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TOPICS IN NONLINEAR DYNAMICS

Lecture 17
Discrete time dynamics (part 1)

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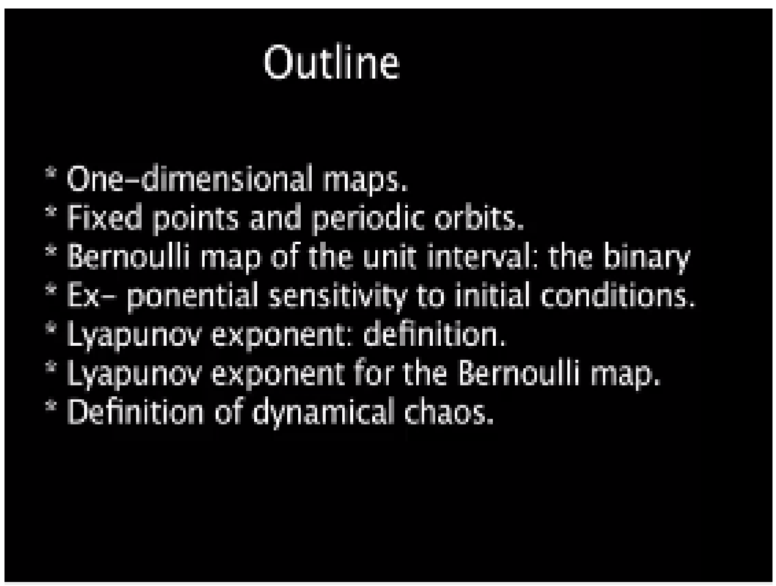

So we begin by continuing with our study of maps and specifically one-dimensional maps so recall that the kind of discrete time dynamics we were looking at was a map function.

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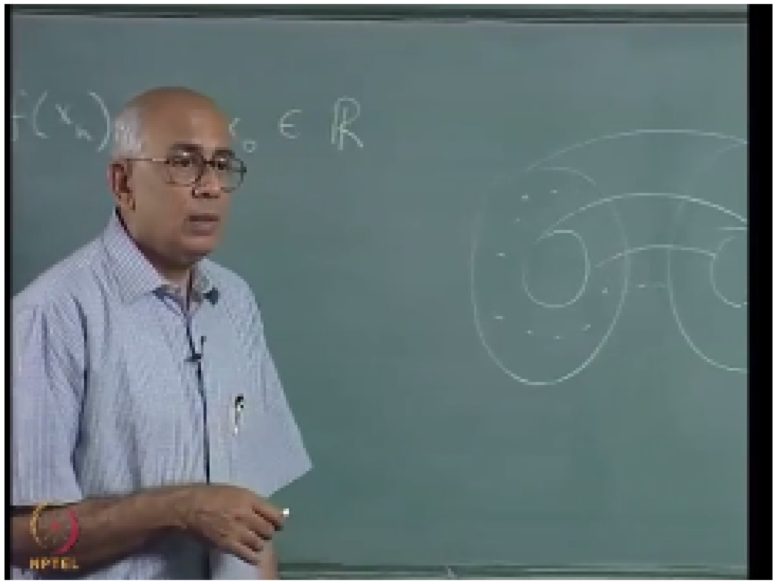

The form Xn + 1 some F (Xn) and X 0 is defined to be in some interval but we started off by defining it to be on the real's and then the question was to study the dynamics if you like implied by this discrete time evolution equation before. I do that let me settle a couple of questions which were asked last time the first of which had to do with what happens if this map function itself changes with time in other words if at the end stage you have a different map function each time you iterate this of course would be the discrete time analog of a non autonomous system where the evolution equation would explicitly involve time on.

The right hand side and we are not looking at such maps to start with although one can in principle do so we are not going to do that at the moment so we are again looking at autonomous systems which do not have this possibility here the second question that was asked was much deeper and had to do with whether in the presence of chaos which we have not defined as yet very clearly you could have order as well from chaos and there the comment that I did like to make is that the lesson taught to us by chaotic dynamics is that simple evolution equations simple dynamics could have very complex solutions.

So very simple looking equations could lead to very complex evolution very complicated evolution in time the solutions could be extremely complicated the other thing is also true namely you could have a very complex system and on the average it could behave in an extremely simple fashion for instance if you took the gas of particles in this room the motion is certainly chaotic at the classical level highly chaotic as we will see and yet there are very simple average laws as the ideal gas equation of state PV = RT and similar laws which represent the behavior of macroscopic average quantities.

Those could under suitable circumstances and special circumstances be actually extremely simple another example is you take a piece of metal and you apply a voltage to it and you have a current in it in a conductor and the current is proportional to the applied voltage very simple Ohm's law this again is in spite of the fact that the individual charges inside the electrons could be doing very complicated things and yet you have a microscopic law which is very simple or Hooks law and elasticity or flicks laws of diffusion and so on all these macroscopic laws are quite simple to write down.

But they refer to average or macroscopic quantities and they come out by averaging over a large very large number of individual microscopic motions which could themselves be very complicated there are reasons for that as well and a little later when we talk about invariant densities I will come back to this and talk about thermodynamic systems and what is so special about them the third question that was asked again jumping ahead a little bit was whether you could have simultaneously coexisting in a system both regular motion as well as chaotic motion and the answer is yes in general this is possible there could be regions of phase space.

In a dynamical system where if you start with an initial condition these regions the motion remains regular non chaotic on the other hand they could be other regions where the system behaves chaotically this is entirely possible we even talked about this in the context of Hamiltonian systems where something very special happens when you have chaotic motion and I will return to this a little later when we talk about chaos in greater detail and if you recall in Hamiltonian systems which are integrable the motion is on n-dimensional in general for an N freedom system and if you took the example for instance of n = 2 then the motion in a four dimensional phase space is restricted to two dimensional tore.

I and they could pictorially. I could draw them as in this fashion perhaps you have one tourist like this and there is another initial condition for which you have a larger torus of this kind and the story are nested within each other it is entirely possible that in the region in between two turbine you have chaotic motion irregular motion but since these tour I split the whole of the space into an inside and an outside remember the energy surface in this case is three-dimensional in a four dimensional phase space and the three-dimensional space is split into an inside and an outside by an object like this torus anything that is inside can never escape outside.

And so the chaos is actually contained in the region between successive tour I if it happens as soon as n becomes greater than three greater than or equal to three then the phase space for three n = 3 for instance the phase space is six dimensional the energy hyper surface is five dimensional and the torah' are three dimensional tour I and these tour I cannot split up this five dimensional space into an inside and an outside.

