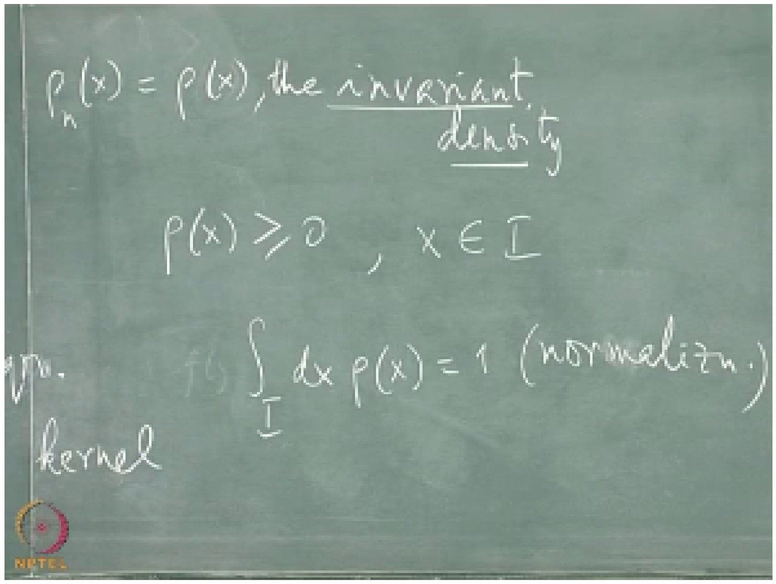

So you have extra conditions on it so we also require ρ of X > 0 for all X in the interval and you also would like to have integral DX of X = 1this is the normalization so if the solution exists that depends on whether this but sort of map function you have then you got yourself an invariant measure.

But you have to satisfy these conditions this Colonel may not have an Eigen function which satisfies these properties may well not have one but if it does then we are in business provided we know that it is the only solution for sure so you also need uniqueness because we need to be sure that this is the only solution there is only one solution which is physically acceptable in this case.

So it is not an altogether trivial matter even in this one dimensional map you have a complicated situation you really have to solve an integral equation it is homogeneous so its overall constant overall multiple cannot be found from the equation itself but that is fixed by this normalization condition this will fix what the constant multiplying this row is the overall normalization constant it should be non-negative could be unbounded we do not care should be integrals.

So that this condition is satisfied and it should correspond to an Eigen vector in function space of that singular kernel with Eigen value +1 there could be other Eigen vectors corresponding to other Eigen values so the spectrum of this operator could be very complicated indeed but the physically acceptable solution is one which satisfies these conditions and there are theorems which tell you that if such a solution exists then it is unique and that is too intricate for me to prove.

So I am not going to do that but let me take it as an assertion here let me make this assertion that if a solution to such an equation exists which satisfies these conditions then it is unique so if you grant me that let us try and see what happens in the simplest cases we know off like the Bernoulli shift for instance.

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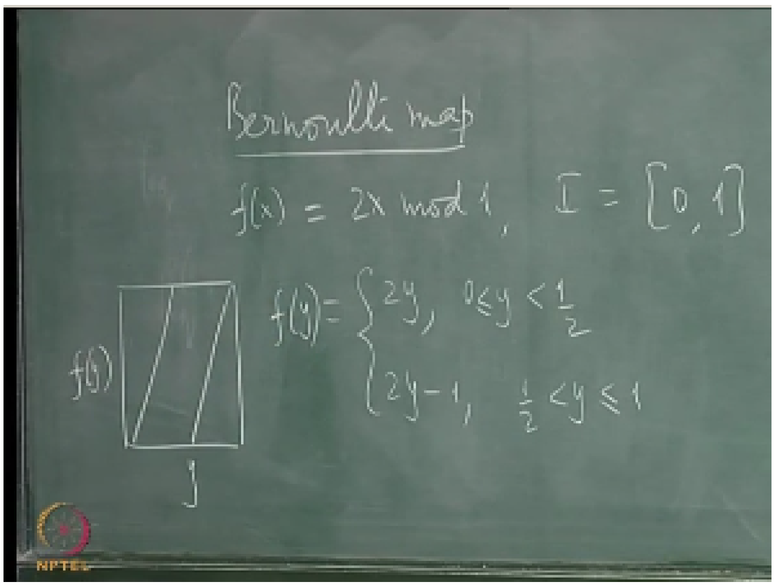

