Indian Institute of technology madras
Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

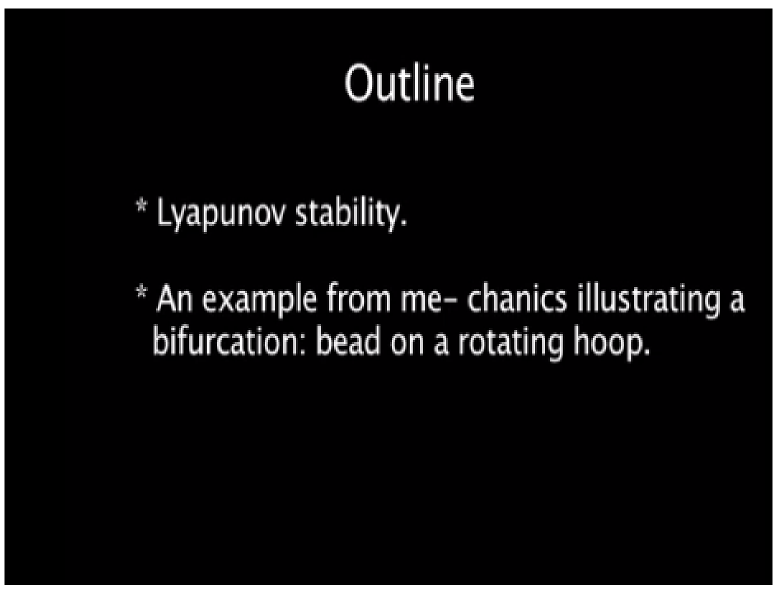
TOPICS IN NONLINEAR DYNAMICS

Lecture 13
Illustrative examples

Prof. V. Balakrishnan

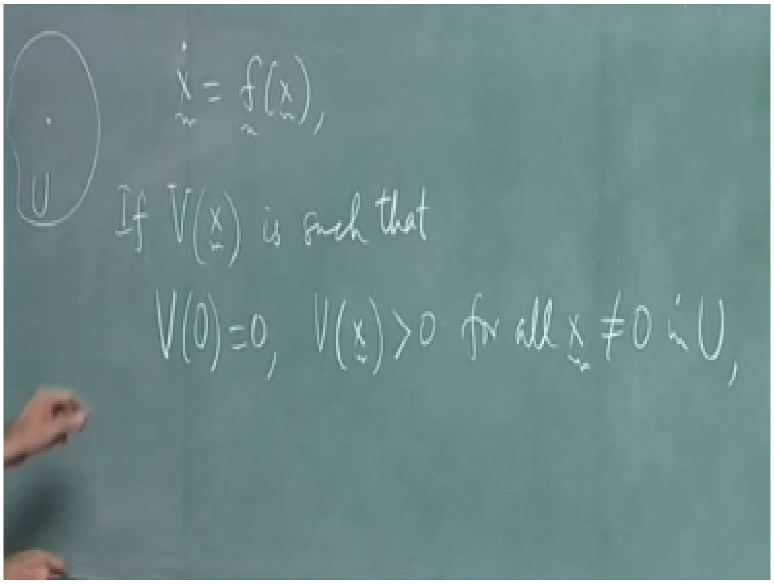
Department of Physics
Indian Institute of Technology Madras

(Refer Slide Time: 00:11)



Now let me repeat this point about Lyapunov stability and the theorems in the simplest form in which we looked at it yesterday last time.

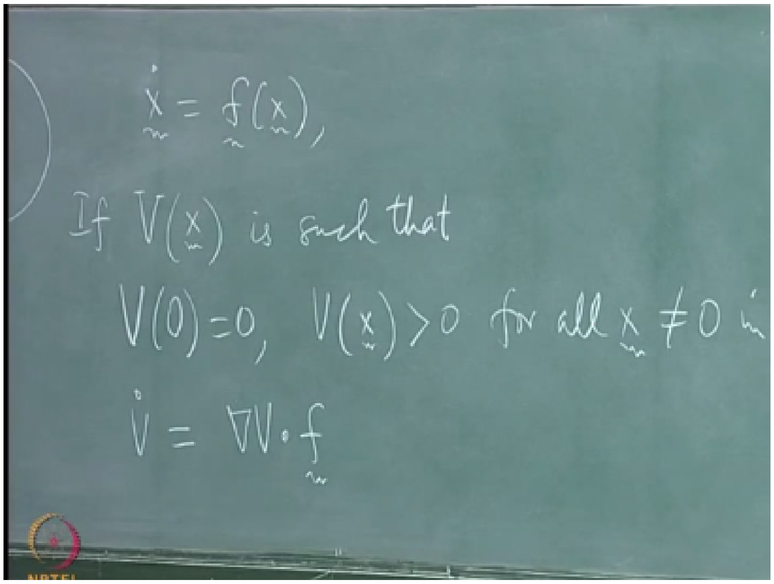
(Refer Slide Time: 00:22)



If this is a critical point say the origin is a critical point of a system and you have some neighborhood of this origin and I would like to understand the stability or otherwise of the system in this neighborhood then the statement was for the dynamical system given by x dot = f(x1) constructs or considers or pulls out of the hat an auxiliary function V (x) called the Lyapunov function which has the following properties and the properties are if V (x) is such that V(0) is 0 and V(x) > 0 ≠ 0 anywhere but strictly positive for all X ≠ 0 in this neighborhood.

So let us call this neighborhood u then V (x) is a positive definite function if it is less than 0 for all X ≠ 0 and U it is a negative definite function if it is greater than or = 0 you cannot find other 0 in the neighborhood U then it is semi definite and if it is ≤ 0 and you cannot find other 0 then it is negative semi definite with these statements with this with these definitions the statements we made or the following.

(Refer Slide Time: 02:00)



So this is positive then V is positive definite and so on now the statements made were the following the rate of change of this function V, V dot turns out to be by a very simple piece of algebra this turns out to be V, V dot F and the statements were if you have a positive definite Lyapunov function and V dot is strictly negative in this neighborhood namely it is negative definite then the critical point is asymptotically stable on the other hand if you have a positive definite Lyapunov function and V dot is only negative semi definite in other words there exist other points other than the origin where V dot vanishes then what you can assert is that the critical point is stable you cannot assert that it is asymptotically stable and if you have a positive definite V and V dot happens to be positive also then it is unstable.

Or a V is negative definite and V dot is negative then it is unstable this is for sure now in the examples we looked at we came across instances we came across one instance where in the neighborhood of the origin for a positive definite Lyapunov function we discovered that in the presence of damping V dot was - γY^2 where γ was the damping coefficient and I commented that this only enables us to deduce that this particular critical point at the origin is not necessarily as importantly stable but nearly stable because we know that γY^2 is vanishes everywhere on the x-axis.

So we can assert that it is stable but we cannot go on to assert that it is asymptotically stable which is what the actual critical point was in the presence of damping so it is not that now it might turn out it might turn out you have a better Lyapunov function not the one that we chose but a better Lyapunov function which is positive-definite for which you can show that V dot is actually negative definite in which case you can assert that this is asymptotically stable as well it is not that this thing exhausts all possibilities it is just that you could choose a bad Lyapunov function in which case you cannot make any statement at all.

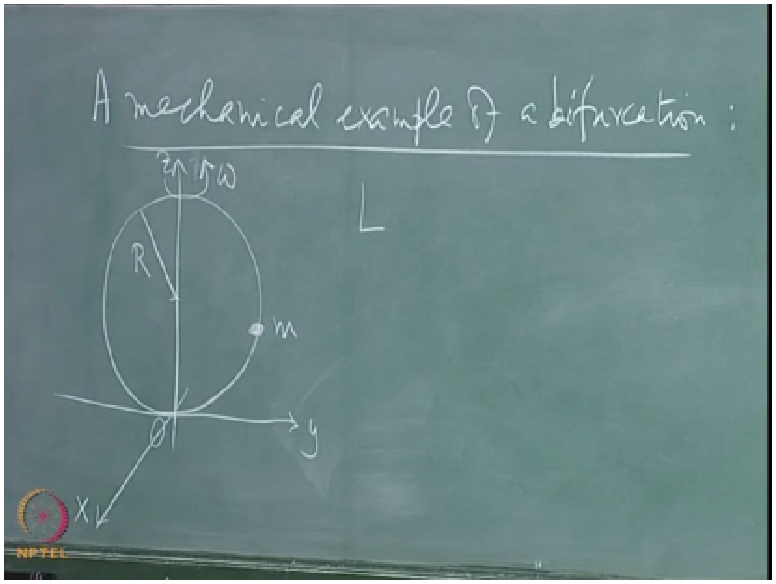
Could change sign or you could choose one which turns out to be only negative or positive semi-definite in which case you can assert something about stability but not about asymptotic stability but the point is if you can find a Lyapunov function which is such that it is positive definite and V dot is negative definite in the neighborhood then you can definitely assert that the critical point is asymptotically stable.

Now we have seen there are critical points which are asymptotically stable but not stable and the critical points which are stable but not as asymptotically stable and certainly there are critical points which are both stable as well as asymptotically stable and the correspondingly Lyapunov functions would reflect these possibilities in various cases.

So these are not mutually exclusive categories in that sense something can be stable as well as asymptotically stable but something which is stable does not have to be as asymptotically stable and something which is asymptotically stable does not have to be stable it is true that you have proved something is asymptotically stable you made a very powerful statement you said that the points were going to fall into this attractor eventually.

So it is a very strong statement in that sense stability is weaker than asymptotic stability it says things do not go away from the neighborhood even if you wait long enough once they have entered the neighborhood but it does not say they are going to fall into the critical point at all so I hope this clarifies to some extent the question that you had so let us take an example now go back a little bit and look at a mechanical example of a bifurcation and then what this problem all fully because it is a standard problem.