So all the chaotic regions can actually be connected to each other and you have what is called a stochastic web or Arnold diffusion which takes you from one point of this stochastic region to another and that goes through all of the phase space which is not striated which is not contained which is not occupied by the tour. I themselves so this kind of behavior can happen could be very complex and certainly coexistence of chaos and regular motion is possible in many systems okay, now let us go back and look at some of these maps in a little greater detail we talked about a fixed point of such a map.

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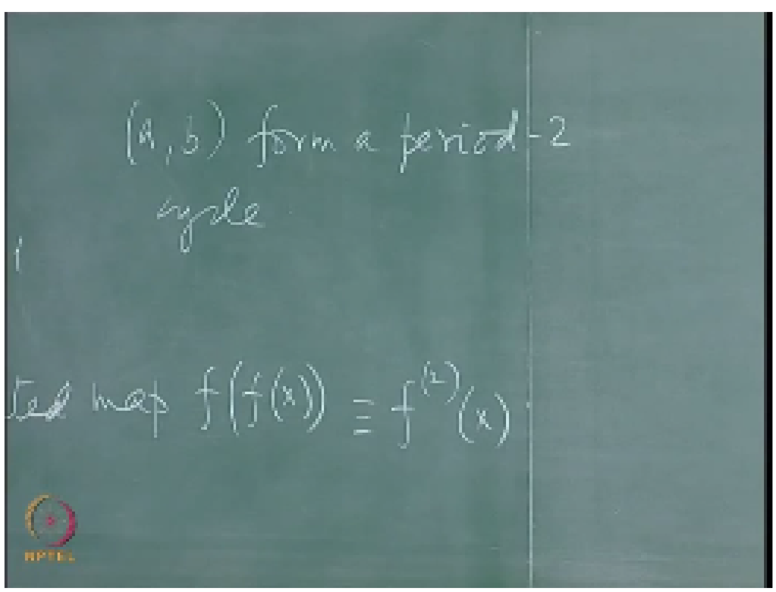

So a fixed point at X *⇒ is a root of x * =f of X * and it was stable if mod f ‘ of x * was less than unity unstable if it is <1 and marginally stable or indifferent if it is exactly =1 so these fixed points are the analogs of critical points for continuous time flows what happens if you have a fixed point of the following kind what happens in a map if the point let us call this point a if you have a = f (b) and b= F (a) what happens if you have a pair of points a and B such that a is f of B and B is f of a so it is quite clear immediately from this ⇒ that a and B are fixed points of the iterated map f of f (x).

Which I will denote by F 2 (x) they need me fixed points of the map itself but they could be fixed points of the first iterated map so you take this initial point you take the value a then you calculate what F (a) is and turns out to be some number B but you calculate F (b) and you are back to a so this is entirely possible what would you then call the pair A and B it is clear that under iteration a goes to B and B goes to a keeps coming back what would you call such an orbit it's the analog of a closed orbit in continuous space it is a periodic cycle and such a thing is called a period two cycle (a,b) we are going to look at examples of this very shortly so in this staircase construction that we had or even this cobweb contract construction.

We had by the method of successive approximations it would turn out that the point the function corresponding to the initial value a is something some B and then the value B is back to a and they form a period two cycle when would such a cycle be stable well it is clear that they would it would be stable if the slope of this function this is a new function now it is the function of the function of X it is a new function if the slope at the fixed point of F 2 (X) < 1 in modulus in magnitude this is exactly when it would be stable and what would be the criterion for that.

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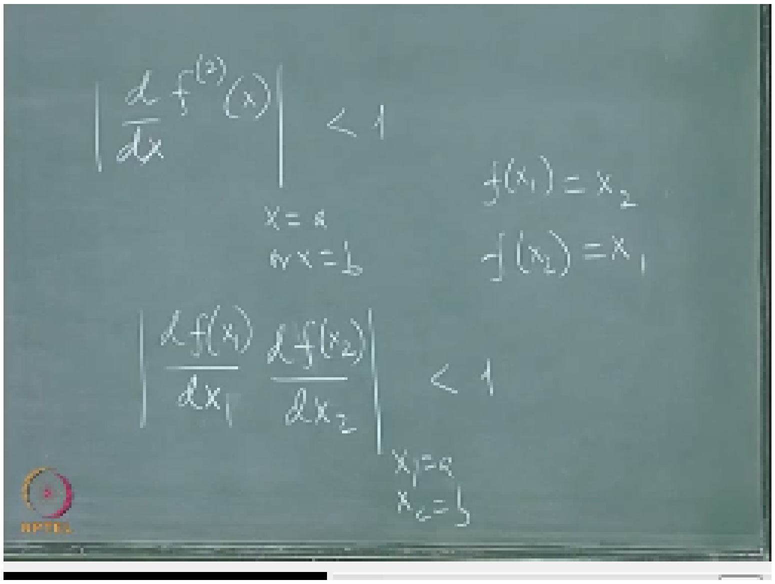

So I did like to have D / DX (F 2) (X) X = a or X =b I did like this to be < 1 in magnitude so it is easy to see that since this thing could also be written as DF f let us call it (x1) so I dx1 so let us so what I have here is a situation where f of X 1 = X 2 and f of x2 is = x1 not again and if you have a situation like this you could rewrite this in this fashion and if the modulus is < 1 the product is < 1 then you have a stable fixed point a stable period to cycle a moment's thought will convince you that this is right all I have done is to write this as f of f of X and I've called f of X some other variable.

So it is the F 1 (F) F X 1 (DX 1) DX 1 (DX 2) so all I do is to differentiate a second time and this is it similarly you could have a period P cycle so an orbit which goes from some value a 1 to a 2 to a 3 to a P back to a 1 and the set of P points at 1 to a P forms a period P cycle which is stable if the magnitude of the slope of the peeth iterate of this function is < 1 at any of these periodic points or written in this fashion if the product of the slopes at all the points on the orbit is < 1 in magnitude then you have a stable period P cycle and if it is > 1 its unstable once again you could have a marginal stability here.

Now let us draw a picture and see when this happens and how it goes on so we need to write down right away some kind of function which would do this and here for example.

