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TOPICS IN NONLINEAR DYNAMICS

Lecture 18
Discrete time dynamics(Part II)

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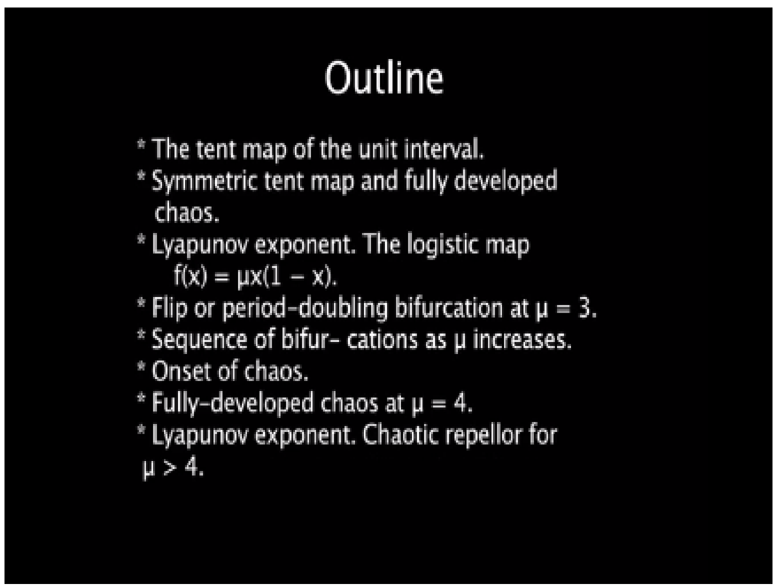

We saw some properties of the Bernoulli map in the Bernoulli shift last time let me introduce you to another map which is similar to the Bernoulli map which has similar properties and is closely related to it and it is perhaps a little easier to understand because it is not a discontinues map and this is the so-called tenth map.

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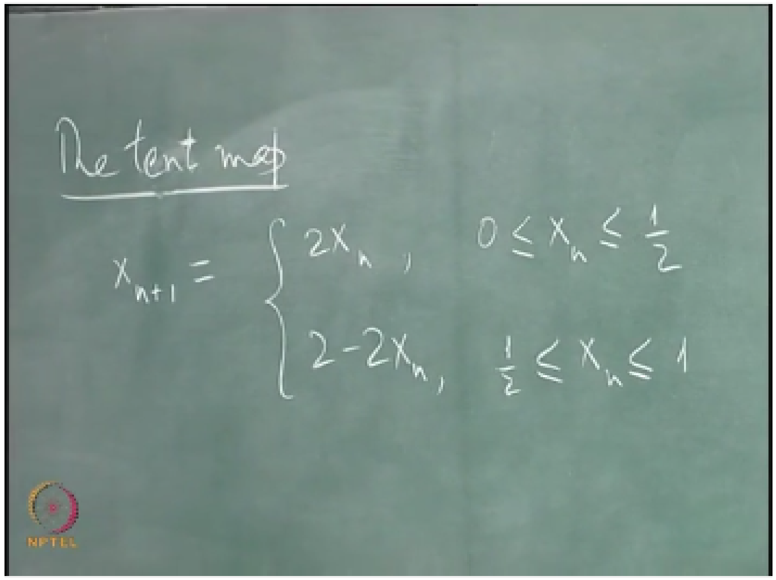

Which looks like this it is given by the function xn + 1 is twice xn doubling map as before provided 0 ≤ s Xn ≤ 1/2 and it is 2 - 2 xn for a 1/2 ≤X n ≤1. So again the map of the unit interval specified by this piecewise linear function we can draw this map very easily.

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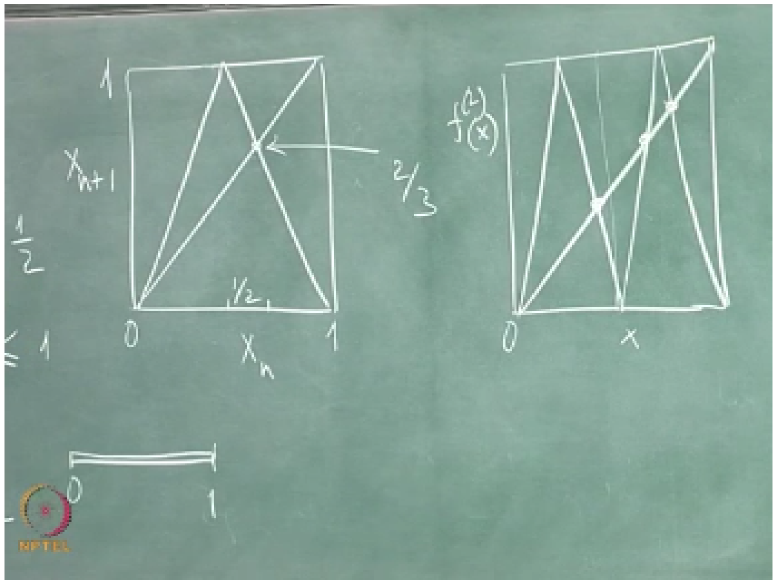

So here we have X n and then xn + 1 and it is 0 to 1 on both sides and this goes up there and comes down here at a point 1/2 the map changes slope from + to 2-2 again it is an on to map of the interval no point is left uncovered, it is not one to one because it is clear the map is not invertible for every value of xn there is a unique x and + 1 but the converse is not true for every xn + 1 there are two values of X sub n and therefore if you give me a final point X sub n there are 2 to the n possibilities for x0, from which this final point could have arisen.

So it is not invertible and it leads to chaos for very obvious reason because the phase space is bounded this is the bisector there is a fixed point here at the point 2 thirds, so the coordinate of this is two-thirds as you can easily check and it is unstable because the slope has magnitude to at this point the fixed point at the origin is also unstable, again the slope is 2. What are the iterates of this map look like well the first iterate of this map if i plot x versus f2 (X).

This would be something like slope 4 goes up and comes down and now you have once again you have, the fixed point of the map and then you have a period to cycle between these two points and then an unstable fixed point here as before. And as you take further and further iterates of this map you have many more spikes up and down and the slope at every one of those points is > one in magnitude and therefore not only the fixed points of this map but also all the periodic orbits of this map are unstable periodic orbits.

It is easy to check once again that the points which lie on periodic orbits and a set of measure 0 they dense everywhere on the unit interval and they are a set of measure 0 once again and the map is fully chaotic, it is completely chaotic in the sense that any typical initial condition from the set of irrational values of x would lead to an orbit that does not settle down to any final point and it wanders uniformly over the entire interval. We still have to find out how a ergodic is this system on the unit interval.

It is an ergodic system that is clear because typical initial points sooner or later go to the neighborhood of every point on the interval. So by our earlier definition of ergodicity this is an ergodic system, it is in fact a system which is exponentially unstable in the sense that it has a positive Lyapunov exponent in this case and the Lyapunov exponent is again log 2 we will verify that will prove that rigorously in a little while but it is exactly the same in magnitude and value as the Bernoulli shift.