(Refer Slide Time: 06:36)



So mechanical we will see there is a bifurcation and I am going to ask you at the end what sort of bifurcation this system displays and it is an extremely simple one but it is an illustrative one because it says something about Lagrangian mechanics as well which we have not really discussed in detail in this course at all but this would give us an opportunity to enough recapitulate our ideas about Lagrangian mechanics.

It is the following problem I have a hoop of wire which is say in the shape of a circle and I have a beam on this hoop of Y a beam of mass M which moves frictionlessly on this ring of wire and let us say this hoop is in a vertical plane and the bead is constrained to move on this circle and let us choose for convenience coordinates such that the origin of coordinates is at the bottom position of this hoop which is in a vertical plane and let us suppose this plane is the y z plane for instance.

And this hoop has a radius R what sort of motion does this constraint system have this point mass under gravity if I take it up here and let go it is going to oscillate without any dissipation back and forth about this point here and it is a constrained system because it is constrained to move on this hoop as it stands how many degrees of freedom does this particle have now how many independent degrees of freedom does it have well I fixed the plane of this hoop and that is the yz is that plane.

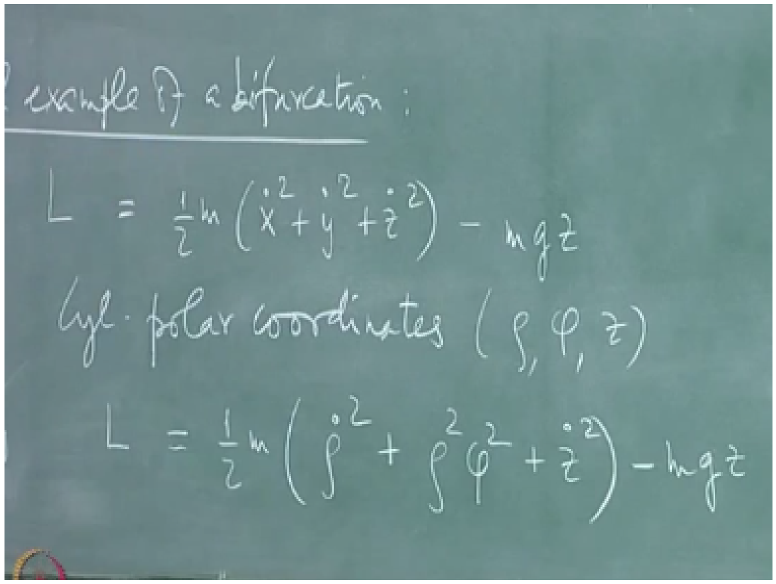
So to start with this particle has only y and z coordinates the x coordinate is always 0 moreover I am saying that it moves on the circle so it is clear on this circle z is a function of Y and therefore it has one independent degree of freedom which you can choose in many ways a convenient one would be to choose the angle it makes with the vertical axis its instantaneous position makes with the vertical axis that will be a convenient degree of freedom.

But now I make the problem a little more interesting by saying that this hoop is set rotate about the said axis with uniform angular speed ω so the entire hoop rotates about this diameter with uniform angular speed ω and so does the beam also rotate and the question is can we do the dynamics can we solve for the dynamics of this point particle of this mass with this time-dependent constraint namely it is sitting on this hoop but the hoop itself is rotating at a constant angular speed ω.

So this is the problem to start with so what do we do we start by writing down the Lagrangian of the particle which is in this case the kinetic energy - the potential energy of the particle and let us do this by saying L which would initially be a function of the xyz coordinates of this particle because now that it is said rotating there is also an x axis coming out of the board you have a right-handed coordinate system here and this L would be a function of all the coordinates as well as the generalized velocities corresponding velocities and what would this be well to start with it would be = ½ m times.

The kinetic energy which I could write as x^2 dot + y dot 2 + z dot 2 in Cartesian coordinates - the potential energy and what is the potential energy of this hoop well it when it is here at z = 0 the potential energy could be taken to be 0 and then when it is at a height z above the xy plane it is potential energy is just mg z so this is - mgz to start with but then the hoop is the beam is constrained to move on this hoop and what sort of coordinate system should I choose when I do that should I chose spherical polar coordinates there is an actual symmetry here it is rotating about the z axis so what would be a good set of coordinates to choose cylindrical polar coordinates.

(Refer Slide Time: 11:13)



So let us choose cylindrical polar coordinates and what are these coordinates well the radial distance the actual distance of this particle from the z axis which is = the ^{2} of the distance is x 2 + y 2 that would be one of them so let us choose row there let us choose the azimuthal angle ϕ so X is ρ cos ϕ and Y is ρ ϕ a plane polar coordinates in the XY plane and of course that itself these would be the cylindrical polar coordinates and what is the kinetic energy in such a case.

So L becomes = ½ m what is X dot ^{2} + y dot 2 become in cylindrical in polar coordinates ρ and ϕ exactly there is a radial velocity ρ^2 dot + yes indeed ρ^2 ϕ^2 because that is what X dot ^{2} + y dot 2 the company you have when you go from x and y to plane polar coordinates or 1 ϕ + z^2 dot is always present this is their - mg z this is what the Lagrangian would be but now we have to impose the constraint that z is a function of the other coordinates because has to remain on the circle.

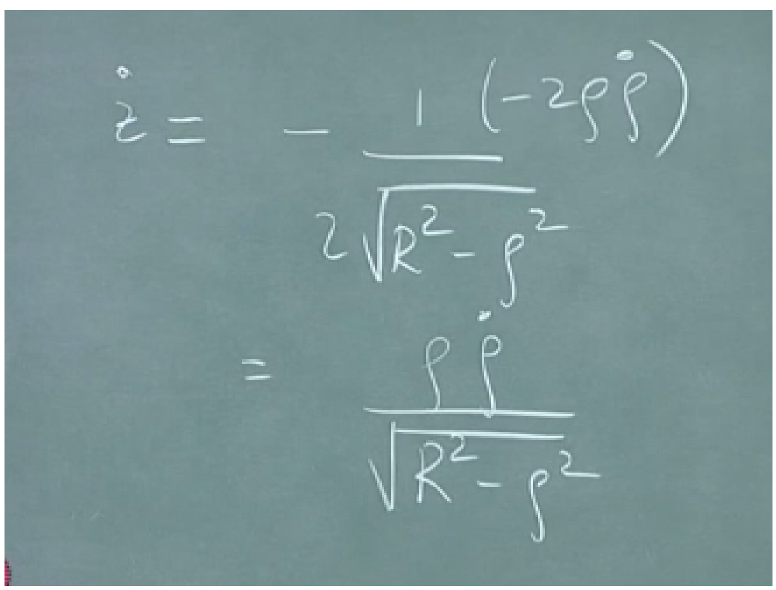
And we need an equation for this circle itself at any instant of time now what is the equation to this circle the circle has a center at z = R and ρ= 0 and in the static case when it was not moving what would have been the equation to this circle it is a circle in the Y z plane with the center at y = 0 z = R and it is radiuses are so indeed this circle would have been for example it would have been (y^2 + z – R) whole ^{2} = R^2 that would have been the equation of this circle when it is static when it is in the Y z plane.

The center of the circle is at y = 0 and z= R now when it starts rotating there is this circle of some axial symmetry about the z axis in the xy plane and therefore what would this equation become instead of Y^2 you simply have to replace it by the actual distance of this particle from this axis it becomes ρ^2 indeed so this is the constraint therefore if you solve for z it says z - R whole 2 = $R^2 - \rho^2$ or z = R +- $\sqrt{R^2 - \rho^2}$ which route should I choose the two routes for z and what do they correspond to here .

Yeah if you fix a value of the other coordinate you could either be here or here and we Are interested in oscillations about this point here so we choose the + √ or the - √ which was the - √ so this is the root we choose that is the physical √ okay because it says when ρ is 0 this is 0 z is 0 and sitting here so let us get rid of this + and write - for the branch we are interested in and now we are ready to eliminate out here this is ϕ dot 2 by the way that is got to have dimensions of angular velocity 2.

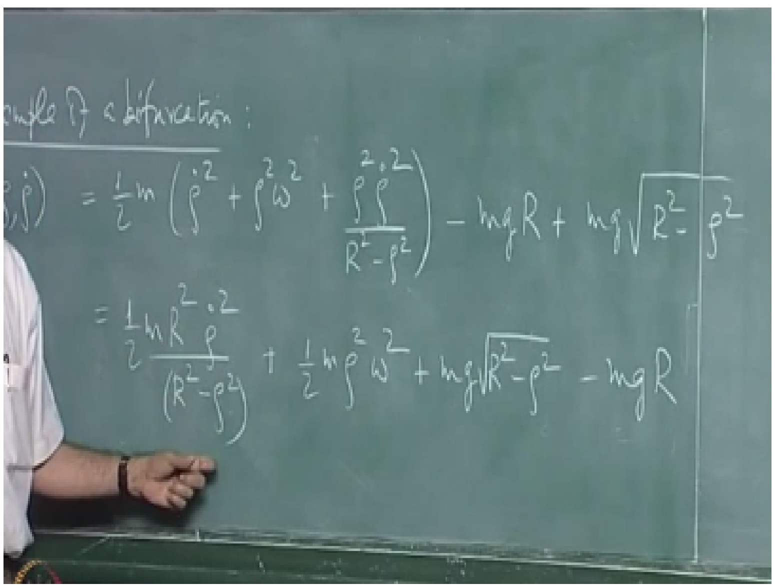
So that is the kinetic energy term and when I rotate it then what does ϕ become what does ϕ dot become if I rotate it at constant angular speed ω ϕ dot is ω itself and my dog is identically = ω therefore we are now ready to write on Lagrangian down but we need to use this in order to eliminate z dot from the problem and z all together.