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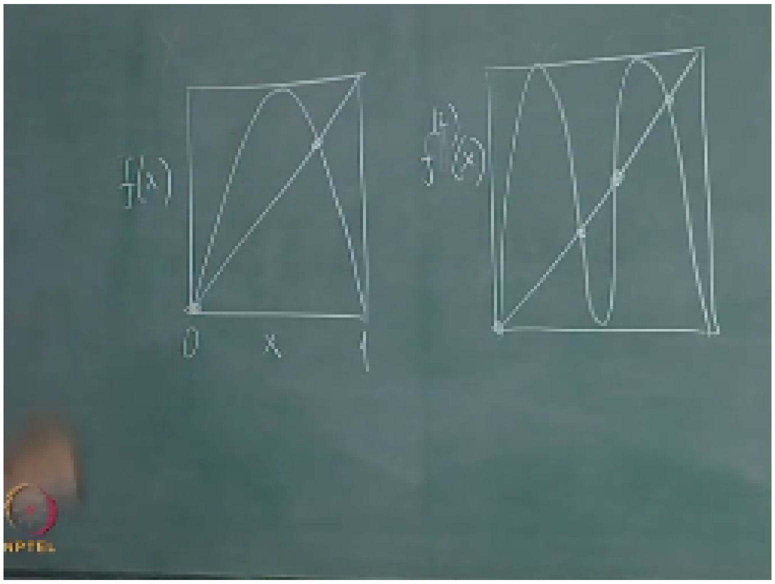

Let us specialize to the unit interval in the x-axis so let me plot f of X here versus X here and we focus on just the unit interval and let f of X also take values in the unit interval and then we ask what does this function look like what are its fixed points and so on so let me take a very typical function which we are going to study in some detail so here is the bisector and suppose this function is like this some function with the single hump on top here is the fixed point and here two is a fixed point and it is immediately clear from this picture that both these are >1 if this is a symmetric parabola for instance this slope is >1 this slope > 1 in magnitude and therefore.

They these are unstable fixed points on the other hand if I iterate this function once I go through the iteration once what would the iterate look like what would F 2 of X look like what would iterate in this function lead to well we can do this in detail but perhaps it will do something like this in this fashion and there are now several fixed points there is 1 here 2 here 3 here and 4 here what could these fixed points correspond to this fixed point remains a fixed point here so the fixed point of a map remains a fixed point of its iterates that is quite clear.

Because if f of X *= X * then F any number of iterates FP (X*) is again = X now under iteration this does not change at all so the fixed point of a map is also a fixed point for all its iterates the converse is not true in general because if you look at this case this is f 2 it has 4 fixed points while f has just two fixed points here now it is quite evident from this picture that this was an original fixed point it remains so this point here is this once again it remains. So but you have two new fixed points here what would they correspond to as far as the original map is concerned what would these two points correspond to that correspond to a period two cycles the new fixed points would correspond to a period two cycle.

So it would simply mean that if you took this value and found the map function it would go to this value and if you took this value and found the map function it would come back to this value so the fixed points the new fixed points of the first iterate f2 they straddle the original fixed point of this map and say evident from the picture in this case at least pictorially it looks like all of them are unstable but that does not have to be the case it depends on the kind of function we are looking at we are going to look at it in some detail but .I want you to appreciate the fact that period cycle period P cycles are fixed points of the peak iterate of the map.

Which are not fixed points of any of the earlier iterates just as this period two cycle is a fixed point of f2 which is not a fixed point of F itself so now let us do some specific calculations to understand what this thing looks like there are several standard maps you should not use the word standard map because that is used in a specific context as well but there are several sort of speak typical prototypical maps which are studied in one-dimensional dynamics which are useful to know about because they illustrate many general properties and let us start with the few of them and go on from there the first of these is the so-called Bernoulli shift or the doubling map or the binary shift and it is as follows so you start with.

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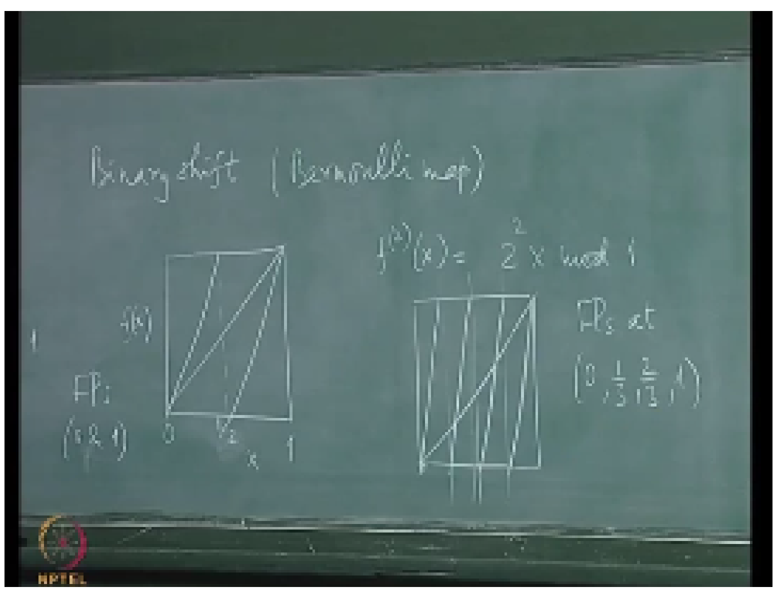

The number X 0 in the interval 0 to 1 in the unit interval and x1 =2 X 0 modulo 1 in other words double the number and subtract 1 if the number exceeds 1 and the general function the map function is xn = 2xn + 1 2 xn mod 1 already this map has many interesting properties and it is called the binary shift or the one-dimensional binary transformation or the Bernoulli shift or the Bernoulli map and so on for ease of notation for convenience let me just call it the Bernoulli map and it is given by xn + 1 is 2x and mod 1 let us write draw a picture of this map so here is the unit interval 0 to 1 and here is X.

And here is f (x) which is 2x mod 1 and as always I draw the bisector first and then I start with this number all I have to do is to double this thing so it is a straight line with slope 2 so at the point 1/2 at the point 1/2 the map function has already crossed 1 and then it would go on like this up to this point up to 2 but then we are told you have to subtract 1 if the number exceeds 1 and that is equivalent to taking that piece cutting it and putting it back here so I should draw it as a straight line so it does not look like it is a curved line the slope is 2 everywhere is this a linear function would you call.