In fact in physical terms in the Bernoulli shift what we did was to take the unit interval 0 to 1 we stretch this interval by a factor of two, so it really got doubled when you do it to X n so it went from 0 to 2, 2 and what did the Bernoulli shift amount to it amounted to taking this part of it cutting this if you like and putting it right back on top of this, so you ended up with something that looked like this, just pictorially between 0 & 1.

So this thing was snipped off here cut and put back onto this in this fashion and this is what led to the non inevitability of the map because you really have to pre images, for every point and then for pre images if you go to iterates backwards and so on this non invertibility is what led to chaos eventually led to the complicated properties of this map. So you μ st not get deceived by the piecewise line rarity of the map it is only piecewise linear but it is a nonlinear function of the same.

Because we saw that a linear map was very trivial it had a single fixed point in general and that was it, on the other hand a nonlinear map has these very strange properties the fixed point is unstable in this current of these conditions all the periodic orbits are unstable and the entire unit interval becomes the attractor in this case a completely chaotic attractor because a the phase space is bounded, B there is exponential sensitivity to initial conditions and C there is a dense set of unstable periodic orbit buried in this phase space.

So all the conditions we laid down for the existence of chaos are met and these are fully chaotic maps both the Bernoulli shift as well as the tent map completely chaotic maps. We could try to trace the origin of this chaotic behavior in the following way we could say well instead of looking at this particular map, the tent map which code which becomes an onto map suppose we did the following.

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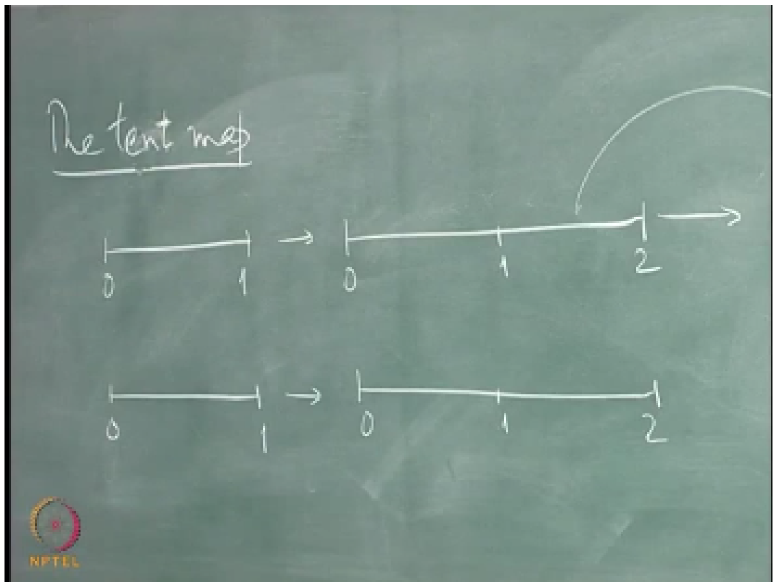

Yeah still in the job yeah it is simply the same thing because what he is pointing out is that in the Bernoulli map you started with this and then you sort of speak stretched it like a rubber band you stashed it all the way from 0 2 to 1 and then you cut and put it on top in this fashion, this is what you did, so this portion was cut and placed on top right on top of the unit interval between 0 and 1.

In the 10th map on the other hand you started with 0 to 1 and then you stretched it by a factor of 2 all the way and then instead of cutting and pasting on top you bent it for backwards. So in that sense what you did was to go here and then bend it backwards, so this was flipped over and bent backwards and again you produce this 221 effect and once again all the periodic orbits were unstable and so on.

So you can do it in many ways in phase space you do the stretching and then you cutting pasting cutting and putting it back or you stretch it and bend it back in this fashion, so these are typical mechanisms by which chaotic attractors are produced we will see if some more examples of this in higher dimensions so coming back to the tent map suppose we look at it as a member of a whole family of maps of the following kind.

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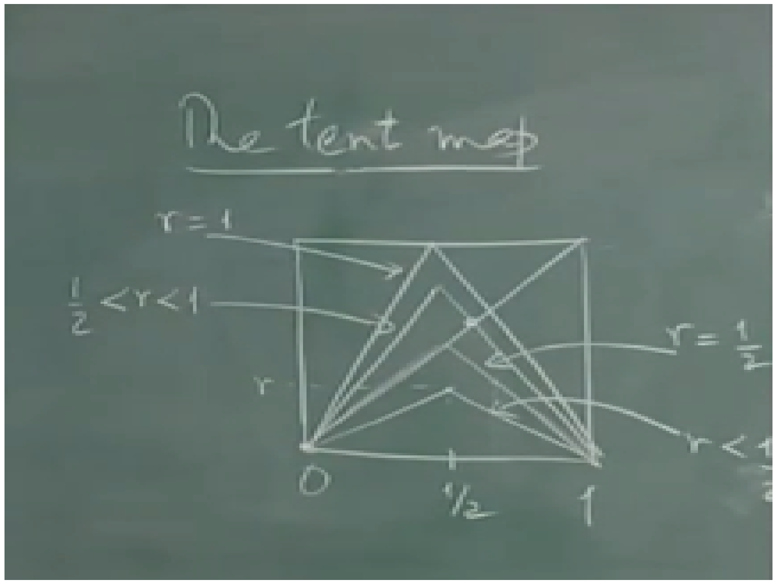

So here is the map function here 0 to 1 there is the bisector and suppose the map function is of this kind it is a line with some slope are up to a 1/2 and then it folds back to one on this side here, let us write this map down so we have xn + 1 or f of X in this case the map function is = depends on what you would like to call our so let us call this to our X with slope to our at the origin for X $0 \leq x \leq 1/2$ and what is the rest of this = you would like it to vanish when X =1 right.

So it is to our x 1 - x for a $1/2 \leq x \leq 1$ and it is continuous because when x is =a 1/2 it is just our the maxi µ m value is just R and that is true here too, so this point here corresponds to our the slope at the origin which is the only fixed point in this case is to our, so when is this fixed points stable? When to R is < 1, so the fixed point at X =0 is stable for two are < one or are < a 1/2 what happens at r =a 1/2 there are =a ½.

Exactly = 1/2 it is evident that this map goes right up to 1/2 and comes back in this fashion, so it falls on the bisector and then comes back here, so this corresponds to our < a 1/2 this map corresponds to R = 1/2 and where are the fixed points of the map at r =a 1/2 everywhere this map is degenerate everywhere all these points remain exactly where they are. What if I started with a trial value ≥ a 1/2 for this map at r =a 1/2 what would happen?

Well let us start here by this staircase construction and in the next step I would go here and that is the end I stay there, so it is clear that this map is degenerate the entire set of points from 0 to

1/2 is a fixed point, so to speak what happens as soon as our exceeds a 1/2 well here is a typical value our exceed > a 1/2 and it looks like this, so in this map 1/2 is < R is < 1, remember the peak is at r.