So let us apply it right away to the Bernoulli shift see where it takes us so we need this equation we retain that and let us apply it to the Bernoulli map and recall in this case the map function f of X was equal to X modulo 1 and the interval I was 0to 1 so to write this out this was equal to what to write it out explicitly it was equal to 2 y 4 y < a 1/2 between zero and a 1/2 and it was of equal to 2y -1 for ½<y < unity and the map was discontinuous at the point half.

So to refresh your memory this is what the map function looked like this is what F of Y was equal to so let us apply and find out what row of X is so ρ of X therefore is equal to an integral over this function from 0 to 1there are two Δ two Δ functions here one of which will be applicable as long as Y is less than a half and the other between a half and one so let us write them out separately so it is 0 to 1/2 dy a Δ function of X -2y row of y + an integral from a 1/2to1 d Y Δ function of X - 2y +10 in order to do this integral we got to convert this to a Δ function / y.

So we have to write Y minus something or the other and then make sure that this Δ function actually fires when y is in this range right so it is these lines on which the Δ function fires and what we have done is to break up the integration from zero to this, this and remember the constraint is that this quantity is equal to X whatever be X and X runs from 0 to 1 so what would this become I have Δ of x - a function of Y.

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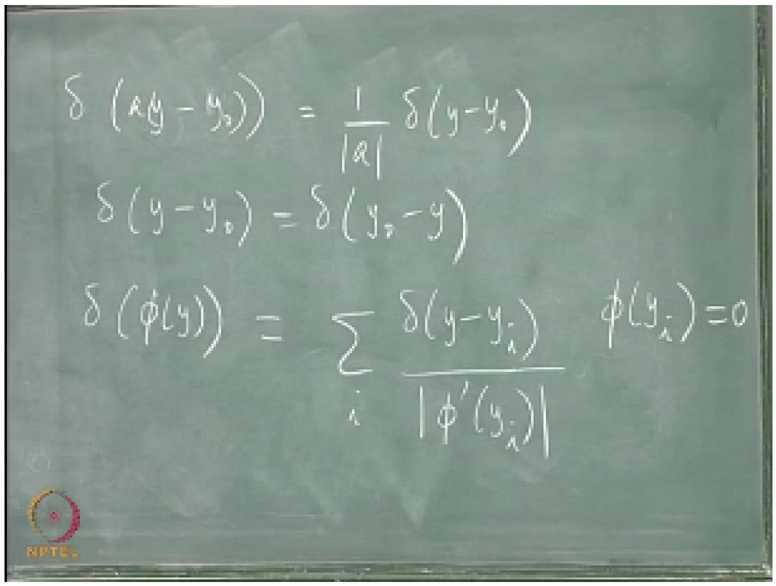

But we use this formula for the Δ function of a function of the argument of the integrant right and if you recall a Δ function of a function ρ of Y can be written down first of all what is the Δ function the certain properties of the Dirac Δ function which will need for this business and I am not sure if you recall them right away but let us write them down first of all Δ of a constant times y minus some why not right is equal to one over modulus a pair of Y - y not modulus is some real constant a.

And then I need a modulus in the denominator right so this makes sure that the normalization is kept correct we also know that Δ of Y -y 0 = 0 ΔY 0 - y the Δ function is an even function of its argument about the point where it files we also know that Δ function of F ɸ of Y where π is a nice function of Y this contributes whenever ɸ of Y vanishes because that is where a Δ function contributes right so suppose Y sub I is a root of π of y = 0 so π of why I = 0 and let us label the roots as y 1 y 2 y 3 etcetera.