This a linear function its piecewise linear it is not linear its piecewise linear ,this piece is linear this piece is linear, is it continuous or discontinuous it has a discontinuity at this point does it have fixed points 0 & 1 are fixed points that is very clear are they stable or unstable they are both unstable because the slope is 1 and gray = 2 in magnitude at each point and therefore it is these are both unstable fixed points what happens perhaps this has stable periodic points we should check that out so what is the doubling what does a double map look like what is f 2 (x) = 2^2 X mod 1 because it is 2x mod 1 and then another 2 X mod 1 multiple it once again.

So it is 4 X mod 1 and what does this function look like. So here is a 1/2 here is a 1/4 here is 1/4 and this function looks like this and this is the bisector where are the fixed points of this map well 0 is a fixed point so here we had fixed points 0 & 1 both unstable the fixed points here of course 0 & 1 are fixed points but you also have two more where are these points the easy to guess here all you have to do is to double this number and take mod1 so where is this point what happens if you take 1/3 where does 1/3 go when you double it 2/3 and where does 2/3 go when you double it back to 1/3 because 4/3 you subtract 1 it is back to 1/3.

So it is evident that 0 1/3 2/3 and 1 are fixed points of this iterated map of which 0 & 1 are already fixed points of the original map and 1/3 and 2/3 form a period to cycle so this point jumps into that that point jumps back into this and it keeps going forever are they stable is this a stable period to cycle it is on definitely unstable because the slope is for the slope in fact is for definitely unstable are there any other period 2 cycles but me well 0 & 1 are trivial period two cycles because there are actually period 1 cycles there are already fixed points.

So I said a period P cycle is a set of P points such that it is a fixed point of the P iterate of the map but not of any of the earlier iterates yes so there are no other fit to period 2 cycles here as far as we can tell okay, what about the next iterate the slope would be 8 and you get a large number of fixed points but you get a new period cycle as well but the fact is that all of them are unstable so it is immediately clear that all the points here all fixed points of the map as well as all its iterates are unstable then the question arises if a point does not belong to a periodic orbit where does it go on iteration .

Where it where does it end up it turns out it wanders forever on the unit interval never leaving the unit interval but in there completely a periodic and irregular fashion in fact in a chaotic fashion so this is our first example of what chaos is we will define chaos much more precisely but the fact is that you have an infinite number of points which sit on periodic cycles all of which are unstable so if you start with a point exactly at one of the fixed by one of the points of the orbit of a periodic cycle this point will move in this periodic orbit.

But all other points the iterates will uniformly and densely fill up the entire unit interval given enough time and this is an experiment you could do with pocket calculators start with the number random number between 0 and 1 and iterate and keep doing this and you will see that gradually its iterates fill up the entire interval you could then ask what are the points that actually fall on periodic orbits we found that 1/3 and 2/3 forms falls on.

A periodic orbit what about the point 1/5 where does it go well 1/5 we will go to 2/5 which would go to 4/5 which would then go to 8/5 which is the same as 3/5 which would go to 6/5 which is the same as 1/5 so you end up with a period 4 cycle the point 1/5 could go to a period 4 cycle and so on so let us generalize this what is the sensible way of doing this a convenient way of doing this is to.

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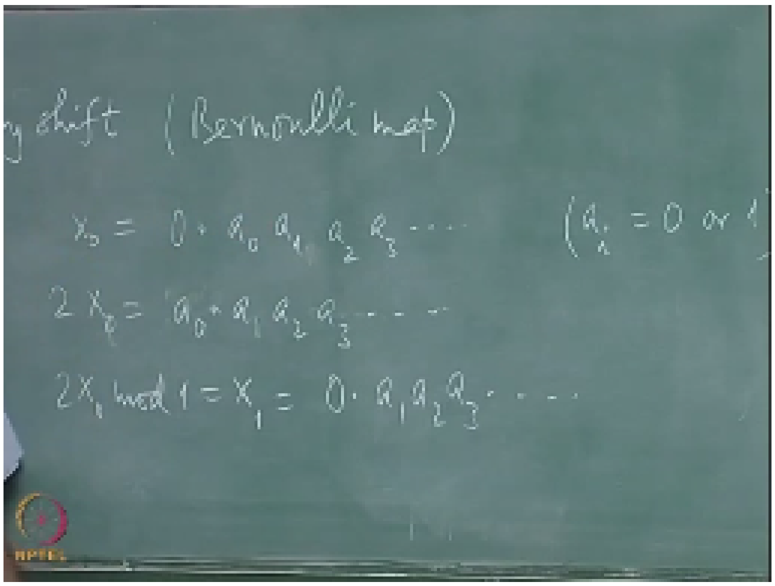

Write this x0 in binary decimal notation using the digits 0 & 1 so it is a number between 0 & 1 so let me write it as zero point a 0 a 1 a 2 a 3 or … but each ai = 0 or 1 if this is X 0 what is X 1 what is x1 = I have to double this number in other words multiplied ./ 2 and if it is > 1. I throw away the 1 part if X 0 < 1/2 then a 0 is 0 because this is the coefficient of 1 ($2^{1)}$ if X 0 is > 1/2 < 1 then a0 is 1 at the next stage when I multiply /2 its equivalent to taking this decimal point and shifting it there that is all it does because. I have written it in binary so it is immediately clear that X 1 on doubling so 2 x0 is a 0 point a1 a2 a3 or therefore 2 x0 mod 1 this is x1 = to 0. a a1,a2,a3… this is the reason for calling it the binary shift all you have to do is to move this decimal point one place to the right and it is whatever is here is.

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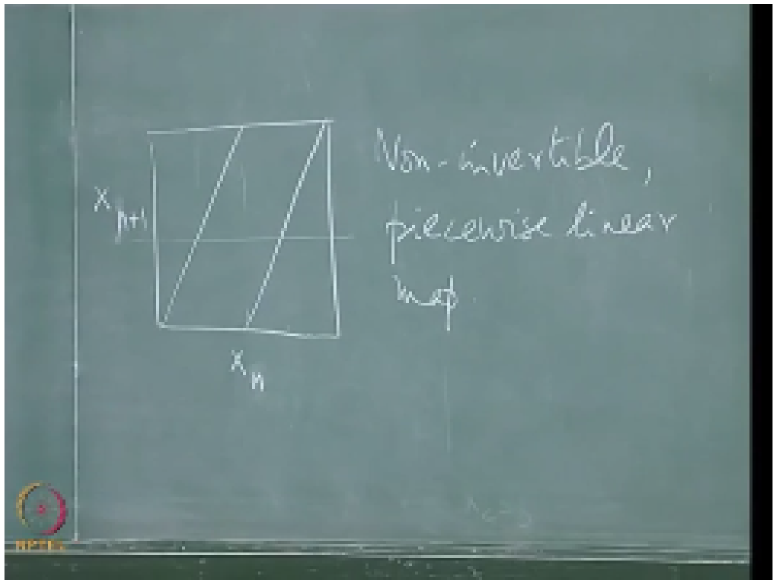

This map invertible is this map invertible in other words if I give you X 1 as a function of X 0 looks like this or xn + 1 as a function of xn looks like this is this map invertible if I give you an X n you can find a unique X 0 xn + 1 but if I give you an xn + 1 can you find a unique X n no indeed because if you give me a value here I find the map function or here I find this function next iterate but if you give me the value of xn + 1 there are 2 possible X ends from which it could have emerged regardless of what at a 0 is you get the same X 1 so the map is not invertible because it is not linear non-invertible although.