What sort of fixed points do you have now you have a fixed point here but you have another fixed point there and the slope in each of these cases is two r in magnitude and that is > 1, therefore these are unstable fixed points definitely and it is easy to see that the iterates of this map would all lead to unstable fixed points and therefore all periodic orbits are also unstable immediately. What happens at r =1 it becomes the original 10th map it is called the symmetric 10th map at fully developed chaos.

Because we will see why it is fully developed chaos in a second, so this is the map for r =1 this is the original 10th map and the entire unit interval is now covered it is an on to map and you have a chaotic attractor, which runs all the way from 0 to 1 but now let us try to draw a bifurcation diagram for this for all the equilibrium points in X as a function of the parameter r.

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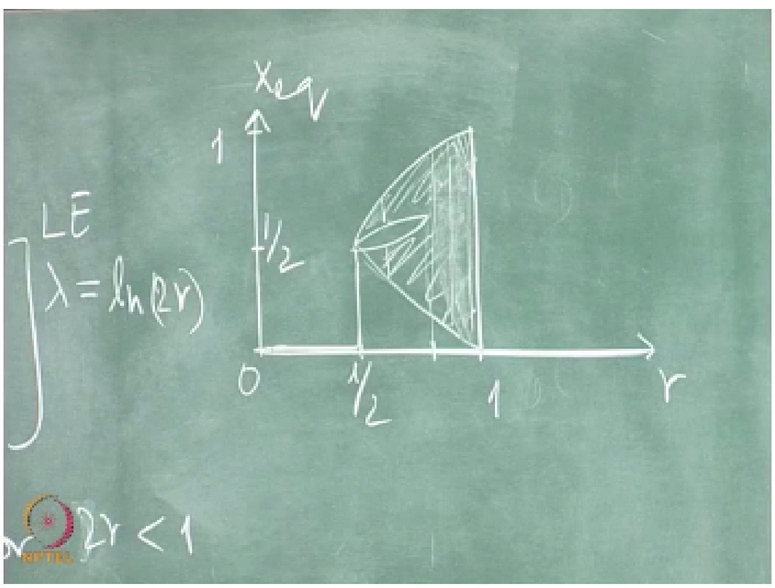

So if you did that so let us call these fixed points let me just call them equilibrium points just to have our idea straight as a function of r here 0 and here is r =1 what would this figure look like till r is < as long as r is < r 1/2 there is only one fixed point and that is at 0 and it is a stable fixed point, so by r normal ways of drawing the bifurcation diagram this is stable here nothing else happens what happens as soon as our exceeds a ½.

Well at r =1/2 it is clear this entire set of points if you like is fixed points degenerate map and notice if our takes on a value between 01/2 and one there is no way you are going to reach any values > R because the function never takes you beyond the point of itself. So the unit interval is not covered wherever you start eventually you are going to fall into a window between 0 and r and in fact what you do is fall into a window and at r =1 the entire unit interval is an attractor.

It is a chaotic attractor but in between in between till a 1/2 this is one in between after this there are no stable fixed points there are no stable periodic points either, the system starts becoming chaotic and the region into which the iterates fall gradually expands till eventually it sort of falls into the unit interval here. But what happens here is very interesting a little band emerges numerically one can explore this and the system falls to this region here.

There is a little window in X which is never covered as importantly you never reach that eventually so that factor is in two bands, there is a band here and there is a band here and beyond a certain value of R which you can discover numerically, these two bands merge once again and the entire you will interval here is a single band which continuous goes on all the way till this point and in between you do not have any periodic cycles of any kind which are stable at all.

We will explore this numerically I will bring a figure which or demonstrate this on a computer which will show you how this attractor gets built up, so chaos actually starts beyond a 1/2 but it is not fully developed chaos because you do not have complete the entire interval is not k a part of the chaotic attractor this portion and this portion those values of X alone correspond to the attractor and as you go along that gets bigger and bigger till eventually there is one continuous set all the way till 1.

These curves are not as smooth as I have indicated here it turns out this curve is a fractal curve in itself one of these is a fractal curve and there is a fair amount of intricate numerical complexity that goes on here, even though the map looks extremely simple. So even in this extremely simple map this bifurcation diagram is fairly intricate we are going to see many more complicated examples but this itself already tells you that very simple one-dimensional dynamics all you are doing is a piecewise linear map of this kind.

In graphical terms all you are doing is to iterate this function over and over again for r > a 1/2 and you end up with this very intricate kind of dynamics automatically. What do you think is a

Lyapunov exponent for this map, we saw its log to when r was =1 but what is the Lyapunov exponent for this map for an arbitrary value of r, exactly it is just two are because the slope is uniform everywhere and in these one-dimensional maps, remember that this quantity F ` of X modulus is the local stretch factor it is the cobian of a transformation as you can see.

If you change from X to f of X this is in fact the air cobian of the transformation you take its modulus and you take the log of this, gives you the local stretch factor the local Lyapunov exponent if you like and of course once you have a constant piecewise linear map with a constant value of mod f ′ of X, everywhere that is the Lyapunov exponent for the map. We are going to shortly come across a map where this is not a uniform it is not piecewise linear this curvature in the problem.

And then this will not any longer be true but right now we see that this is in fact the Lyapunov exponent everywhere so let me write that down for this pardon the log yes Lyapunov exponent is log so my statement was mod F ′ of X gives you the local stretch factor and its logarithm gives you the Lyapunov exponent. So this map has λ the Lyapunov exponent λ =log to. What happens if R is < ½?

If R is < 1/2 you have this figure yes the Lyapunov exponent is in fact negative what does that suggest to you that is going towards the entire phase space is shrinking towards the fixed point. So even that works out as you can check and when this chaos happen when the chaos or start off at what is what is the onset of chaos in this problem at exactly a 1/2 slightly infinite simply to the right of a 1/2 you have a positive Lyapunov exponent because log to r when r becomes > 1/2 becomes log of a number > 1 and becomes positive.

So chaos is characterized by a positive Lyapunov exponent that is what I meant by exponential sensitivity to initial conditions if the largest layup and off exponent in a system happens to be 0, you have no chaos in this problem it has to be at least one positive Lyapunov exponent. Now we talked about one dimensional phase space X is a single scalar variable, so there is only one way up and off exponent but in a D dimensional phase space or an N dimensional phase space there are n directions.

And therefore in principle you could have n lay upon of exponents but you need to have at least one of them positive in order to have chaotic behavior you could have more than one positive and this can happen even in cases where the phase space is bounded, even in cases where the volumes are preserved because there could be some directions in which you have stretch and some directions in which of contraction and as long as you have a direction which you have a stretch you have chaotic behavior under these conditions have specified.

You could even have a system in which there is an attractive and you could have chaos in the sense that you could have a three dimensional system, in which the attractor falls into maybe a two dimensional manifold or even some fractal manifold dimensionality < 3 but they could always be a stretching direction.