(Refer Slide Time: 16:00)



So if z is this then it implies that z dot is = - 1 / 2 $\sqrt{R^2 - \rho^2}$ it is ρ that changes with time R is a constant and I am differentiating this with respect to time so you differentiate this and then you differentiate - ρ^2 which is = - 2 $\rho\rho$ dot smashing so that gives us = $\rho\rho$ / $\sqrt{R^2 - \rho^2}$ you can put all that in here and we finally have our Lagrangian which is a function of ρ and ρ dot alone.

(Refer Slide Time: 16:39)



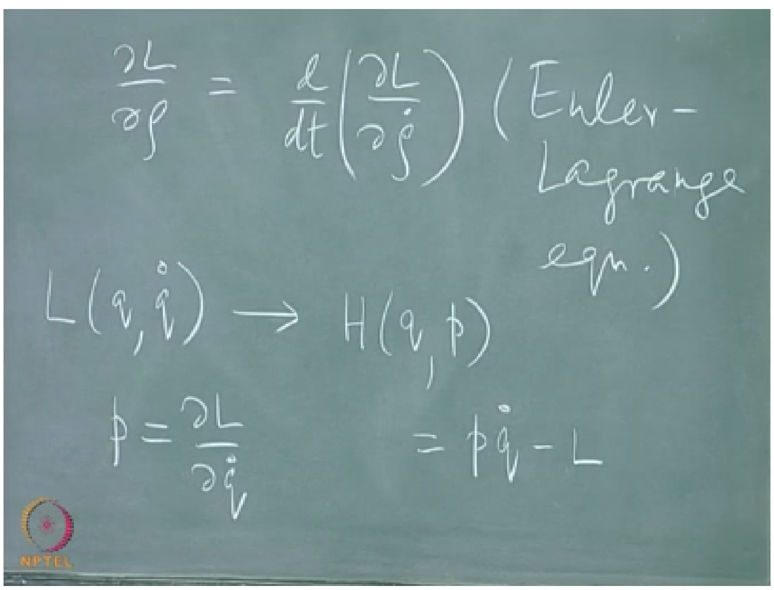
Just the radial coordinate ρ and its corresponding generalized velocity ρ dot and this becomes = 1/2 m ρ dot 2 + ρ^2 ω 2 because I get rid of ϕ^2 dot + z^2 dot which as you can see is ρ^2 ρ^2 dot / R^2 - ρ^2 this term - mg times z which is mg R + mg times $\sqrt{R^2 - \rho^2}$ and we can simplify this a little bit and get a Lagrangian which is = 1/2 m we combine these two terms here and it is clear that you get just m R^2 R or R^2 / R^2 - ρ^2 here + ½ m ρ^2 ω^2 that is this term + mg ^{2} $\sqrt{R^2 - \rho^2}$ – mg.

This is just a constant in a relevant constant that is our Lagrangian with one degree of freedom independent degree of freedom ρ and it is a function of ρ and the corresponding velocity ρ dot the constraint has been taken into account automatically and notice that in the so-called kinetic energy term you have a coefficient which depends on the coordinate this is typical of a constrained problem eliminated the constraint.

And it is become a function it is true that it is quadratic in the generalized velocity but it is got a coefficient which depends on the coordinate itself and there is this extra term which has emerged which looks like part of the potential what would this be due to physically what is happening what is that going to give us this gives us the effect of the quote-unquote centrifugal acceleration that is exactly what it is as you can see because it is a rotating coordinate you end up automatically with the term which will give us this pseudo force once you write the equation of motion.

Now and that is one of the advantages of the Lagrangian formalism where constraints can be taken into account easily and non inertial forces and automatically included it is emerged automatically once we wrote down what this thing was the kinetic energy was now what is the equation of motion we know that in a Lagrangian system of this kind the equation of motion if you have L as a function of q and q dot.

(Refer Slide Time: 19:54)



Then ∂L / ∂ρ the partial derivative of L with respect to the coordinate is = d / dt ∂ L / ∂ ρ dot this is the Euler Lagrange equations for this simple system and that is the equation of motion which is supposed to give you the correct counterpart of Newton's equation of motion well all we need to do is to apply that here so this would imply if I differentiate I get ∂ L / ∂ ρ there is a term here it sits here and differentiate this you have to differentiate this you have to differentiate this and equate it to d / dt ∂ L / ∂ ρ dot which means twice or whatever ρ dot divided by this.

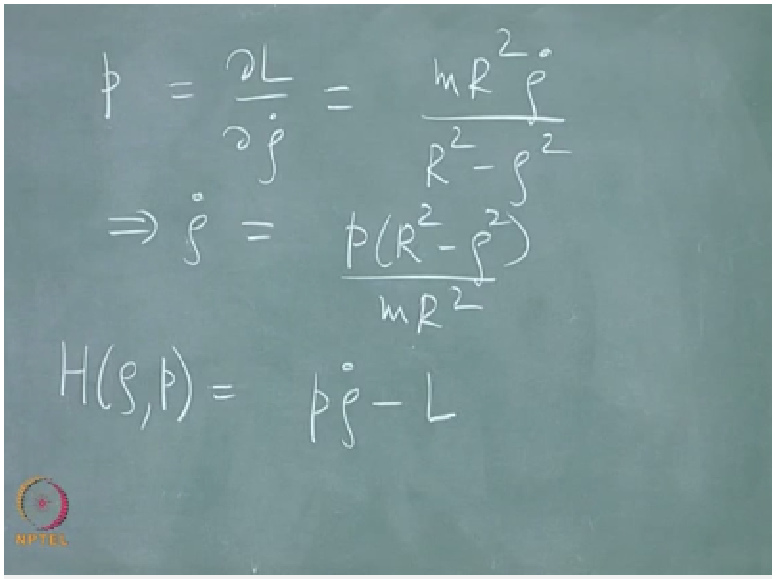
And you get the equation of motion but we are interested in writing things down in the Hamiltonian framework because we would like to write everything as first-order equations in this case and so we make a change from the Lagrangian to the Hamiltonian formalism and if you recall if you start with an L which is a function of q and q dot then the way to go to the Hamiltonian formalism is to go from this function to a Hamiltonian which is a function of q and p by making the following transformation.

You first define a generalized momentum which is ∂ L / ∂q and that is the definition of the generalized momentum given a Lagrangian the momentum conjugate to a particular coordinate q is the partial derivative of L with respect to the q and then once you have done that with respect to q dot sign with respect to the velocity then this Hamiltonian is = pq dot – L.

And it is a transformation which takes you from the Lagrangian to the Hamiltonian however you have to be careful that you can that you invert this equation this in general gives you p as a function of q and q dot because L is a function of q and q dot but in going to the Hamiltonian which is a function of q and p you must eliminate q dot and you do that by solving this equation for q dot as a function of q and p.

So this gives you p as a function of q and q dot you must now write q dot as a function of P and q substitute that here and wherever q dot appears in L substitute for q dot in terms of P and q and you have a Hamiltonian and you are guaranteed after that Hamilton's equations of motion so this is the procedure for going to a Hamiltonian let us do that here let us apply that here so what we need to do is to first find.

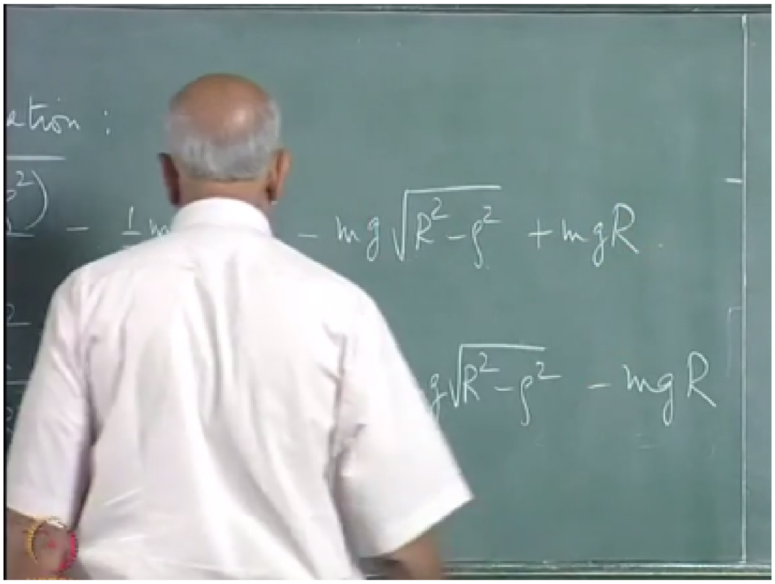
(Refer Slide Time: 22:55)



The momentum conjugate to the generalized coordinate ρI should really write P sub ρ it is a radial and obscure momentum but since there is only one degree of freedom let me just call it P and this is = ∂ L / ∂ ρ dot and what is that = it says differentiate this pretending everything else is a constant except ρ dot the 2 cancels and you get mr^2 ρ dot / R^2 - ρ^2 please notice now that we are not turning out it is not turning out that the momentum is some mass times the velocity not at all it is function of the coordinates times the corresponding generalized velocity this again is typical of a problem which involves constraints.

But that is the definition of the canonical momentum conjugate to ρ once you have this of course you can invert this trivially in this case this implies that ρ dot is P times R^2 - ρ^2 / mr^2 and we can write down therefore the Hamiltonian as a function of ρ and p which is = P ρ dot - L we are ready to write down then a Hamiltonian.