It is piecewise-linear this non-inevitability is crucial because it means if I give you xn and asked what X 0 could it possibly have emerged from there are 2 to the N possibilities all of which would lead to the same xn so you can see at each iteration I am losing a piece of information. I am losing one bit of information because this gets erased at the next stage a one would get erased and you would have just 0.3 and so on so you have no way of going back and recovering what a 1 was or what a 0 was so this plays the role of this is this is responsible for many of the properties of this map.

That it is not invertible and moreover the number of pre images of this map the number of possible x-0t switch lead to a particular xn is actually increasing exponentially within its 2^n the now let us see what are the points that lie on periodic orbits can we say that from here what points would lie on periodic orbits if it is part of a periodic orbit it is clear after some time this pattern should repeat what is the necessary and sufficient condition for that well either this number terminates either this expansion here in binary decimal form terminates at some point after which you just get 0 0 0 everywhere or it repeats it is a periodic pattern by itself.

What do you call those numbers rational numbers all rational numbers between 0 & 1 would lie on periodic orbits and all irrational numbers for which this never repeats itself would lie on the single chaotic attractor here so that tells us at once at all.

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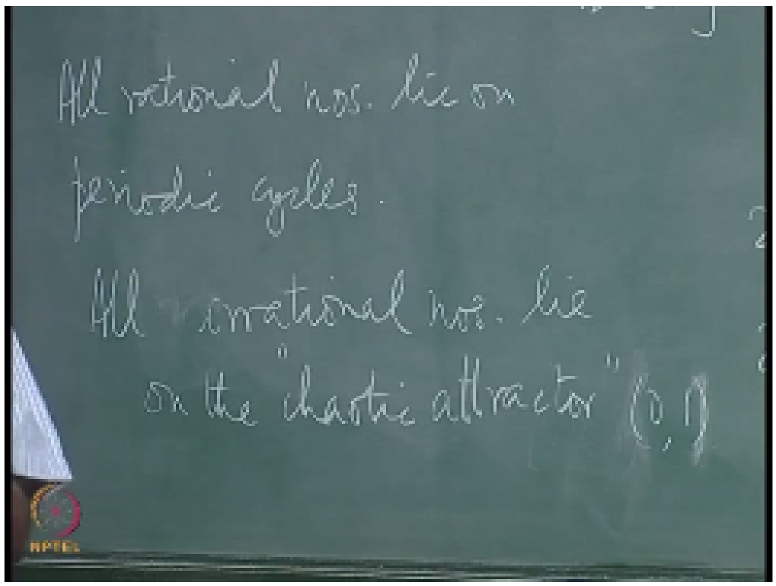

Irrational in this case it is the unit interval itself the full unit interval is the chaotic attractor in a sense which will become precise in a moment the rational numbers are dense on the unit interval arbitrarily close to any point in the unit interval you can find rational numbers so they are dense on the unit interval but they form a set of measure 0 the total length of all the rational numbers is 0 you can count them off their denumerable they are infinite but denumerable you can count them off you can put them into one-to-one correspondence with the integers yeah. So it is clear that is a good point he says that what if you have a system what if you have a number where the first K digits are some specified digits and after that a pattern starts.

What sort of object would that be what sort of point would that be it would be a point which after a certain number of iterations falls into a periodic orbit so it's a pre-image of a periodic orbit would you call such an initial point rational or irrational national absolutely once again it is rational is just that it does not start off right away with a periodic pattern yes why not why not it is a rational number.

Why do you say that why do you say it is not ah okay, we will come back to this question and answer what I want you to think about the answer to this question what if I have K digits here which are completely arbitrary and after that you have a 0 1 0 1 0 1 etc after this many the next number is etcetera this is certainly possible forever what kind of number is this and what kind of orbit does it belong to so what would you say is this a rational number or not it is a rational number therefore the correct statement is that all rational numbers either lie on periodic orbits or on pre images of periodic orbits a finite number of pre images of periodic orbits.

In other words after a finite number of iterations all rational numbers would fall into periodic orbits but the number of irrational numbers is infinitely larger than the number of rational it is unaccountably infinite you cannot count you cannot enumerate all the irrational numbers between 0 & 1 and those numbers the fate of those numbers is that they uniformly and densely fill up this unit interval never hitting a rational point and never settling down to anything specific any specific part of this unit interval but always moving about always wandering on the unit interval back and forth so that is one aspect of this quote/unquote chaotic behavior.

Which we still have to define precisely which I will do in a very short while the other aspect is that if you start with two numbers which are close to each other then here is an instance where the error after n iterations becomes as large as the unit interval itself and this is exactly what I said when I said we have exponential sensitivity to initial conditions so it is very obvious that if you start with an X 0.