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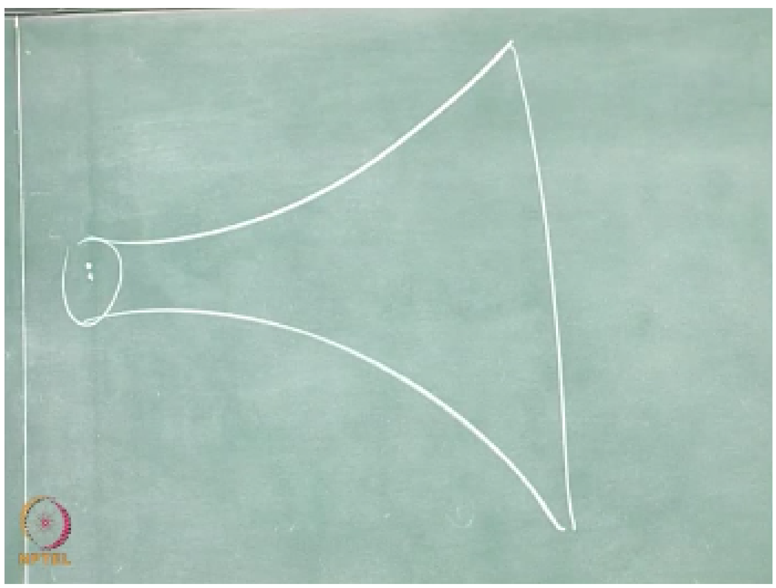

So in sort of heuristic terms think of it in this fashion if I start with points which are kind of close together in a circle like this and suppose this way space area shrinks to a line but suppose it shrinks in this fashion and becomes a line finally. So in this direction things are shrunk we have gone to a line but in this direction they have expanded to system size in this fashion and therefore initially neighboring points could have expanded arbitrarily far to system size itself, that itself implies loss of information and possible chaotic behavior.

So this is typically what happens even though the phase space is bounded, even though the whole thing is compact even though the system could be dissipative. So that space volumes actually shrink with time they could still be chaos in the problem because you still lose information initially arbitrarily close points could diverge exponentially fast with a positive Lyapunov exponent, at least one positive Lyapunov exponent that sufficient to produce chaos.

So in this figure in this particular map we see that the system proceeds very rapidly to chaos it goes straight away from a stable fixed point to some kind of degenerate map and followed by chaos at once no periodic orbits this is not very generic this happened because of the particular shape of this map that we happen to take, we can take other maps where this will not happen and you might expect a slightly more gradual approach to chaos.

And there are several routes to chaos and dissipative systems, is this system dissipative or conservative? How would you classify this what would you say I would classify it as a dissipative system we make will come back to this and I will point out why this is really a dissipative system in that sense in us in a certain specific sense. We look at conservative systems conservative maps which still have chaotic behavior yeah.

I really verified as far as the system itself it has to be dissipated absolutely this kind of thing has to be dissipated but we will look at a map an artificial map no doubt we will look at an example of slightly higher dimensional Bernoulli shift in which you do not have dissipation, in the sense that the map is invertible, the area is preserved and yet you have chaotic behavior so we will get back to this okay.

So now the next thing I want to do is take another prototypical map where things become a little more complicated and this has to do with the logistic map.

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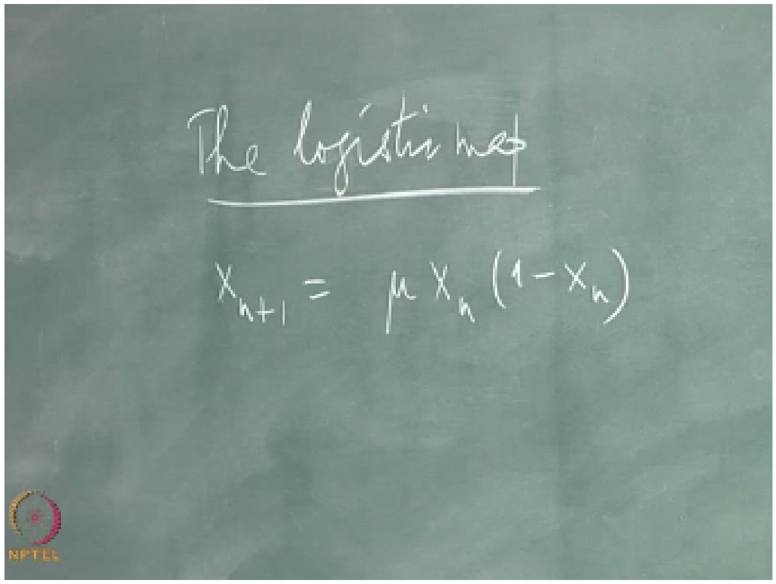

This was one of the first maps but perhaps the first map, where many of these features were elucidated to start with. It is a very simple-looking map but at the same time it can become very intricate indeed let me show you what happens here this map is parabolic, it is just a parabola and it looks like this. So the map function is given by xn + 1 is =xn x 1 - x n μ ltiplied by a certain constant here and many names for this constant let me call it μ.

μ is a real number a positive number and when are the fixed points of this map, where 0 is obviously a fixed point and this perhaps one more fixed point we have to draw this thing here, so let us look at it. One is not a fixed point now 1 - 1 over μ is a fixed point clearly one is not because at one this vanishes but this side does not here.

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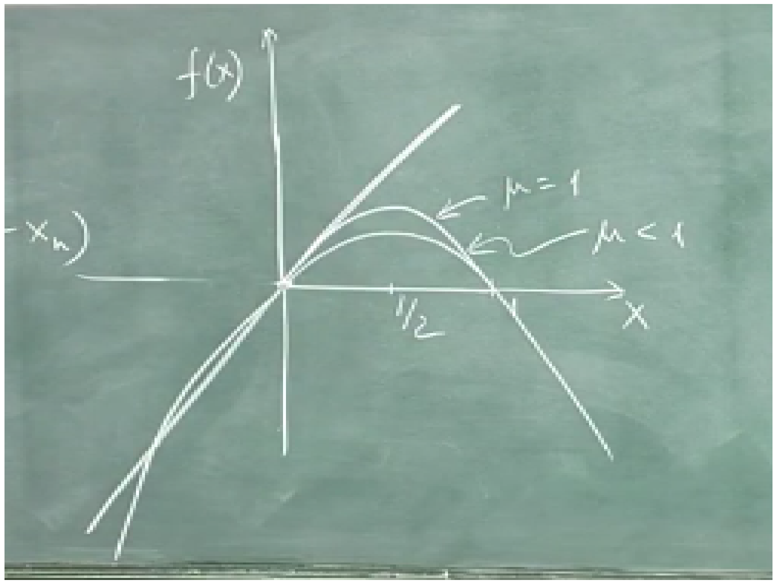

So the map looks like this in fact we should draw the full map and then we will see why I am going to restrict myself to the unit interval, so let us put x here and f of x which is μ times x times 1 - X μ is positive, the map vanishes this quantity vanishes at both 0 and as well as one. So here is one and this map perhaps looks like this it is a parabola it goes up and comes down in this fashion.

The largest value is at a 1/2 that is obvious because x times 1 - x has the maxi μ m value at a ½, now I start with μ a small positive number and if this is the 45^0 line here it is evident that this is a fixed point and it is stable because the slope near the origin is just μ and as long as μ is <1 this slope this magnet this fixed point is stable every point is going to get attracted to it by the way this map it goes off like this and then quadratically.