Then this fires whenever you have such a route but near one of these roots if it is a simple root then it is clear that π of Y could be written as ϕ of why I = 0 by definition + y -y I times π ′at why I but that is like a constant and you bring that down so this is equal to modulus x prime at why I and you have to sum over all the roots so these are the properties of the Δ function that we need here and is very simple in this case.

Because by symmetry I write this immediately as Δ2y -X and then I take out the 2 so this becomes equal to 1/2 because I take out the two here and then after that I can do the integral because this becomes Δ of Y -x over 2and you got to wait it with row of Y and integrate over all values of Y from 0 to half and please notice that going back to this graph whatever be the value of x between 0 and 1 when y is between zero and a half there is a contribution from this Δ function it does fire at one unique point and fires at a point which is given by x over 2.

Therefore this integral contributes for all x between 0and 1 that is important to remember so you have to draw a picture of this kind to find out where these Δ functions contribute and in this case this is the picture we have to look at so as a function of Y this is a half this is 0this is 1 this is 1 you have X on this axis and wherever X is between 0 and 1as long as y is between 0 and a half there is a point where this graph is intersected and therefore there is a point where the Δfunction contributes.

Therefore I can write this first integral down immediately as ϕ of X over 2 + once again I write this as Δ of 2y - 1 +X and I bring out the 2 which comes out here and this again becomes rows of 1 +x over 2 again as before for every y between a half and one no matter what the X is between 0 and 1 this Δ function does contribute is one unique point where it contributes for every x and therefore for all X in this equation X element of 0 to 1 you have this equation we have therefore converted this equation.

Here for the invariant measure from an integral equation to what kind of equation what kind of equation is that it is a recursive equation but it is not a difference equation it is for a function in which the arguments are different it is not a differential equation it is not a difference equation it is a functional equation this sort of thing is called a functional equation this integral equation here is very important this thing.

Here for the invariant density this is called this is the Fresenius Peron equation for the invariant measure it is an integral equation with a singular kernel and we have reduced it in the case of this one dimensional map to this functional equation and for every X this must be true does not look like a very straight forward equation to solve because functional equations are not easy to solve there are no standard algorithms in most cases to solve these equations at all there are some techniques but they are very intricate.

And in general there are no systematic algorithms as you have for the case of integral equations or different differential equations or recursion relations difference equations or integral differential equations and soon functional equations are difficult solved but we have this uniqueness criterion that I have mentioned earlier as a very big help in other words if you can find a solution which satisfies this functional equation.

Then it is got to be the unique solution provided it is normalizable and provided it is nonnegative now of course one can immediately see from this that if you put X = 0 then it says row of zero must be equal to row of a half then you put X = 1 and it says row of 1/2 = ϕ off one and then you put values in between and you discover that all those rows must be equal to each other so what would be a guess it is a constant right.

And what kind of guess the constant should it be it should be normalized 0 to 1 row of XDX must be equal to 1 1 itself 1 itself so if ϕ of X is a constant equal to 1then it says 1 = 1 +1/2which is true right so in this case the unique solution to this density is in fact a constant so the unique normalizable normalized non-negative solution ϕ of X = 10 > to x > 1 kind of establishes.

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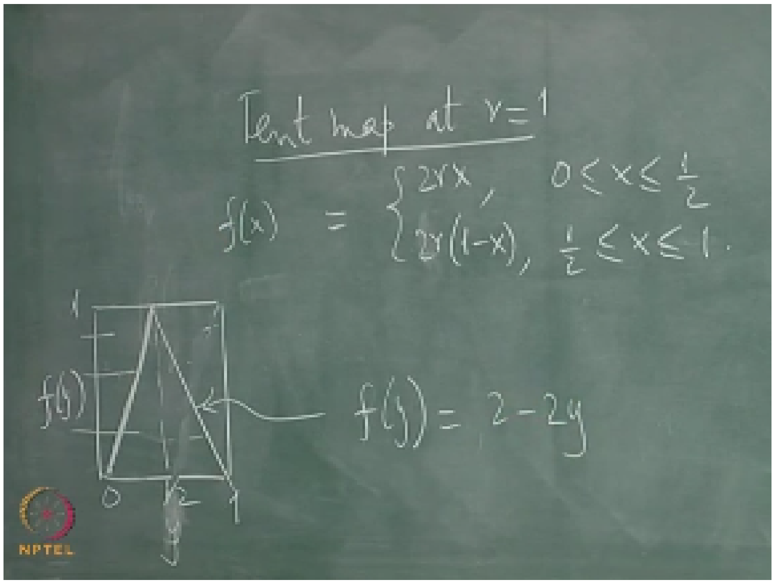