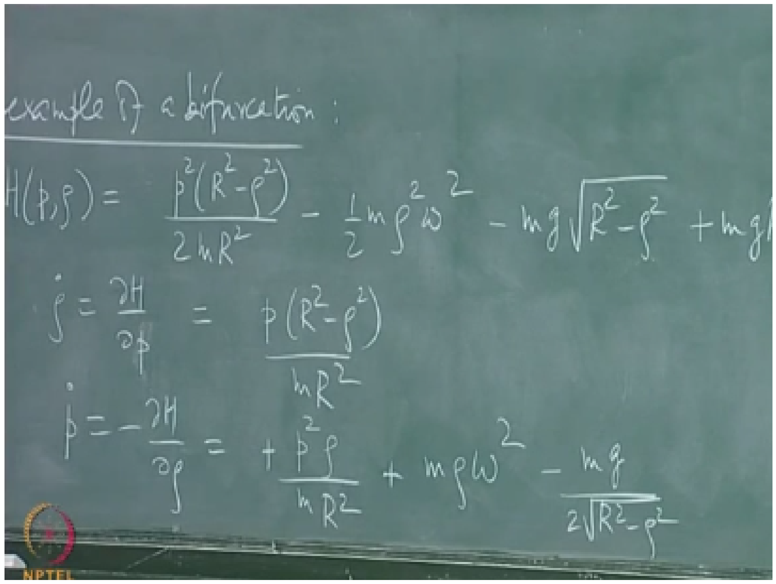
(Refer Slide Time: 24:29)



Therefore H of P, ρ is = P times ρ dot and ρ dot is given by this so it is P^2 times $R^2 - \rho^2 / mr^2$ - the Lagrangian but in the Lagrangian you must replace ρ dot in favor of P since it must be a function of ρ and P alone if you did that you get P times ρ^2 dot so it is going to give you when i^2 this I get an mr^2 whole square that cancels against this and then $(R^2 - \rho^2)^2$ 1 power cancels against this and as a factor 1/2 .

So it is cleared we have 1 of this – 1/2 of the same and therefore it is just this now that takes care of the kinetic energy part 1 - 1/2 m ρ^2 ω^2 that is this quantity - mg ^{2} √ $R^2 - \rho^2$ + mgr okay I hope you kept track of all the - signs because if not we are gonna get wrong answers so let us see what happens that is our Hamiltonian we can get rid of this now we are ready to write down Hamilton's equations of motion. Now it is a nonlinear Hamiltonian it certainly does not look like simple harmonic motion or anything like that and notice the signs of these terms.

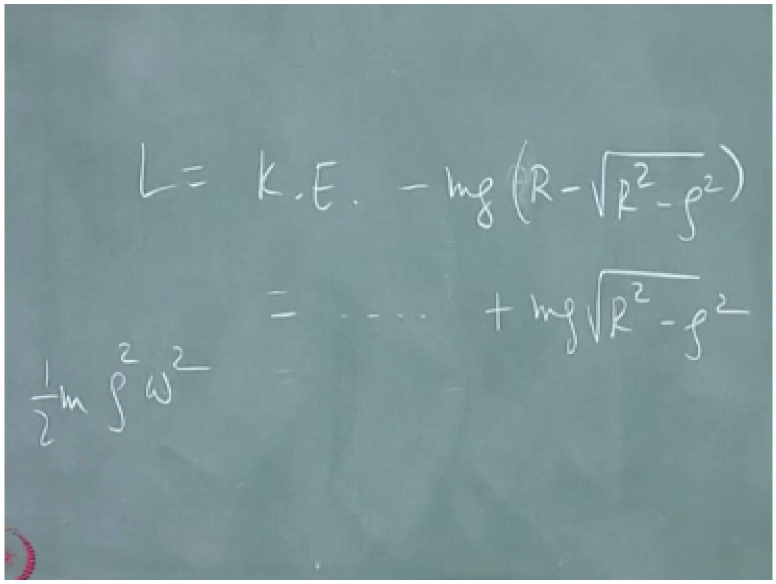
(Refer Slide Time: 26:34)



Now what are the equations of motion well we know that ρ dot is ∂ H / ∂ that is the first of Hamilton's equations and that of course is going to give us something we already knew because ∂ H / ∂ P and this is something we already knew because if I differentiate this with respect to P it gives P times R^2 - ρ^2 / mR^2 but we already knew that P was ρ dot times mR^2 / R^2 - ρ^2 that is how we got P so this is not telling us very much nothing new in any case it is just the old relation between the generalized momentum and the generalized velocity.

But the other equation says something interesting and that is P dot is - ∂ H / ∂ ρ and what is this = and now we have to differentiate everything okay I differentiate with respect to ρ and I get - P^2 ρ / mR^2 it is a + sign because I already have a - sign here and I have another one here so it is a + sign and then + m ∂ ω^2 that is this term here - mg /2 $\sqrt{}R^2$ - ρ^2 not very happy with what happened I hope we did not leave out any - sign anywhere in between because we have then we are going to have an unphysical result here. So I need to differentiate this and it came from - should be a + should that be a +.

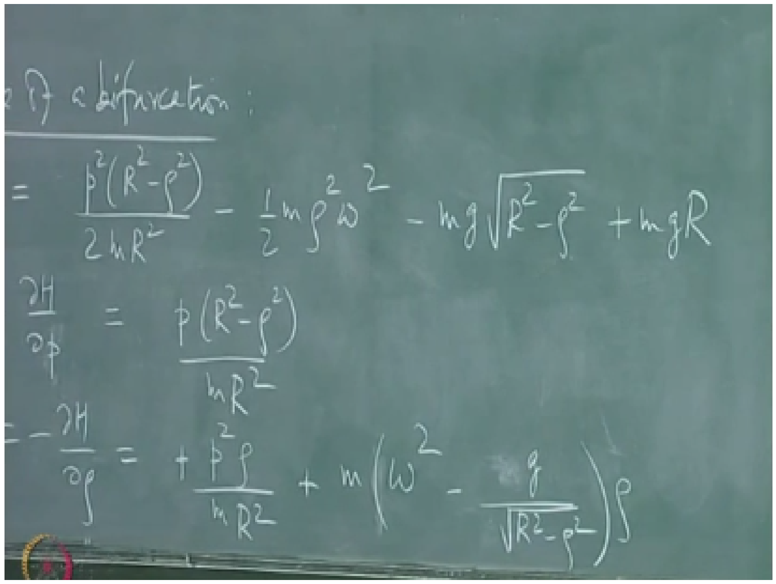
(Refer Slide Time: 28:50)



We started off by writing L = the kinetic energy - mg said that was the potential energy but if you recall z itself was R - $\sqrt{R^2 - \rho^2}$ and so it gives us this is = various terms + mg $\sqrt{R^2 - \rho^2}$ in the Lagrangian so when I wrote the Hamiltonian I put a - sign there so that gives me a - similarly the original term in the Lagrangian was of the form ½ m ρ2 ω 2 the ϕ dot 2 term in the Lagrangian with a + sign and when we write a - for the Hamiltonian it is p q dot - L that 2 appears with a - sign so we are okay.

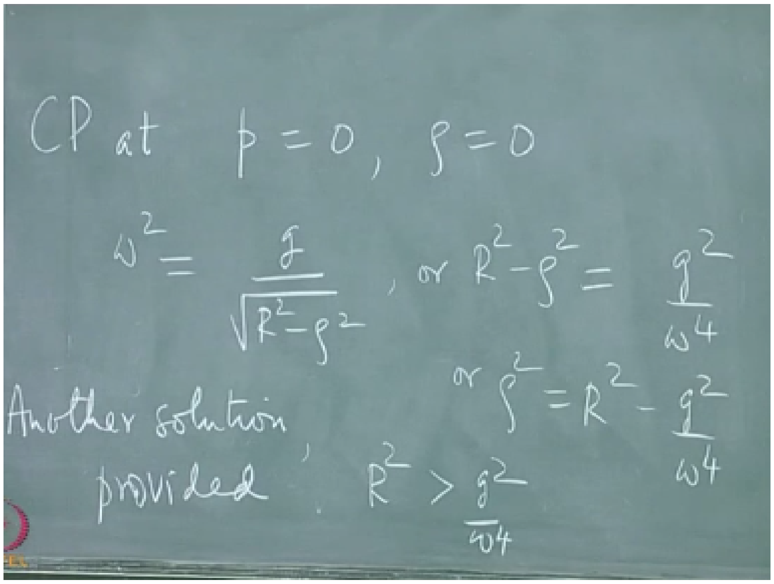
All right now let us differentiate this so we were in the middle of that and then I need to differentiate it and change the sign so there is a - here and a - here both these things go away and you get a + and then I have to differentiate this with respect to ρ so that is = 2ρ and therefore the 2 goes away and the - turns out.

(Refer Slide Time: 30:22)



And let us take the row out of the bracket and the M as well in this fashion so these are the two equations of motion they are the Hamilton equations of motion again you see they are badly nonlinear equations because you have all kinds of powers here you have a cube here and then you have a third power here and then you have this 1 over the √and so on so it is really a nonlinear problem in this sense now the question is where are the critical points of the system. They did not happen when the right hand side is = 0 of course you put R = ρ you end up with an infinity here so the critical point would occur at P = 0.