So it is quite clear that is going to intersect at some other point here and this fixed point is going to be unstable and this is the figure from μ < 1, we are not really going to be interested in points outside this unit interval because if you start with some point here, it is going to get flow into this and if you start with points out here beyond this fixed point they are actually going to disappear to ∞and similarly on the other side things are going to escape to ∞.

What happens when μ becomes =1 at μ exactly =1 this fixed point becomes tangential in this fashion, I have drawn this badly this value is actually 1/4th because that is the maxi μ m value of x times 1 - x between 0 and a ½, so it is not drawn to scale but this fixed point becomes marginally stable at μ =1 what happens beyond μ =1 it crosses. So let us draw that separately and now let us start focusing on points in the unit interval.

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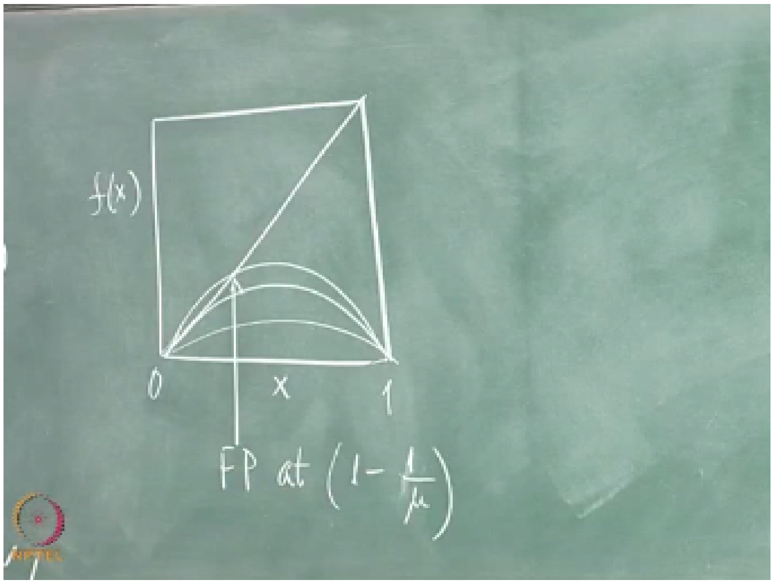

So here we r 0 to 1 that is the bisector and I am plotting f of x versus x and f of x is μ x times 1 – x 6, so originally from μ < 1 I had something like this perhaps at μ =1 I have this is a marginal fixed point and it is immediately clear that as soon as μ exceeds 1 it does this the slope here has exceeded 1 but the slope at this fixed point is <1, so it becomes stable and this fixed point at 1-1 over μ that becomes stable we can easily compute what the value of that slope at that point is.

Now what is the value of the slope at this point let us compute that, so f ′ of x is = μ - 2μx so what is the value of the slope at this fixed point. The slope at the origin is μ of course as we know very well what is the value at that point it is 2 –μ.

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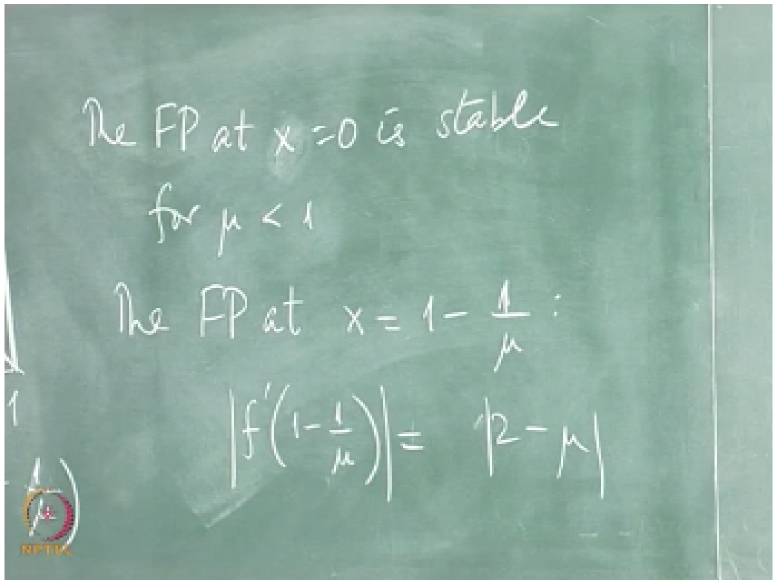

So it is evident that the fixed point at $\mu=0$ is stable form you <1 at $\mu=1$ it becomes unstable and a new fixed point gets in which is given by this as soon as μ exceeds 1 from you <1 remember this followers on the negative side. As soon as μ exceeds 1 you get a new fixed point in this unit interval and it is slope the fixed point at sorry $X =0$ at $x =1 - 1$ over μ f' at this point $1 - 1$ over μ is $=2 - \mu$.

So the mod of this is this prime, so where is this thing stable till what value of μ is this table till three because we want the mod of this, so from $\mu=1$ $2\mu=3$ this fixed point is stable right at $\mu=3$ this map is out here till three quarters. So you have this and the slope here becomes $=1$ at $\mu=3$, in fact we can start writing down the Lyapunov exponents now because if you have a stable fixed point then the Lyapunov exponent is simply the log of mod F' at that fixed point because if you wait long enough the iterates all the iterates fall into this point and therefore if you look at the definition of the Lyapunov exponent.

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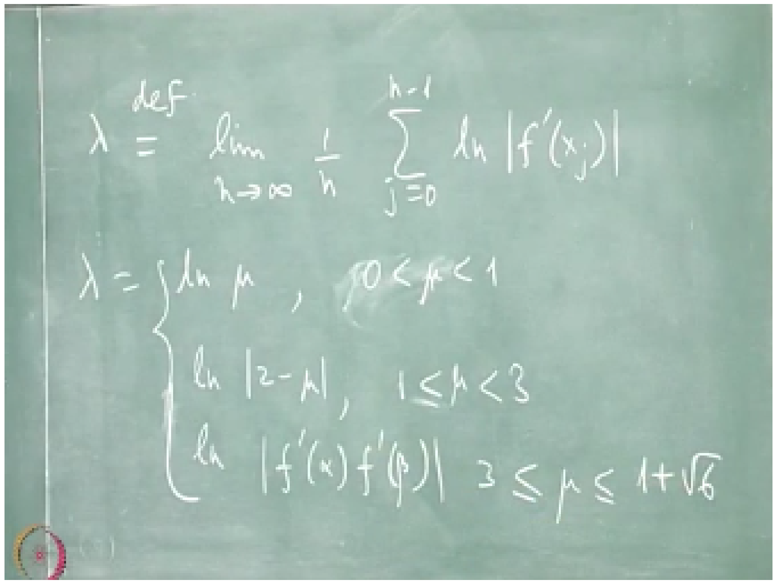

Which remember was = λ by definition was =limit n tends to ∞1 over n summation j = 0 to n - 1 log mod F ′ of the iterates XJ and now it is clear that if there is a fixed point then all these logs are going to this XJ is going to be dominated by the value at the fixed point and that is going to come out and that is going to divide this n is going to divide that and give you that as the fixed as the Lyapunov exponent itself.