Why I said that in the case of the Bernoulli shift the system is erotic and the invariant density is unity in other words all irrational points on this line their iterates will uniformly and densely fill up the entire interval uniformly with a very simple invariant measure just a constant the moment you have that under your belt then you ask what about the other maps that we looked at for instance.

Let us look at the 10th map at fully developed chaos and see what happens in that case so for the 10th map at r equal to i use to our so this must be equal to 1 10 map at r = 1 recall that this map itself was equal to 2r x 40 > x less than equal to a 1/2 and it was 22 r into 1- x for a half less than equal to x >1 but me yeah it should be one is anticipated us so it should be one it turns out to be exactly equal to one.

Once again we can check, check this out right away this is what the map looks like in this case right and this function here this f of y equal to when r = 1 it is 2 -2y right when you put that equal to X so what happens what kind of functional equation do you get this portion remains the same what happens here so you do not get this so if 2-2 y = X on this side right so what you get yeah.

So what is y equal to but what do I get here plus row of what ya 2 - x over2 which is 1 -x over 2 again so it says row of x is equal to this quantity i put x = 0 and it says r 0 of 0is =ϕ of one when I put x =1 and it says row of ½=ϕ of 1 and so on yeah, yeah there is sort of yeah because after all what I did was to take this map going up that way I cut and pasted it in one case and I folded it in the other case in some sense you expect.

The distributions would not change at all and indeed that is true the difference of course is that in one case you have a discontinuous map and they are the case you have a continuous map this is complete continuous here although its slope is not defined at this point it really does not matter and you end up with once again $\phi = 1$ again by inspection I put ϕ=1 and it satisfies this equation on both sides so for both these maps so for the this is the 10thmap the symmetric 10th map at fully developed chaos.

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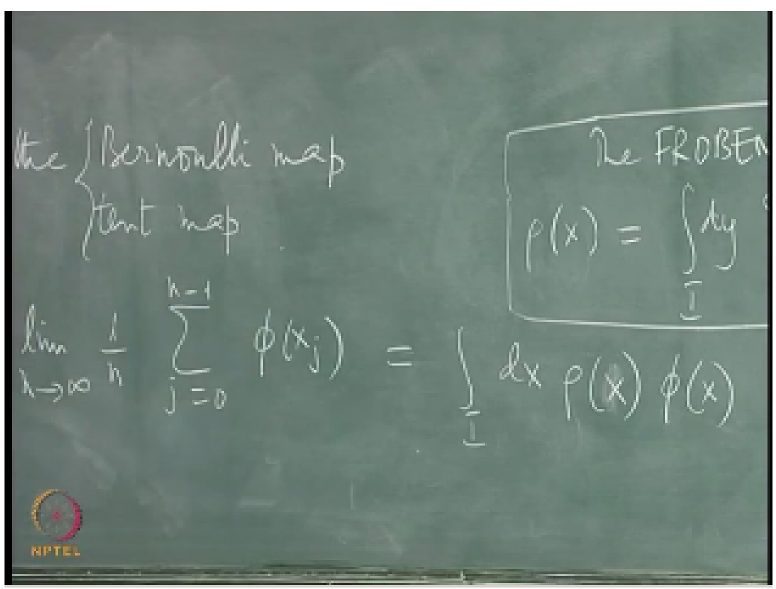

So in both these cases I have ϕ of X =1 for the Bernoulli map for the Bernoulli map or the Bernoulli shift can plan not true for the 10th map for our less than to less than one between half and one that is not true because then it is not the unit interval at all it is much more complicated than that and the invariant measure is not findable by analytic means and find it numerically because the equation is not easy to solve in that case but here for this fully developed cases for these on two maps this is certainly true.