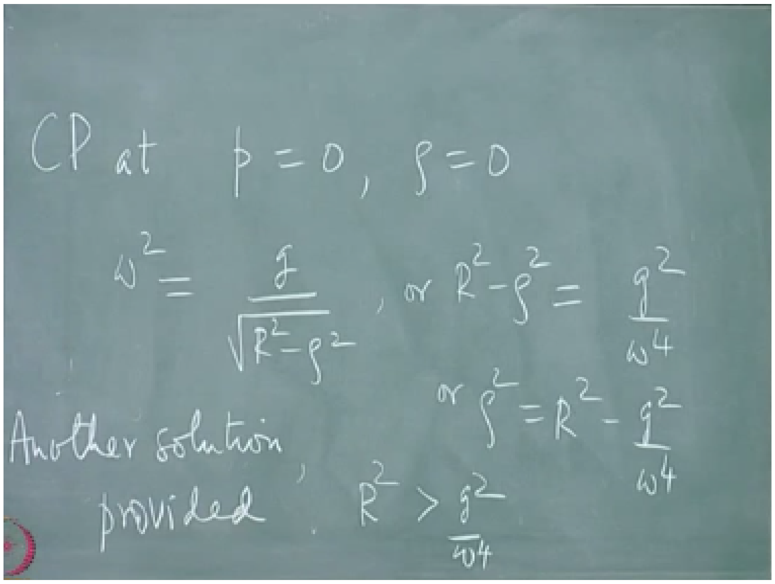
(Refer Slide Time: 31:16)



And let us write that down $P = 0$ and certainly row = 0 is a critical point because if I put $\rho = 0$ this vanishes and this vanishes as well that is always a critical point that corresponds to this point the system at rest here which we know is an equilibrium point whether it is stable or not we do not know is there any other critical point in the system so you set $P = 0$ and this term goes away and you see this could vanish this bracket here could vanish and when does that happen.

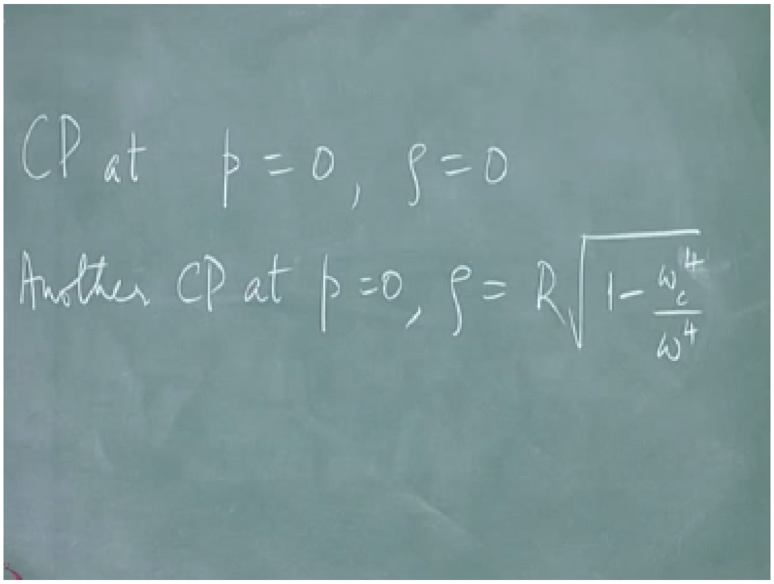
So this would vanish provided ω^2 - g / R^2 - ρ^2 is 0 or ω^2 = this or R^2 - ρ^2 is = $g2$ / $\omega 4$ that is the root here or $\rho 2$ is R^2 - g ^{2} to the $\omega 4$ provided this is positive provided this quantity is positive otherwise it is not going to happen it is not a real root what would this imply when is this going to be positive so another solution non-trivial solution for ρ at the $^{\surd}$of this value provided R^2 was greater than g^2 / $\omega 4$ for or since ω is the quantity that you have in control R and g are constants this implies that you have a second solution I need this equation.

(Refer Slide Time: 33:45)



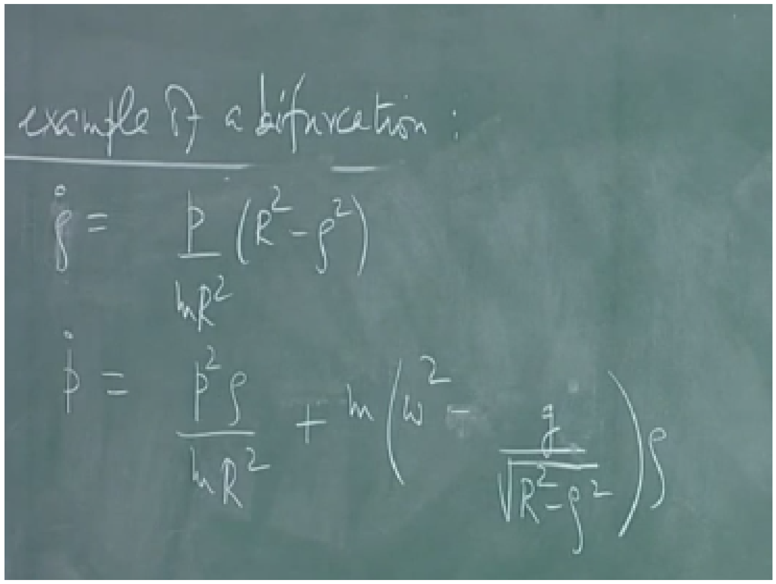
So I am going to ask to dictate it to me afterwards so you have another critical point provided ω to the 4 is greater than g^2 / R^2 or ω is greater than ω critical which is = $\sqrt{g}$ / R then you have another solution it gives you something else altogether we need to compute what that is so that route is given by this let us write it in reasonable form so you have a second root at ρ^2

(Refer Slide Time: 34:34)



Another CP at P = 0 and ρ= the √of this so let us take out an R completely R times √ 1 - g^2 / R^2 /ω4 but g^2 / R^2 is ω critical to the power 4 right so this is = ω critical square ω critical fourth over ω4 some constant its √ g / R now what kind of stability do we have here what can we say about the stability of these solutions so we have to go back and write the equations of motion down and see what it does for linear stability.

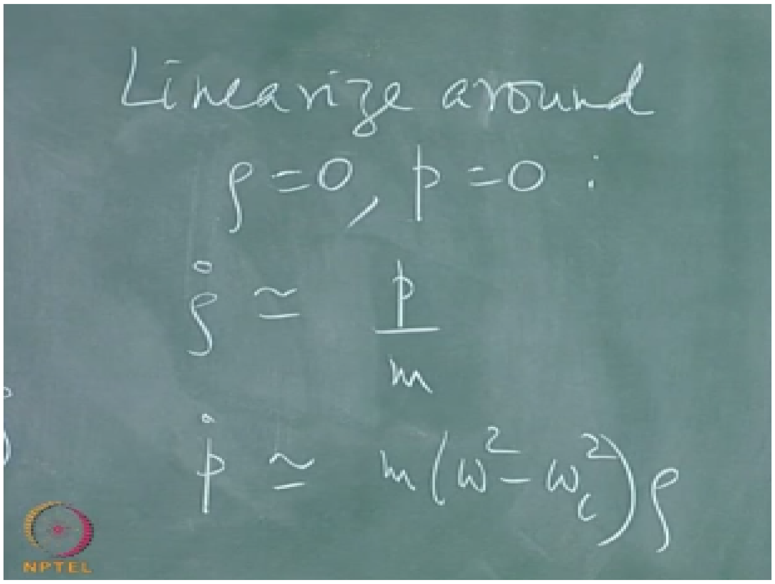
(Refer Slide Time: 35:34)



And if you recall we had P we had ρ dot is = mR^2 P / R^2 - was this right what was what was wrote out P over m R^2 was this that was the equation and the other equation was P dot is = now you have to tell me what it was I believe there was a P^2 / mR^2 + $m\rho\omega^2$ + - mg∂ √ R^2 - ρ^2 and that was it so what we did was to take the ρ out and the m out so we wrote it in this form so you could rewrite this as in terms of ρ / R and the factor g/ R you could write in terms of ωC but anyway let us leave it in this fashion.

So we have one critical point here and another one there and let us look at the stability of these critical point.

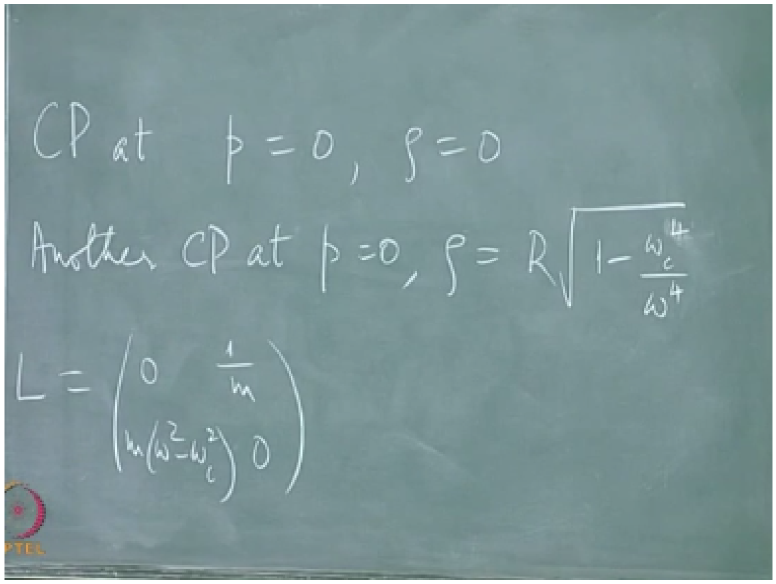
(Refer Slide Time: 37:15)



So let us linearize around ρ= 0 P = 0 and then the equations of motion are ρ dot is approximately = the R^2 cancels P / m which is what you would expect because you would expect very close to this for small oscillations you would expect that ρ dot is just P / m or PS and ρ dot as usual the non-linearity does not play a role there and then this equation this term is already third-order we are going to look at it near ρ= 0.