So the Lyapunov exponent is log μ as long as μ is <1 between one and three the Lyapunov exponent is log mod 2 –μ and it is again negative showing that there is a stable fixed point, so let us write this down λ for the logistic map =log μ four μ <10 <new <1 <1 it is =log mod 2 –μ again for one <μ is <3. So you can see what is going to happen when μ becomes =1 then this lay open of exponent vanishes and you think the fixed point has become unstable and therefore the system is going to go chaotic like the earlier 10th map but that does not happen.

This fixed point takes over and it is log is <1 here and there for you again have stability, now let us draw the bifurcation diagram and see what happens.

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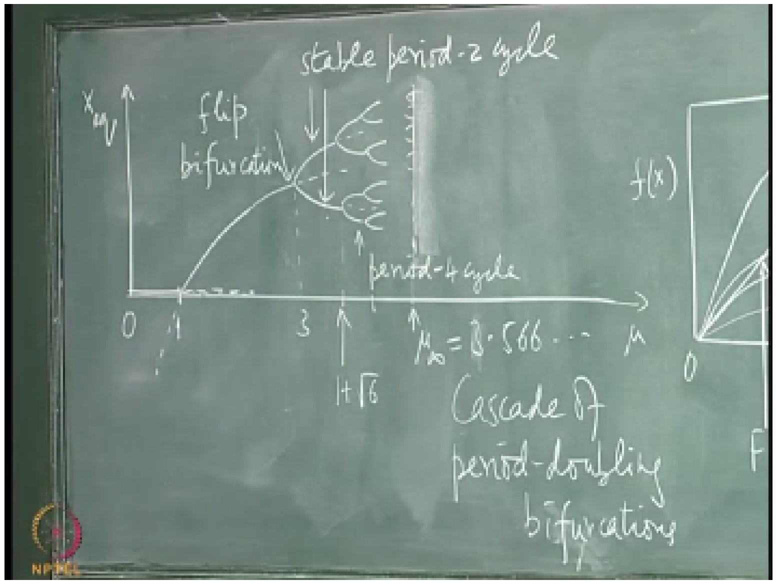

So we need a long graph here μ eventually I will run out of space but here is 0 till one which is not very interesting view itself is the fixed point is at 0 and it is got a Lyapunov exponent is negative, so you have this as a function of X equilibrium and then this fixed point becomes unstable so you should draw the bifurcation diagram the dotted line here and you have the other fixed point which is that one - 1 over μ, which turned out to have negative values at μ=1 it is 0 it crosses this.

And then takes over from here and this fellow becomes unstable earlier that was unstable what kind of bifurcation do we had at μ=1 exchange of stability bifurcation right and then you have a fixed point which goes along and this goes along till you have till you hit the value free. At three this guy here becomes 0 and you would expect okay maybe now we are going to have a chaotic behavior. B

But what happens at three is that the map looks like this but what is the iterate of this map look like what would the first iterate look like if you trade this map this map here at μ=3.

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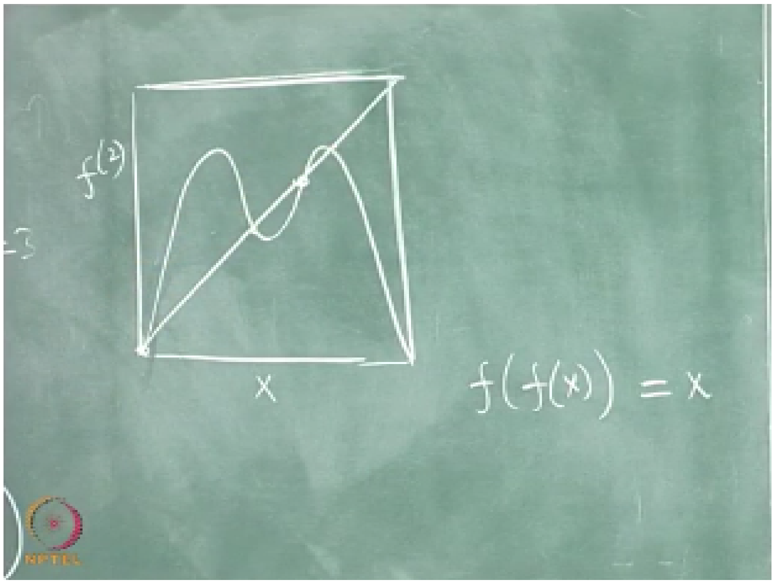

I am not going to be able to draw it to accurately but it starts looking like this let us draw it a little better than this starts looking like this and you have this behavior this was the fixed point at 1 - 1 over μ, but then you have two other fixed points for the iterate this is F_2 the fixed point of the origin continuous is unstable of course and this fixed point also becomes unstable μ=3 but then you have this and that and that is what sort of point orbit is that?

It is a period to cycle it is a period to cycle, so the system bifurcates to a stable period to cycle which you discover by solving the equation f of f of x =x this is a fourth order equation because f of x is quadratic and therefore f of f of x is a quarter and you have a fourth order equation but you can solve it easily because x =0 is a root which you do not want it corresponds to this and x =1 - 1 over μ is this route and you do not want that.

So factor out x times x - 1 + 1 over μ and the rest is a quadratic which will tell you what these two roots are simple exercise and these two points form a period to cycle which is stable and the system falls into that or into this flip-flop between these two. So the bifurcation diagram now has a new kind of bifurcation this becomes unstable but then it bifurcates into a period to cycle this is not a pitch fork bifurcation, looks like a pitch fork but it is not.

Remember in a pitch fork bifurcation a stable fixed point became unstable and created a pair of stable fixed points or critical points, here a stable fixed point bifurcated by exchange of stability to another stable fixed point and now it bifurcates into an unstable fixed point and a stable period two orbit cycle. So this guy here this and this is a stable period to cycle and the system asymptotically flips between this value and that and this kind of thing is called a flip bifurcation.

So at μ=three you have a flip or a period doubling bifurcation because a period has doubled what would the λ be here suppose these two points which are some functions of μ. Let me call this alpha of μ and this point here made of new the functions of μ of course what would the Lyapunov exponent be in this region? It would start at 0 but then it would become negative because you have now found a period to cycle which is stable.

And what happens to the Lyapunov exponent by this definition you have two values here in this summation is not it. So what would the Lyapunov exponent be in this case, so it would just be log modulus F' of α F' of β will just be that right but me it is corn because eventually wait long enough this thing becomes a constant at this value superior to cycle and then the end here divides this under the limit it would just be this product of logs that is <1.

So it is stable mod this product is <1 because the period to cycle is stable once again the Lyapunov exponent drops to negative values and this happens for three $\leq \mu \leq 1 + \sqrt{6}$ and that is easy to verify all you have to do is to find out when this number here hits one and it hits one at root $1 + \sqrt{6}$ here. So that happens somewhere here this is no longer to scale this is $1 + \sqrt{6}$ when matters begin to happen very fast as you change μ at this point this period to cycle becomes unstable.