Now what is the great advantage of finding the invariant measure well any physical quantity whose average you want time average you want can now be replaced by an ensemble average so all we have to do once you have an invariant measure is to say that if I have the iterate x n and I take any function of it here and I would like to compute the time average of this quantity then I would write the following π of X.

The time average of this observable is equal to limit n tends to ∞ 1 over n a $\sum$ from j = 0 to n - 1 let us see π of x j where XJ + 1 = f of X T that is the rule for iteration i iterate each time I get a new variable each time I get a new value of the argument and I put that in here and I some of this take the arithmetic average and I have got a time average but if the system is argotic on some interval I can replace this by an ensemble average weighted with the invariant measure therefore.

This can also be written as equal to an integral over the interval DX ρ of X where this is the invariant measure or the invariant density rather x π of x itself if I know ρ of X this is just an integer now what was our definition of the Lyapunov exponent remember the Lyapunov exponent λ was defined in these cases as equal to limit n tending to ∞ 1 over n summation from j = 0 to n - 1the log of the modulus of F prime this right.

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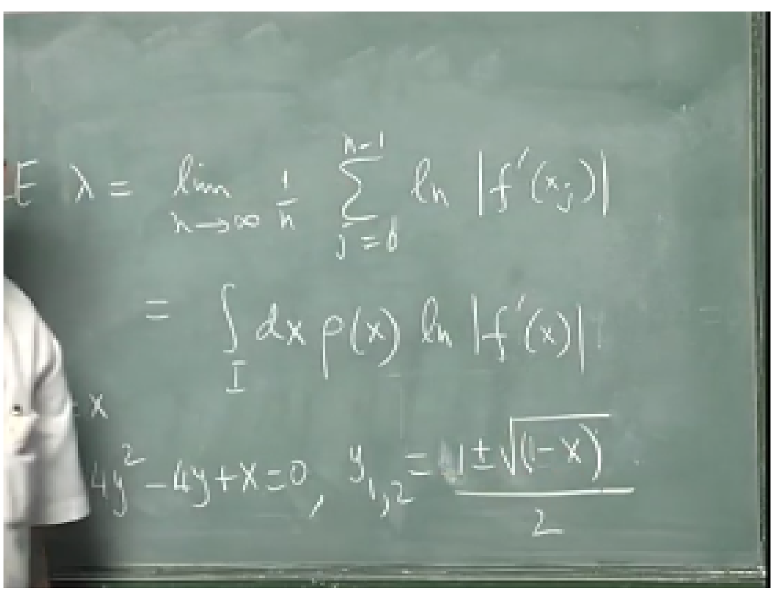

So instead of iterating the map and finding the slope at all the iterates and taking the modulus and then the log and summing over the whole thing you can write down now where God I city you can also write this down as equal to an integral over the interval DX invariant density ρ of X log of the local stretch factor which is log of mod f ′ of X so you give me the map and I compute this number for you and what is it for the Bernoulli map log 2because ρ of X is 1 yeah we implicitly use.

This we actually implicitly use this although in that case it was trivial because I know this thing is independent of x j it was piecewise linear with a constant slope so I this was rigorously equal to log 2 times 0 to n -1 of 1which was n and it canceled on both sides and you ended up with log to explicitly but now it is consistent with the fact that ρ of X = 1 so this immediately implies that ρ of X = 1 and λ= log 2 independent of the starting point.

So the system is uniformly hyperbolic everywhere there is a stretch factor which is essentially log to the log of the stretch factor is essentially the Lyapunov exponent and it is uniform everywhere it is log 2 in this case we are not going to be able to say the same thing for the logistic map at fully developed chaos because for one thing this is no longer constant so you certainly cannot do this and we do not know what the invariant density is we need to discover what the invariant density is for the logistic map.