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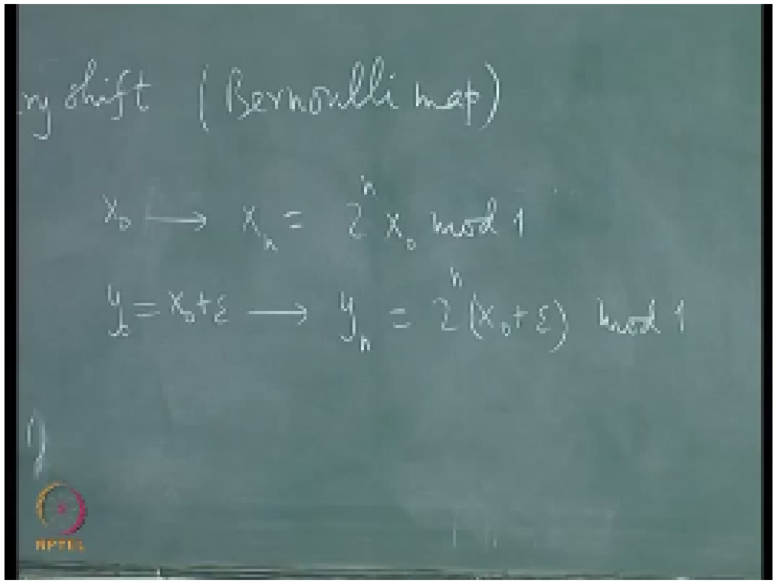

So I start with an X_0 and that leads after n iterations to $X_n = 2^n X_0$ modulo 1 and I start with a Y_0 which = $X_0 + \varepsilon$ and that goes to a Y_n which = 2 to the N X $0 + \varepsilon$ ε modulo 1 so the separation between the two is 2^n ε mod 1 and if n. Becomes large it is clear this number for arbitrarily small ε could become as large as the unit interval itself so this was the statement I made that there is exponential sensitivity to initial conditions that the separation between neighboring trajectory bring points.

In fact exponential in time remember that n is tiny discrete time and 2^n N to we did like to make that a little precise we did also like to find out how do we how are we sure that the iterates of these irrational points fill up. This entire space and finally we did like to ask what is meant by exponential sensitivity so let us take that up next.

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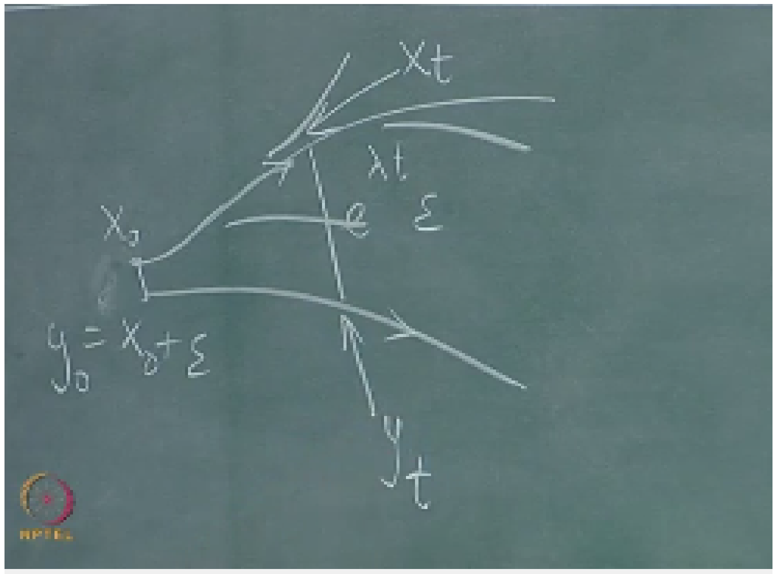

So this is at the root of chaos now what. I mean by exponential sensitivity to initial conditions pictorially it means that if you start with an initial point here and let me do this in continuous-time dynamics just for pitching pictorial for illustration sake and suppose this is the trajectory of this part of this point and I start with a neighboring point here and let us suppose I is the trajectory here then. I did like to find out if points arbitrarily close to this point spread away from it exponentially fast or not as time goes on so I did start with this initial distance here between them after time.

T I did find this distance and if this is exponentially multiple X to the power some λ T times this then I would say there is exponential sensitivity but I want to do this carefully so how would I do this and let us do this in terms of iterations either discrete dynamics or continuous dynamics does not matter how would I do this how would I write this carefully if this is ε and this is e to the λ T ε and I want to extract this λ clearly I take this distance divided by this distance take the log and then divided by 1/2 in order to extract the λ so I define yes there was sanctity behind - what - the exponential sensitivity okay.

If the separation is a power loss separation in time like √ of time or time T itself or T^2 or T^{20} or whatever then you can actually compute the difference you can calculate the way the error amplifies in polynomial time it is completely computable but not possible if it is exponentially fast in other words no matter how small your initial error was eventually things would blow up exponentially and we saw in our study of dynamical systems continuous-time dynamics that except that points.

Where the vector field vanished everywhere else the flow was actually exponential the solutions locally were always of the form Exponentials of time multiplied by some Eigen values of some liniearized matrix so that is generic always besides that is precisely the sort of separation which leads to amplification of uncontrolled amplification of errors of initial uncertainties or imprecision if they yes it means that there is exponential instability however for chaos you need something more than that you need to have that in whole regions of phase space not at individual points.

So certainly we can control what happens in most cases in regular behavior but here it becomes uncontrolled the error will actually grow till it fills up the system size itself that does not happen in normal systems okay , the linearization is an approximation of a nonlinear system yes the reason it became exponential there was because we had first-order dynamics we had first order exponential we had first-order dynamics but the point is the system exponentially unstable in whole regions of phase space or only at isolated points just the fact that you have an e to the LT times x0 locally.

When you rectify a vector field does not imply chaos at all you need many other conditions which is yes the solution is exponential but the errors do not multiply exponentially in a linear system they do not multiply exponentially in a linear system at all right you need non-linearity for this sensitivity to happen it is not enough to have non-linearity but a plain linear system it does not do that it does not do this at all okay, because we are going to put down conditions for chaos you have in mind.

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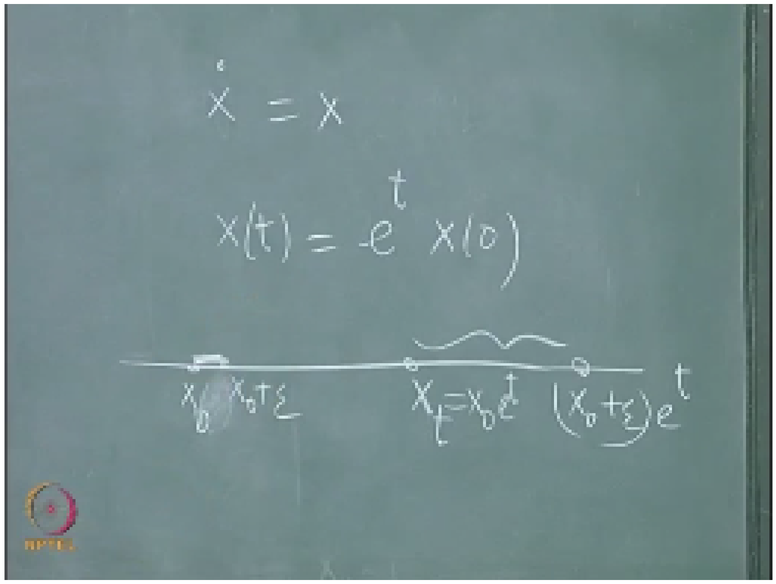

The system of the following kind $\dot{X}$ = to X on the real line the solution here is X (t) is e^{t} X (0) therefore if I start with an X 0 here X 0 here then suppose here is X_{t} and I start with an X 0 +ε here that of course will go to X 0+ ε e to the T whereas this is XT is X 0 e to the T and this separation is an e to the T times this original separation this is not chaos this just says that the solution is an exponential this is not chaos because the phase space is unbounded for chaos we are going to define it for systems for which the phase space is actually bounded and yet you have amplification of errors such that it mixes up in the space.