And you have a period for cycle coming out that becomes stable, so it is not this iterate but you have the iterate with a period for cycle as well as this these points and these are unstable now that becomes stable so this is a period for cycle a little more change in μ and it becomes a period 8 cycle. So the next bifurcation happens here and this period doubling cascade of bifurcation starts happening, so you have cascade as μ increases and it happens for smaller and smaller intervals in μ.

So eventually what happens is that you have 2 to the N period cycle where n becomes unbounded and that happens at a finite value of μ this is called $\mu\ \infty$and it is =0.5 66 etc what 3.56 this is known to 15 decimal places for this map one can compute the value of $\mu\ \infty$ numerically there are lots of scaling properties that happen here which I will not go into right now. But you have the end of the period doubling bifurcation cycle this one at any rate etcetera.

So you have a whole lot of points which form part of this our tractor not a full interval it is not a continuous interval at all, the limit points of all these period 2 to the N cycle out here is a set of points which is got a fractal dimensionality between 0 and 1 it is of the order of point 57 or something like that. So it is a dust to set of particles a set of points on the unit interval yeah I will come back and explain what fractals are so when I do that to remind me to go back and tell you this.

It is not a continuous interval at that point at that point the Lyapunov exponent again hits the value 0 and then after that it has nowhere to go it is actually exhausted stable fixed point, theory a two cycle period for cycle period eight cycles etcetera all these are exhausted and the system becomes chaotic. You might ask it never went through period six it never went through a period nine and so on it went period 1 to 2^{2} 2 q^{4} and so on up to 2^{∞}.

So it still has surprises in store at this point the system becomes chaotic and after that you have a whole band in which the system dress exactly as in the case of the tent map beyond r =1 but it is not the full interval as yet however. In this case the system has further surprises in store and now let us draws the Lyapunov exponent and sees what happens.

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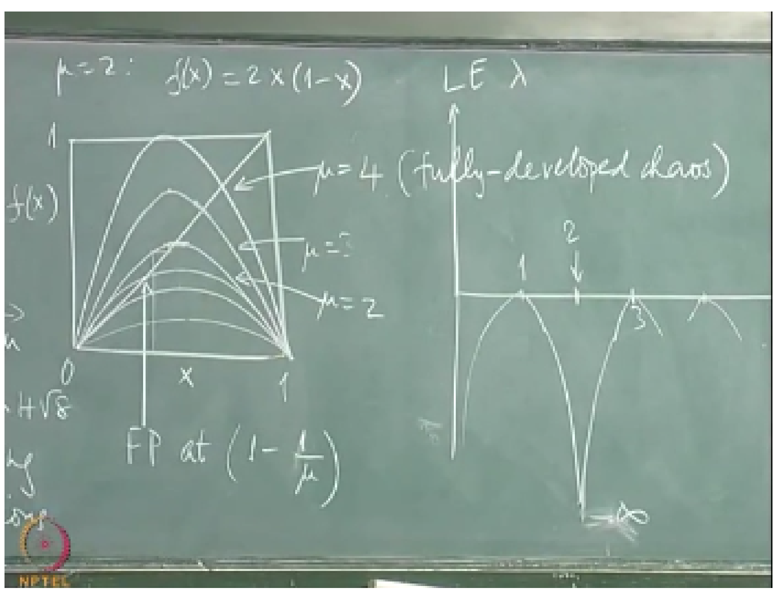

So I plot as a function of μ I plot the Lyapunov exponent lambda there is no chaos as long as μ is <1 there is a fixed point here and it is some negative value it is at log μ and as μ=0 of course is that - ∞ but then it comes up and does something like this it is negative. At μ=1 it is hitting the value 0 but then it starts becoming negative once again because it is now given by log mod 2 –μ and this goes on till the point 3 where again it hits 0.

And incidentally in between one and three at μ =2 something interesting happens, let us draw the map for μ=two. So I would like to draw you = 2 the map function f of X is twice X into 1 – x. The fixed point is at a 1/2 because it is at 1 - 1 over μ and what is the slope at that point it is two –μ right, so the slope is 0the log of that is - ∞. So what has happened is that exactly at μ=2 the map looks like this and the slope here is 0.

Remember that the stability of a fixed point was determined by the modulus of the slope at that point if that slope was <1 in magnitude then we said it was stable and if it was > 1 it was unstable =1 it was marginally stable but at μ=2, the slope here is 0 that is the least value the mod of the slope can take and it is evident whenever you start that in this case how would you get to this point I mean it is quite clear that this is the smallest magnitude you could possibly have and in fact the map becomes super stable.

At that point this fixed point becomes super stable the log of that plummets to - ∞, so this goes off to - ∞ and that is because at μ=2 the fixed point is super stable, so the Lyapunov exponent goes all the way to - ∞but me the significance is that it is the it is the best stability you can have cannot have anything beyond that because in this map if I start with any point here I hit this and then I hit this the approach to this is extremely rapid not here because it does not really wander around at all.

That was simply degenerate that was simply degenerated yeah that was simply degenerate I mean the entire map function lay on the bisector itself, so nothing moved in that case yeah I agree that is the sort of degenerate case is very unusual but here when the slope now I am not saying this is more stable or less stable i am simply saying it is a matter of terminology that when the slope becomes 0 then the map is the fixed point is said to be super stable. It is stable if the slope is <1 in magnitude and when it is 0 which is the least value it is super stable the significance is that the corresponding Lyapunov exponent goes to - ∞tends to - ∞.

So the local stretch factor or a contraction factor is the biggest it can have yeah yes I am not saying lambda is an indicator of stability and saying λ if it is positive as an indicator of instability but when you have isolated fixed points then λ tells you in some sense what the contraction rate is and the contraction rate here is a largest. And that is because it is directly measured by the log of this modulus of this slope here.

There is another measure it says more precisely how contraction occurs in phase space as you move forwards in time or expansion occurs we move backwards in time and that is measured by something called the kolmogorov entropy, I have not introduced that as yet and when the system is super stable then the cone maker of entropy also behaves has a very specific kind of behavior that is the reason I called it super stable at the moment, but for the moment let us leave this as just a matter of terminology we'll come back and see what it is significant signifies.

But it is easy to see that in this case the Lyapunov exponent we actually go to $-\infty$ here then it comes back crosses this at free and then because now at this point a period to cycle takes over it falls back once again. And then there could be possibilities that these the periodic cycles themselves become super stable if any one of them if the point 1/2 if this point becomes a fixed point or part of a periodic cycle then you can see immediately.

Since the slope there is 0 if the peak becomes always any of the peaks becomes of the extreme a of this map or its iterates becomes part of a periodic cycle you have super stability there immediately. So this could happen over and over again and then again at the point $1 + \sqrt{6}$ it climbs up to 0 but once again a period for cycle takes over and it keeps doing this till it hits $\mu\ \infty$ $\mu\infty$ it is exhausted this and the Lyapunov exponent crosses over finally to positive values.