Before we can assert that the Lyapunov exponent is log to it turns out to be locked to four fully developed chaos I mentioned this last time but then it is not obvious unless you know what to put in here and this quantity here of course is x dependent because the map is four times x times 1 - x and its slope changes from four at the two end points right to zero in the middle and you have to do this integral in order to compute what the Lyapunov exponent is but.

Before that you need to know what row of X itself is but what is the Fresenius Person equation in that case for the logistic map what you have to do is to take this function this is f of y and this is why and we know that F of Y in this case equal to 4 y times 1 - y you must set that equal to x and solve now for y to find the two roots of this equation and certainly there are two roots because for every given X there are two values of Y and one of them isy1 of X.

And the other is why 2 of X at every level X so you need to find that and then this equation changes and becomes a functional equation and it looks like ρ of X = ρ off sorry equal to ρ off the Δ function fires at two points now in each case so for a given X you have a y1 of X which is this route and you have a y 2 of X which is that route and therefore the density is evaluated at y 1 of x divided by the slope f ′at y1 of X modulus plus row of Y 2 of x divided by the modulus of the slope of the map at the point y 2 of X where.

This is why one and this is why two or the left branch and the right branch of the inverse function you have to compute this by solving y1 and y2 are the roots of this equal to X and it is fairly messy it is a functional equation which is fairly messy because you have some complicated function of X here which involves square roots and so on because this is a quadratic function of Y you solve it there are two roots you have to write down explicitly so let us do that so it says $4y^2$ take it on the right hand side - 4y + X =0.

So y 1 , 2 = +- = 4 +-√ - 16x / 8 which we can simplify so this becomes 1 +- √1 - X / 2 that is y1 and y2 as functions of X you have to plug that in here put in the values of the slopes of the map at those points and then you have a functional equation whose solution is by no means obvious but it is immediately clear that because of these guys sitting here that a constant is not a solution you put exit row of x = 1 it is not a solution to this equation so guess work does not seem to work here so what does one do.

But now I am not going no, no I am not going to find a rover there is no reason why it should be logged to do not yes we know I said that it is locked oh so now you want to work backwards but it is not so easy guessing solutions to this because as you can see immediately you are going to have something which is row off and look at it is not trivial1 +root 1 -x over 2 when that is horrendous row of 1 -√- x over 2 and then there are these factors as well it is not by any means trivial this itself involve square roots.

And so on by the way since its modulus you expect both of them will come out because it is clear that the map is symmetric so the magnitude of the slope here is the same as the magnitude of the slope here so this factor will come out but it is a function of X and constant is not a solution so what we need is a clever trick in this case this clever trick was found by 491 and alarm long ago in 1947 and they solved for the invariant measure invariant density of this map and this turns out to be a very well-known solution turns out to be very interesting solution also with many, many ramifications.

But the idea is the following and let me give you a hint as to how one goes about solving this problem at fully developed chaos at parameter value for this map is solvable in the sense same sense in which the Bernoulli map was so solvable we could write the solution X n = 2 to the N X 0 modulo 1that is formally in exactly the same way.

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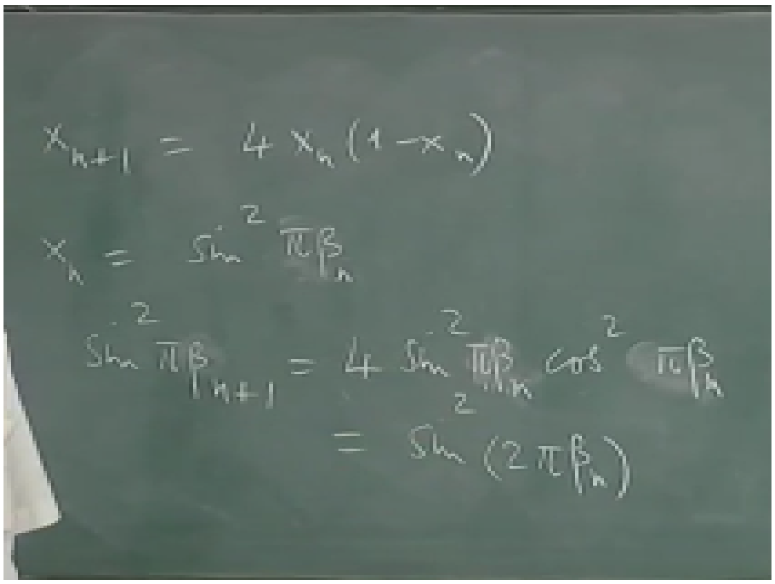