So badly that the error finally amplifies to the system size itself the size of the phase space itself that is not happening here this is integrable completely integrable there is a concept of chaos but it is not very interesting if the exponential instability is because of an unboundedness in some phase space in some axis that is not very interesting because the system itself is integrable so you will see what else is needed in order to produce chaos yes though they would be chaotic as difficult yes .I agree entirely but they are not very interesting because there we know the system is unbounded and that is the reason why things are becoming exponentially large.

But it is not very interesting we are looking at systems which are not integrable that's very clear chaotic systems are not integrable in general okay again I need to qualify that we will come back to this in the case of discrete maps so I want to define here this multiplier λ in a careful way I want it such that it is a property of this trajectory I want to see how things are thrown away from it.

As you go along so if you have a neighboring point here that's thrown away if you have a neighboring point here that is thrown away another one here is thrown away and so on and I want to probe the rate at which it is being thrown away so let me define it in the following way. I start by saying I have an initial point X 0 and an initial point X 0+ ε here say that is my initial distance and if after time. T this point x 0 finds itself at X T and the point why not finds itself as at Y T then.

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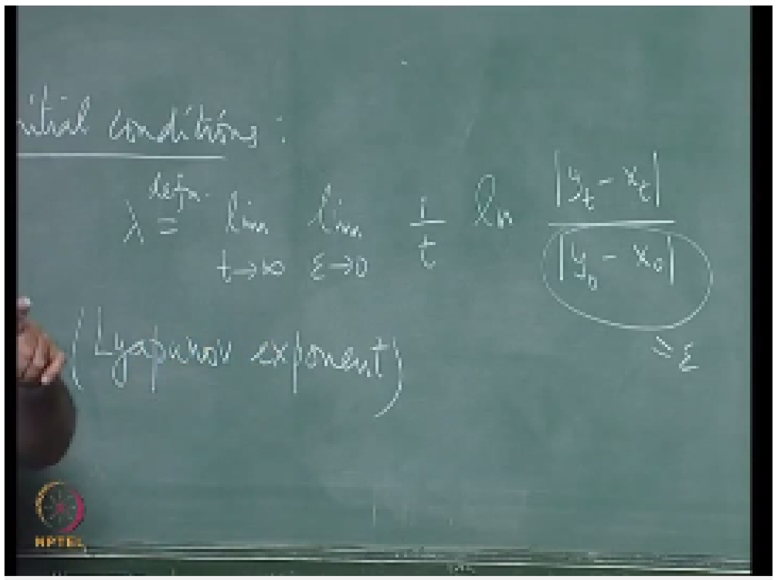

I compute modulus of YT - X T / Y 0- X 0 in this case and in doing so I have got this e to the power whatever it is I have got this, I take the log of this that brings down this exponent and I divide by a 1 (t) and that is supposed to give me this λ but I need to make sure that I am on this trajectory back here so I go to the limit in which ε goes to 0 recall that this is = ε and I need to do this and this is supposed to be the asymptotic behavior so I take the limit as T tends to ∞ of this quantity and that is my λ this is the definition of the Lyapunov exponent .I want you to pay particular attention.

To the order in which these limits are to be taken if I do not do that then it is clear that if the phase space is bounded this number here is always finite and then I have a whole lot of points it does not diverge or anything like that and then. I take its log divided by 80 and as long as this is finite the whole thing goes to zero on the other hand if ε goes to 0 first you may have a very non-trivial λ this limit may exist and give you a non-trivial number and that's the one we want to calculate but for maps we could write a similar thing this was for flows for maps what would you write again.

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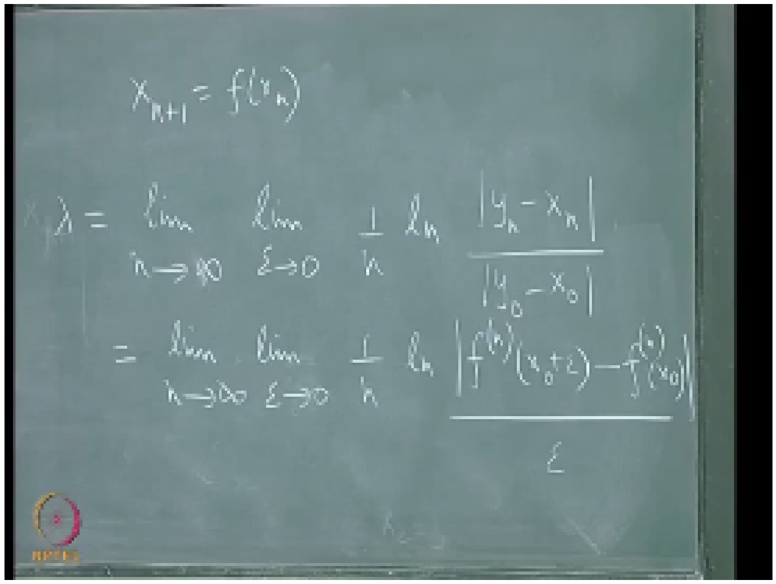

I did write λ = limit n tending to ∞ limit ε tending to 0 1 /N the log of the nth iterate / that would be my definition of the Lyapunov exponent if it exists now our map function says xn +1 is f (X n) so what does this become this becomes = limit n tends to ∞ limit ε goes to 0 1 / N the log of f nth iterate of X 0+ ε because that is the entity rate of y 0- f nth iterate of X 0t / ε but what is this equal to we could simplify this it is very clear that if this map is assumed to be differentiable then what happens so coming over from.