And the system becomes chaotic, so this here is the start of this chaos here at $\mu =4$ that is the largest you can have here, so let us jump straight to $\mu=4$ which is here so the maxi μ m occurs exactly at one it becomes an on to map then it displays properties very similar to that of the 10 pipe. Because now at this stage you have 0 to 1 and 0 to 1 here this fixed point is at three quarters and is unstable all the iterates of the map also lead to unstable fixed points there are no more stable periodic cycles possible.

And this map becomes fully chaotic and the entire unit interval becomes part of short $\mu =4$ the entire unit interval becomes part of the attractor. But in between this end of this first period doubling cycle to the reaching of this chaotic attractor you have many intricate phenomena that go on here between $\mu\ \infty$ and 4 because what happens is although all the 2 to the N period cycles became unstable there are many other integers and we have not exhausted them.

So it turns out that this map exhibits cycles of all integer periods 1 2 3 4 5 etcetera etc not necessarily all of them stable but eventually what happens is various complicated things happen here which will describe by and by including a phenomenon called intermittency. So there are long regular verse of the iterates followed by chaotic intervals followed by regular bus and so on and eventually at the point $1 + \sqrt{8}$ a period 3 window takes over.

So for a little bit of time there is a period 3 cycle and the period 3 cycle happens at $1 + \sqrt{}$root of 8 it is still <four a set of tangent bifurcation occurs. So you have a stable fixed point an unstable a stable unstable stable unstable and the system flips between these and that remains stable for a little while. So you have a stable period three window the chaos disappears and then once that disappears that periodic window disappears again chaos takes over and you have a chaotic behavior.

Here finally till at $\mu=4$ you have what is called fully developed chaos and we will have more to say about this and the question is what is the Lyapunov exponent at $\mu = 4$ when the entire unit interval is completely chaotic, it turns out that the Lyapunov exponent is can you guess what it would be because it now has property is very similar to the tent map in some sense. It is locked to it again becomes log 2.

Even though in this case the slope is not uniform, so you really have complicated behavior the system is a gothic on the unit interval and ends up with a limiting value of the Lyapunov exponent which is log to once again that stage. So it hits something here limiting value at $\mu=4$ this value is log 2 in between in this chaotic region the Lyapunov exponent has only to be found numerically, there are very few analytic expressions what would be the value of the Lyapunov exponent in this periodic window.

It would become negative in general if there is a stable periodic window it would simply become negative, so it is not as if this stays and goes up monotonically to log 2 there are still complications here goes up and down and eventually hits the value log. Yeah this is a numerical result here as to where the period 3 window emerges, so the statement I made was after the two to the N cycles ended after that set of period doubling bifurcation you had the onset of chaos at this point.

Exactly at μ ∞ the Lyapunov exponent is 0 and right above for any infinite decimal value of μ beyond μ ∞for an ∞ as many larger value you have a positive Lyapunov exponent the system becomes chaotic but in between in this chaotic region it is interspersed with periodic motion and the last of these the very last of these periodic windows happens at 1 +√8, as a value of μ when the chaotic attractor disappears and the system falls into a stable period 3 cycle. So there are three points and it flips flops between these three points.

That continuous for a while a small range of μ and eventually that periodic cycle becomes unstable all the other integer periods also become unstable and the system has no recourse but to become fully chaotic and this continuous still μ =4 okay. Now the question is where does this come from because we only looked at the iterates of this map we did not ask what are the other possible periodic points.

It turns out that in this map for good reasons the route to chaos is via period doubling, so we start with period one that is double 2 that is double 24 which is double 28 and so on but this is only one set of possible periodic orbits you could still have period 357 or any other number which is not of the form 2 to the N. All those periodic orbits appear in this region most of them are unstable when they are here but occasionally you could have stability once again simply because of the dynamics which is not trivial at all it is extremely complicated.

And finally the last window that appears where you have a stable periodic orbit is a period three window right here and once that two becomes unstable you have full chaos complete chaos. The attractor at this point is not an interval it is called the π attractor at μ ∞ λ is just about to cross over from 0 and the attractor is called a π inbound it is not chaotic because chaos has not set in yet the lambda is still =0 it is not taken off to positive values.

This is the limit set of this set of bifurcation points and that is a fractal object it is a set of points with a certain dimensionality called a fractal dimensionality, which is between 0and one and it is non chaotic and because it is not an interval but a set of points which has a certain structure it is called a strange attractor but it is a strange non chaotic attractor. I earlier introduced the idea of strange attractors in three-dimensional or higher dimensional flows which are chaotic with at least a positive single positive 1 positive Lyapunov exponent.

But here is the case where the Lyapunov exponent is dead 0 and yet you have an attractor which is not a periodic attractor of it or anything like that but it is a strange non chaotic attractor in this stage. But immediately after that you end up with a chaotic attractor we will come back to some of these points in particular you want to study this we want to study fully developed chaos and see what happens at this point but first one quick question what happens.

If I have a map in which μ becomes > 4 what would that look like well it is evident that this would go like this then if you took this map seriously what would happen? It is quite clear that what could happen is the following remember this graph goes on both sides. So if I start with the point here it goes here I go there and I am out of the unit interval and it leaves the unit interval till μ=four points which started inside the unit interval remained inside there.

But now things have started escaping out of it all points do not escape all the free images of this interval would escape the pre images of this interval would escape and so on, so you have a very complicated set of points which would escape and another set of complementary set of points which would remain inside and these would form what are called cantor sets. So we will talk about this in the context of the tent map.

So what you have here is something called a chaotic repelled because things are moving out of this interval they are getting out here and this too has its uses but we are interested when you are studying chaos it is up to four that you would like to look at yeah the unit interval yard then the guava will be linsley bounded between the 0 to one interval rate but what is going to be the question is if points leave the interval yeah then those points will learn to.

I am not going to be densely moving across the 02 an indirect way exactly, so now what will happen once things he's got a point he says if you have a situation like this and some sets of points leave the interval then what remains as an attractor is no longer the unit interval and that is absolutely true so what remains here is what would I would call a cantor set we will talk about this it is silk chaotic.

Because they still could be exponential sensitivity to initial conditions between what remains that could still happen does not matter does not matter does not matter no but if I start with the point in whatever is left as an attractor you still have, you still have a Lyapunov exponent you still have unstable periodic orbits you have all those points a certain set of points has left. So you are attracted and so being the unit interval has split up into many disjoint pieces yeah going to hold within this region is going to reduce yes.

So it again goes towards stability yeah in some trivial sense I mean once I get out up there everything has gone off to ∞it is no longer stability everything is just moved off to ∞could be chaotic while it is showing that it could certainly be chaotic whatever is left inside okay whatever is left inside on yeah will reduce. So can you say that things are now become unbounded right this motion is now going off to ∞.

So I know I understand that fully completely things are just escaping to ∞there is no longer of direct interest what is of interest is the following which is to say that it does not really go off to ∞ is to take this map and make this put is not on a square but on a square lattice repeat this map over and over again, so I say when something goes out into power goes into the next square and that is how I steady scattering and it is in fact a model for studying chaotic scattering for real physical systems again as a toy model we will get back to this next time.

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