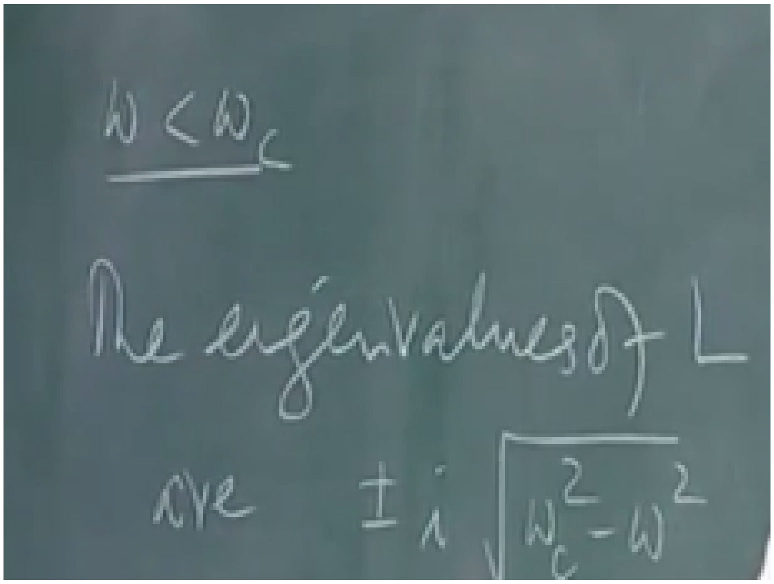
So this can be got rid of and then what you get P dot is approximately = so this can be dropped because it is second order and ρ this can be dropped because it is got third order terms and then you have ω^2 - g / R times ρ but what is g / R ω C^2 so you have m times ω^2 - ωC^2 times.

(Refer Slide Time: 38:42)



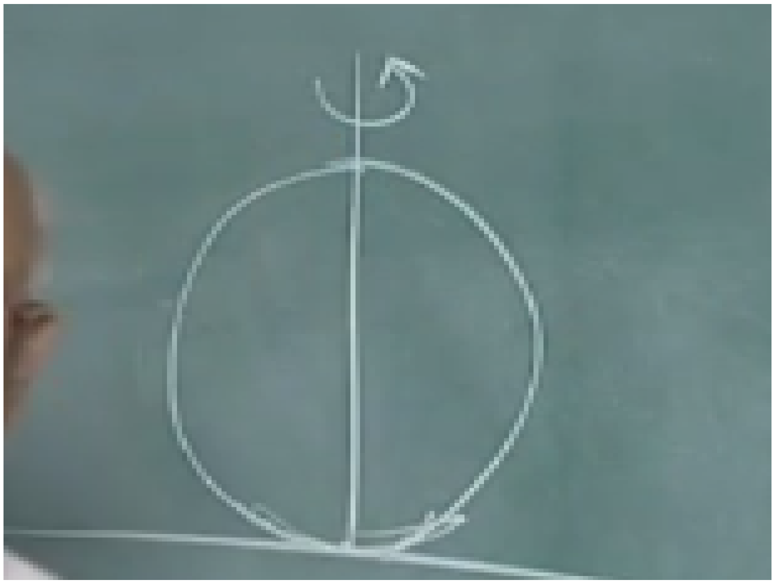
What kind of critical point is that well the linearized matrix L is of the form 0 1 / m m times ω^2 - ωC^2 and then a 0 here. What kind of critical point is that it depends on the value of ω whether ω is greater than ωC or less than ωC it depends on that completely this be comings now the stability at the origin so let us look at the case ω less than ω C first so you start rotating at a small speed first and see what happens then so I think we need to retain this so let us do this.

(Refer Slide Time: 39:36)



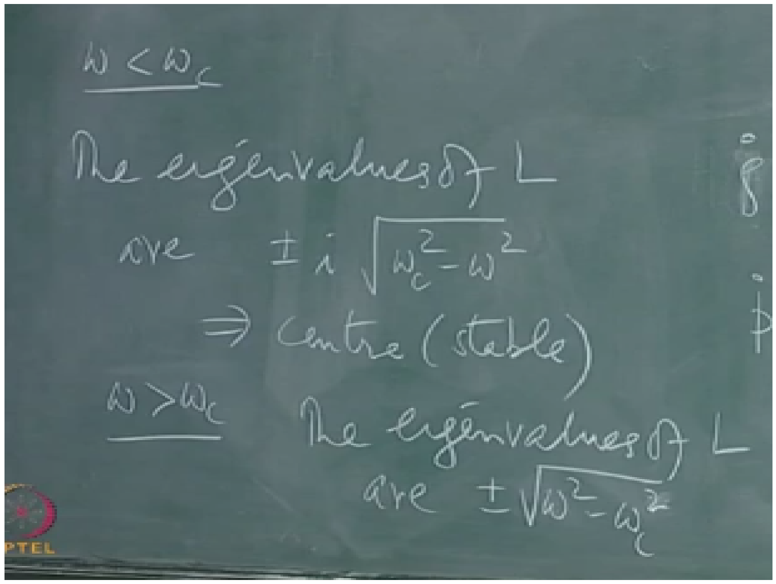
Therefore ω less than ωC the eigen values of L well what are the eigen values of this matrix ω is less than ωC + or - with a R with an either right because ω is less than ω C so what kind of roots are these pure imaginary so it is Center absolutely so you have a Center which is stable at the origin this means that if you rotate this hoop at sufficiently slow speed.

(Refer Slide Time: 40:42)



Then as it rotates this point is a stable Center so if you start here it oscillates about this point while rotating so it comes back and you have undamped oscillations about the bottom that is your critical point which is stable and you can show simultaneously that the other critical point is unstable on the other hand what happens if ω is greater than ω C.

(Refer Slide Time: 41:11)



So this is stable and if ω is greater than ωC then the eigen values of L + - √ ω^2 - ωC^2 what kind of critical point is this it is a saddle point and therefore unstable the origin becomes unstable you would have to do the same thing about this point to discover that this becomes stable so this implies yes there should be no critical point that second critical point does not exist you are right second critical point does not exist at all if ω is less than ωC because this is a pure imaginary absolutely right yeah so it says wherever you are it is going to oscillate about the bottom most position and these are stable oscillations on the other hand the moment you have ω crossing the value of ω C the origin becomes unstable and a new stable critical point emerges.

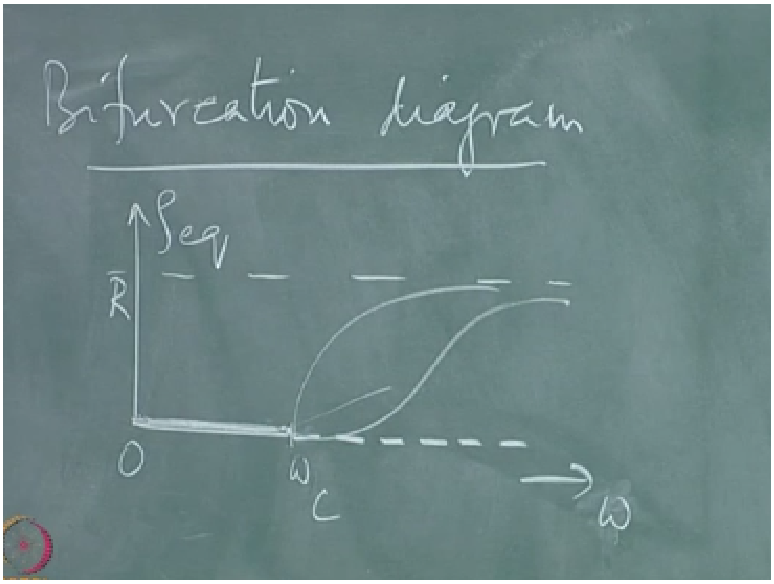
So this is going to help us find what kind of bifurcation this is and this is the interesting question but before we do that let us see what happens let us draw this bifurcation diagram and see what happens so here is the bifurcation diagram.

(Refer Slide Time: 42:45)

So it is not an exchange of stability transition at all it is not an exchange of stability bifurcation because this one does not exist only if ω is greater than ω C bifurcation diagram is as follows we plot ρ equilibrium the equilibrium distance radial distance actual distance from the z axis as a function of ω which takes on only positive values and here is ω C, P is always 0 so I do not plot that it comes out of this axis but P is always 0 at equilibrium and then it says this is your stable root and we know it becomes a saddle point as soon as ω exceeds ωc.

So the rest of this diagram should really be a dotted line and the new root that comes out is this what is the graph here what kind of behavior does it have at ω = ωC it is 0 its bifurcating out of this point that is clear what happens as ω tends to infinity what happens to that equilibrium value of ρ it tends to R so it is very clear physically that this is what is happening.

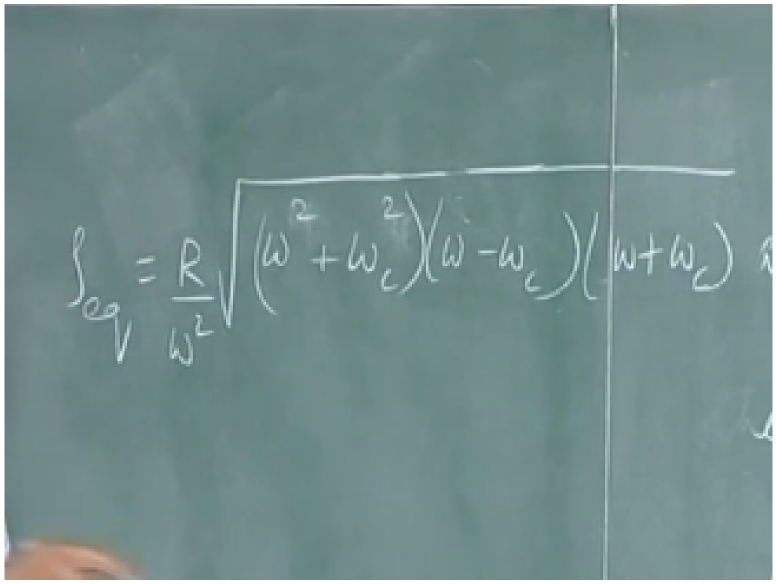
(Refer Slide Time: 44:38)



It is clear that this hoop as you rotate it faster and faster with some angular frequency ω this is the farthest it can actually go it cannot go beyond that it is constrained to remain on this hoop so once it zooms up to this point it has to remain there and this is a stable oscillatory point so here if this is a equilibrium value then if you displace the hoop about it as it is rotating it would actually do this about this point so it gets shot out a certain distance from the bottom as long as you can start exactly at this point and of course it remains at that saddle point.