If you have something which says xn +1 = 4x n x 1 - xn and xn is between 0 and 1. So it suggests very strongly it is like the sign of something or the other but the sign runs from - 1 to 1 perhaps the sine ^{2} would be a good idea so suppose I put X n = sine ^{2} θ sub n what happens to this map so it is a sine^2 θ n + 1 = 4 sine ^{2} θ n cos ^{2}θ n = sine ^{2} to see the N so as long as we define the range of θ properly it looks like θ n + 1 is 2θ n we could put in a pie to make sure everything runs from 0 to 1 instead of 0 to whatever right but that is solvable that is the Bernoulli shift if you to put your putting a factor of pie.

So let us put sine ^{2}πsome β n so it is a sine ^{2} πβn + 1 is β sine pi β n / n and this whole thing becomes square to then simply say is β n + 1 is twice β n modulo 1 but that is just the Bernoulli shift for which we know the solution so it suggests strongly that I can actually write down the solution to this guy and in fact it is of the form X n = sine 2 π times β n which is equal to but we also know X = sine ^{2}πβ 0 so this becomes √ of x ^{0} here and then a sine inverse gives you this and then a 1 over π gives you this.

So this x 2 to the n over π that cancels off sine inverse root x 0 please check the factors of π and so on what is something like this is the solution to this map at fully developed chaos so at the parameter value for things become a little simpler and it looks like it is essentially the Bernoulli shift in some transformed variables that is the reason why it turns out to be log 2 and incidentally that also tells us that if the measure in βn is uniform this has a Bernoulli shift.

So it is constant then the measure in X is just the accordion of the transformation that takes you from β 2 X once again and I will write this down and we will verify this subsequently for the logistic map ρ of X for the map the fully developed chaotic map so for f of X = 4xtimes 1 -X the solution to this equation is ρ X = 1 over the normalized solution x times 1 - X this is due to a l am this map is called the coulomb map very frequently this was used this thing was used as a random number generator by them originally by for reasons which I will explain later on it is like the Bernoulli shift in transformed variables and you actually have this as the invariant density notice it is not bounded at X = 0or X = 1 goes but it is integral just square roots in the denominator and gets integrals.

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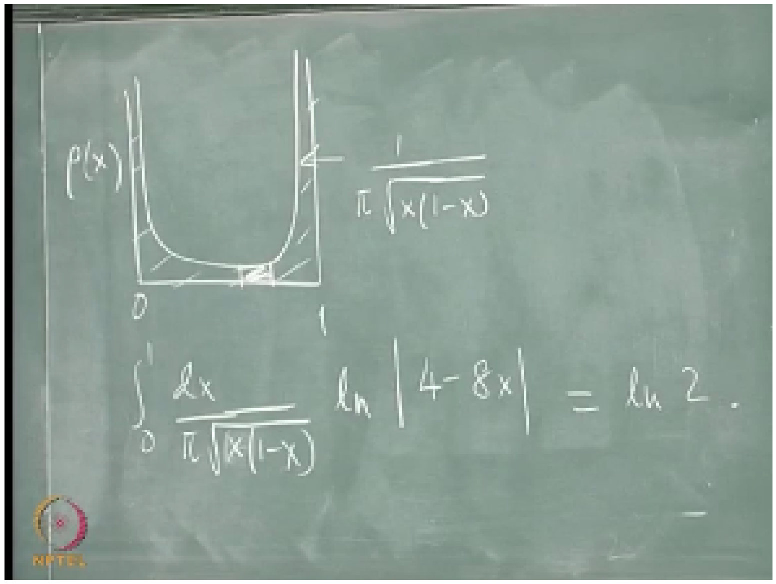

So if you draw a picture of this invariant density then it looks like this here 0here is one it is a very symmetric thing which looks like this and this curve is1 over π √ x times 1 - X the total area under the curve is unity so the system predominantly likes to stay near the origin or at the other end and it is very rare that it comes in here so the total area is actually one is completely symmetric the cumulative distribution is some sign in versus you can easily see you have to do an integral over this, this is the density the probability density.