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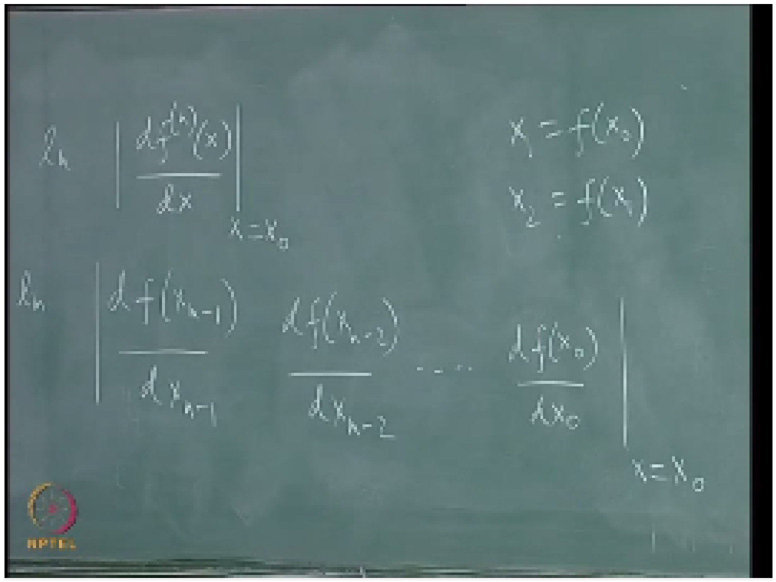

There therefore λ is equal to limit n tends to ∞ 1 over N and then what do we write here yeah it is just the derivative at the point X 0 so it is equal to log of the derivative would be X at X = X 0 that if it exists is the Lyapunov exponent and it is a function of the starting point so it is in fact definitely a function of the trajectory on which or the orbit on which X 0 finds itself could change from one orbit to another most certainly but what do we have here remember that X 1 = f (x) not X 2 is f of X 1 and so on X n is f of X n - 1 and so on so could we not write this in a simpler form you write this as limit n tends to ∞ 1 over n the log off.

So let me write this D F and X what is F n of X it is just X sub n right so this is = D f of X n - 1 over the X n - 1 times DX n - 1 which is d f of xn - 2 over the X n -2 and keep going down all the way the f of X 0 not evaluated at X = X 0 agree but that can be simplified a little more and you can write.

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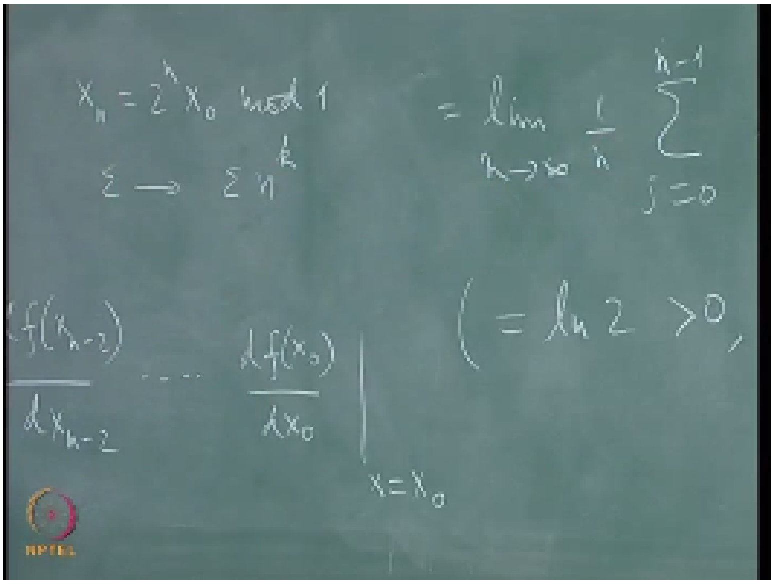

This as equal to limit n tends to ∞ 1 over n a summation from J = 0 to n = 1 F ' of X J lock agree so it is just the time average this is an average over time it is the long time average of the log of mod f ' of x at all those points on which the initial point x 0 falls as you iterate in time over a very long period of time what is this for the Bernoulli shift that we just looked at what would this be what would this be for the Bernoulli shift it could become an integral yes we are not sure yet but we have it yes we should be able to convert this to an integral but then you have to say over what and so on.

But this can be computed very easily for the map we just looked at the doubling map the Bernoulli shift well the map was just 2 X mod 1 and what is the slope it is 2 everywhere the slope is 2 at all points on this map and therefore you had 0 to n - 1 log 2 and you have got to divide by this so what is = 2 and this system is exponentially sensitive to initial conditions could turn out that λ is zero in some cases could certainly turn out that λ is zero incidentally if you had a power loss separation between two trajectories what would you get here what would you get.

Suppose you had an error which increased like a power law remember in our map here X n was 2 to the N X0 mod 1 and X Y n y was = 2 to the N X 0+ ε mod 1 and what we found was this amplification of this error 2 to the n ε and we got this we essentially found this. We wrote it as e to the N this quantity is e to the N log 2 and we detected this log 2 there if instead of 2 to the N X 0+ ε.

Suppose this went like N squared what would happen if you went through this process we have a power law separation now and instead of errors amplifying n fold suppose the initial position imprecision ε goes to ε times N^k yes when. I take this and divided by ε this goes away when I take the log I get K log N and then I have log n over N and the limit is 0 so this method this definition of the Lyapunov exponent is geared to find exponential instabilities.

It does not mean that trajectories have to stay close to each other they could separate sub exponentially and if they did that you detect zero Lyapunov exponent Lyapunov exponent would just turn out to be zero in those cases could even turn out to be negative we are going to see instances of that we will see the physical interpretation of what happens if things become negative but you can guess suppose I find a λ to be negative what would you conclude I died yes absolutely right.

I had conclude that things are converging exponentially rapidly either falling into some periodic orbit or into some fixed point which is stable exactly so I did immediately detect stability if I had negative Lyapunov exponents but if, I have a positive Lyapunov exponent it certainly implies chaotic behavior it will define chaos carefully but now time the time has come for me to say what the definition is I will repeat this again we want a finite phase space we want exponential sensitivity to initial conditions.

In the sense of one or more non zero Lyapunov exponents and finally you want a dense set of unstable periodic orbits buried in this phase space that is what throws things aside on both sides exactly like a separatrix bus the statement was and I will repeat this again because we are over today.

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