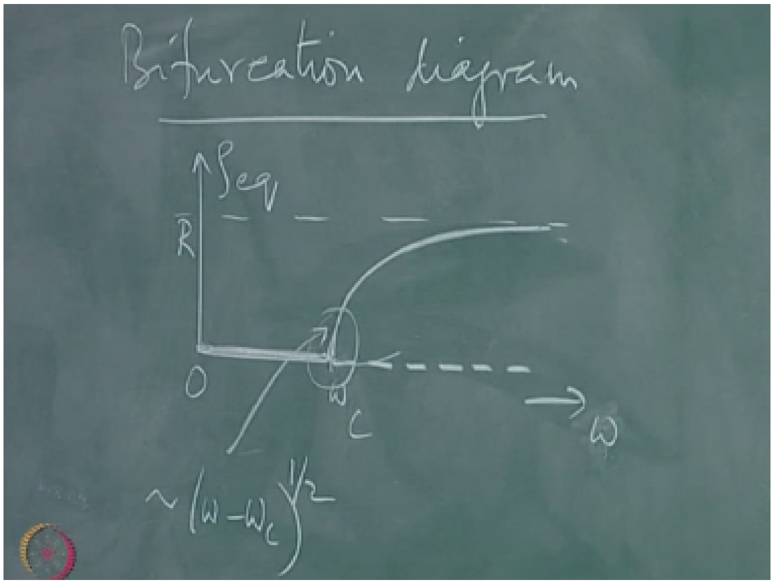
It is an unstable equilibrium point but a little bit of perturbation puts you away and it zooms up crawls up to this point and oscillates about this point so this goes off and as asymptotically hits the value R as ω becomes very large now what kind of behavior does it have at this threshold with the slope infinite with the slope be finite would it do this would it do this or would it do this what would it do that is crucial to know yes we are indeed we need to know exactly what the behavior is like and that is easily seen because you can see that.

(Refer Slide Time: 46:01)



This ρ equilibrium is in fact = R times √ ω 4 - ω C4 / ω^{2} take it out of the √and near ω = ω C it is clear you can write this as ω^2 + ωC^2 times ω^2 - $\omega C2^2$and near ω = ω C you can replace this with ω C that with ω C and you basically have this - that which in turn you could write as ω - ω C times ω + ω C and this harmless factor you could write as twice ω C so the whole thing goes like the √of ω - ω C so what is the shape of the curve here what is the slope what is the slope of y = √(x1) at X = 0 it is infinite.

(Refer Slide Time: 47:09)

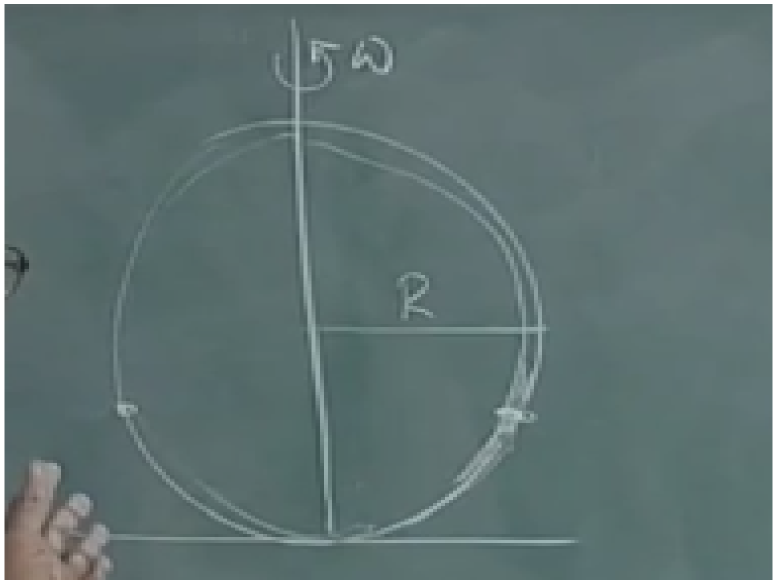


So this is in fact a thing like this goes off and this thing here in this neighborhood it goes like ω - ωC to the power ½ so that is the exponent and this is the stable hook what kind of bifurcation is this is it an exchange of stability bifurcation no because there is not a critical point at all to start it the other one does not exist for ω less than ω C is it a saddle node bifurcation.

That means this pair a stable unstable pair of critical point emerges as you cross the bifurcation value but that is not true because you already had one here so it is not saddle node it is not exchange of stability it must be a pitchfork bifurcation but then a pitchfork bifurcation is one where a stable point bifurcates into an unstable point and a pair of stable equilibrium points where is the other pair one where is the other member of this pair pardon me.

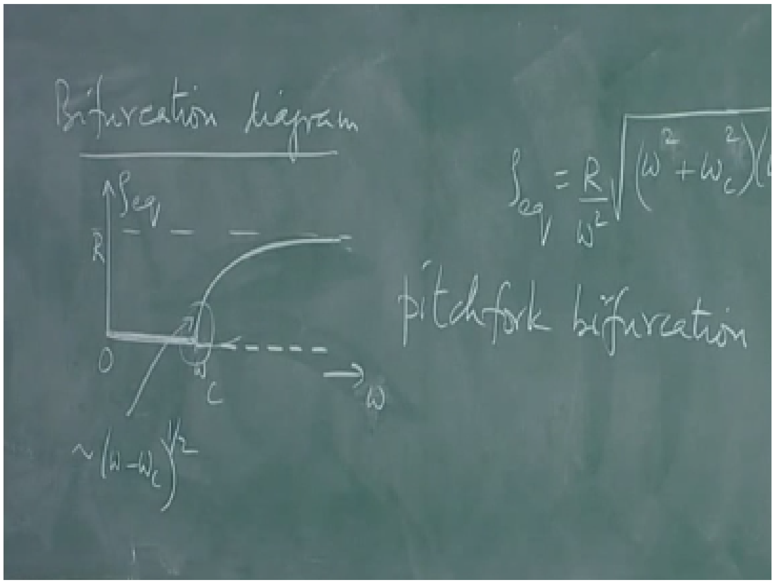
Why is it not physically realistic because we because ρ cannot be negative we plotted a coordinate where it cannot be negative but it is indeed a pitchfork bifurcation because there are indeed two possible states of this suppose you color this part of the hoop green.

(Refer Slide Time: 48:27)



And this part of the hoop read certainly something that sticks here and something that sticks here can be distinguished between depending on what kind of infinitesimal perturbation you had here it would either zoom up to the right-hand side or the left-hand side the green side or the red side and therefore it is in fact a situation where you have two different equilibrium points one on one side of the hoop and the other on the other side of the hoop and you do not see that in this picture because this is just the actual distance we need there are two critical points that I Have emerged from a single one.

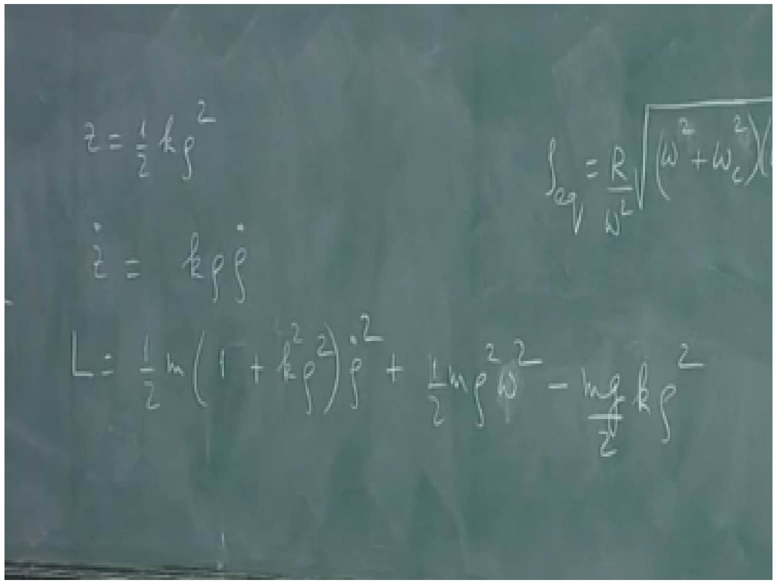
(Refer Slide Time: 49:07)



So this is very much a pitchfork bifurcation you could use this to measure little g to measure and everything else and you are sure that you do not have friction you could use this actually to measure little g because you could measure where the equilibrium point is and this will tell you directly in terms of R and g and ωC it tells you g so that is one possible thing you could also put this in a bowl in a spherical ball and rotate it rotate the ball you would still have this problem but this is the way a constrained mechanical problem is simplified when you use the Lagrangian framework.