So you could ask how much time does this iterate spend in any sub interval then that is just, just the ratio if you want this sub interval the ratio of this area to unity which is the total area by a god I city once again so you can calculate the fraction of the time a very long trajectory a typical chaotic trajectory spends in any given region sub region of the full interval now I leave it to you as an exercise to verify that for this map the integral from 0 to1 DX over π root 1 - x times 1-x times the log of the modulus of the slope of this map.

But the slope of this map is 4 -8x this integral can be done without too much difficulty and I leave it to you to verify that this is equal to log 2 so that rigorously establishes that the Lyapunov exponent is locked to for the chaotic for the logistic map at fully developed chaos at parameter value for this trick does not work for any value less than 4 of course when there are fixed points which are stable or periodic orbits we know the Lyapunov exponent X explicitly but In the chaotic region in general 11 only knows it numerically.

But it reaches a maximum value at fully developed chaos for all mu between 0 and for the largest value of the Lyapunov exponent is log 2and it occurs at four parameter value for this transformation is part of a more general relation called topological contumacy the, the maps be relating to each other they are related by his transformations of variables that is the reason why this trick works in this case I will come back and talk about topological contumacy little more because.

It says given one map you are able to find things for other maps to which this initial map is related but right now the point I wanted to make was that even though you do not have a uniform density the lyapunov is still exponent is still locked to and that these points are preferred for this map rather than anywhere in the middle in this case there is one more value of the parameter where this map is explicitly solvable in the sense that you can find X and as a function of X not explicitly and that case does not have any chaos unlike this case and that is the case when you have a super stable fixed point.

So in that case it was twice in between and recall this was the case where the fixed point was at X equal to a half and the slope there was equal to zero so the Lyapunov exponent went to minus infinity at that point but you can also solve this you can solve the equation xn+1 2 X n 1 - xn by a similar trick you can write down X n as a function of X not explicitly what would you do what would you what does it suggest you do let me give you a hint suppose you multiply both sides by 2 what happens multiply.

This by two becomes four what do you what is what does it suggest for the right-hand side what should one do let us write this out for xn - 4xand square what does it suggest no such as something very strong I mean does not this suggest that you, you complete squares what should I do to complete squares so I do -1 there and I get a -1 here so 2 to xn is this so to xn + 1- 1 is equal to that but what is this equal to minus yeah to xn - 1 the whole square is not it so let us write this right.

So what does it suggest we do next bring the minus to this side so it is a minus this is equal to that so this is 1 -2 xn+ 1and this of course is 1 - 2 X n the whole square that is now trivial because it immediately says define a new variable u n = 1- 2x so it says you n plus 1 equal to minus goes away so what is the solution what is the solution to this recursion relation yeah each time you square right.

So what is you, you n as a function of u 0 sorry, sorry I am sorry u 0 to the power so now go back to X and the problem is solved these are the only two cases which can be solved analytically the only cases where the map n hu x times 1 -x for x in the unit interval can be solved in closed form explicitly the recursion relation can be solved are at parameter value 2 & 4 it is lucky for us to is not very interesting.

Because it is just a fixed point which is very stable no problem there but for is fully chaotic so we have like the Bernoulli shift we have a simple model which is not so simple it's got intricate properties as we saw but at that value at four things become analytically tractable you can solve things you can write down the Lyapunov exponent explicitly and we can form a relationship with the tent map and the Bernoulli shift so that is why this map again is used as a kind of paradigm for chaos in many cases.

So let me stop here today and resume from this point onwards we are going to look at a few other maps which have some other interesting properties and then I comeback and talk about the property some more properties of the logistic map you.

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