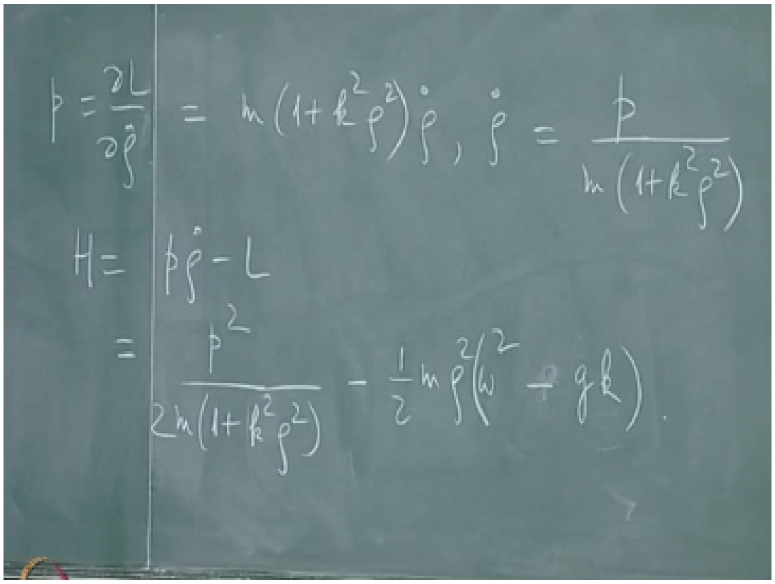
Aand it is stability is easily analyzed when you use the Hamiltonian framework in this fashion you could now ask what happens if I had different shapes I do not necessarily have to have a circular hoop so let us see what happens and quickly see what happens if you have a slightly different shape of hoop.

(Refer Slide Time: 50:32)



One possibility is to write is to use for instance a parabolic hoop so here is it and once I said it rotating in this fashion a parabolic hoop if this z is ½ K Y^2 for example then on this parabola z is = 1/2 K ρ^2 because that is Y^2 is replaced by $x^2 + y^2$ once you set it rotating my cylindrical symmetry and then we see that z dot is = K ρ dot in this fashion we run through our steps quickly then the Lagrangian becomes = ½ m there is certainly a ρ dot 2 + this is n dot 2 but the z dot 2 is $K^2 \rho^2$ if I pull the ρ dot 2 out this is 1 + this and ρ dot 2 in this fashion + in this case ½ m ρ^2 ω^2 that came from the ρ^2 ϕ dot 2 tongue and the kinetic energy - mg times z but this is over 2 K ρ^2 this is what the Lagrangian is and what is the Hamiltonian well to run through these steps quickly.

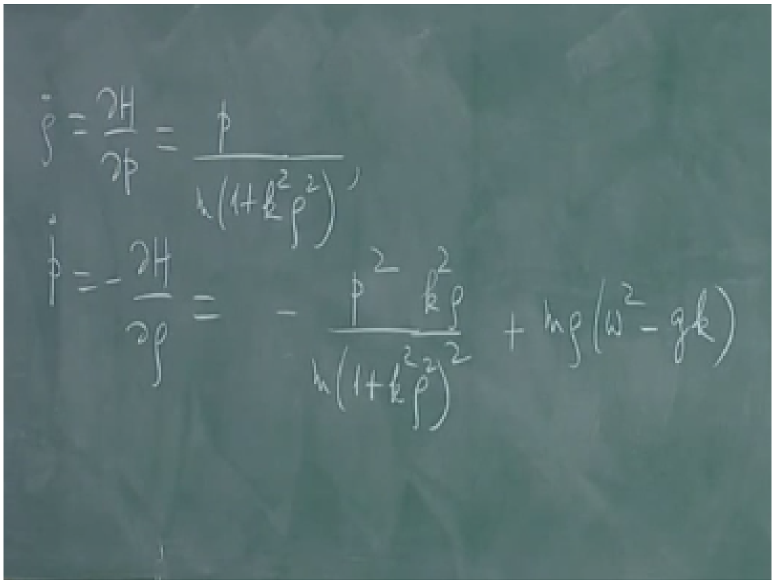
(Refer Slide Time: 52:02)



The conjugate momentum P is ∂ L / ∂ ρ dot and that if I differentiate here is m times 1 + K^2 ρ^2 ρ dot or ρ dot is P / m1 + K^2 whole^2 that is the conjugate momentum again it depends on the coordinate so what is the Hamiltonian this is = P ρ dot - L P ρ dot gives you P^2 / m times 1 + K^2 ρ^2 here is the first portion - L and therefore you got to subtract this - this term times the 2 of this so you have a P ^{2} over m ^{2} that gives it to M and therefore it is clear that you just get twice as before + 1/2 $m\rho^2$ ω ^{2} - thank you - this + mg /2 Kρ2 like so in this fashion.

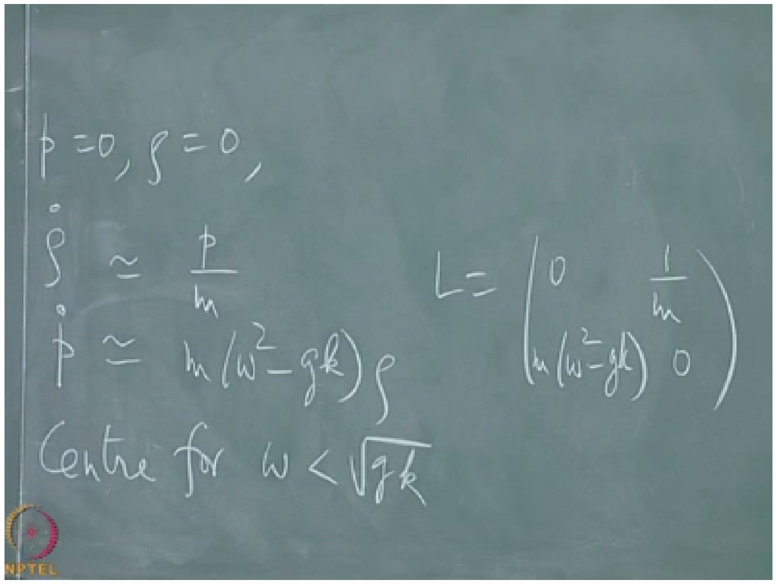
So we can simplify this a little bit take out the M ρ2 /2 and then write this as ω^2 - gK that is it and what are the equations of motion.

(Refer Slide Time: 53:49)



So again we have ρ dot is ∂ H / ∂ P let us not gonna give us anything new except this whole thing P over m into $1 + K^2 \rho^2$ but then P dot is - ∂ H / ∂ ρ and that gives you something non-trivial and this is = $- P^2 /m (1+ K^2 \rho^2)^2$ multiplied by the derivative of this quantity which is = $K^2 \rho$ this term if I differentiate this and then I differentiate that and that gives me - m $\rho\omega^2$ I need a - out here if I differentiate so this becomes a + what do you conclude what kind of critical points do you have and what is your conclusion do you have a bifurcation here at all what happens now clearly P = 0 ρ= 0 is a critical point.

(Refer Slide Time: 55:34)



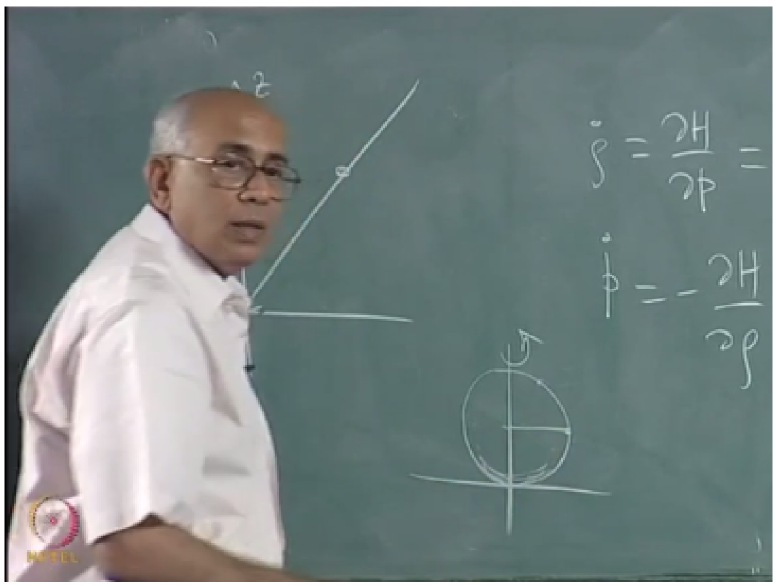
so the linear matrix about that point so near huh uh-huh so what happens well I mean it is clear this is going to this a linear term and it is going to change sign depending on ω is less than $\sqrt{}$of gK or greater than $\sqrt{}$of g K so the linear term the linear linearized problem is P dot = P / m to first order and repeat ρ dot sorry ρ dot is this and let me raise this problem write it properly ρ dot and P dot is = m times ω^2 - gk times ρ.

So what you conclude from this so L0 1 / m m times ω^2 - gK and a 0 here yes if ω^2 if ω is less than so Center for ω less than $\sqrt{gK}$ you could have guessed that by dimensional argument $\sqrt{gK}$ is the only quantity of dimensions frequency so other than ω itself so for ω less than $\sqrt{gK}$ you end up with a center which means you have stable oscillations what happens if ω is greater than gK the $\sqrt{}$og K becomes a saddle becomes unstable but you do not have an alternative after that and what does the system do goes off to infinity.

It goes off to infinity exactly this critical point has becomes unstable and that is it was because you had the curvature in the circle was appropriate that you ended up with another stable point so it deeply depends on the function that you had it depends on whatever was multiplying it depended on the shape of the wire the constraint machine which gave you which depended on the shape of the wire told you whether you had another solution or not yes well if $\omega = \sqrt{gK}$ this goes away the problem is not linearizable anymore it is intrinsically nonlinear as you can see.

So that is the marginal point at which the center is losing its stability at this point infinitesimally greater value of ω and you are off it goes off so you can only have stable oscillations about the center as long as ω is less than √gK I leave you to play with this and figure out what happens if instead of this kind of situation you had for example you took a stick and you put it on a stick just a rigid rod you put a bead on that and you rotated this about the vertical axis so this is a linear function so the circle is special of course there are other functions for which this would happen but the circle was the simplest case this and we did not look at what would happen in the hoop problem.

(Refer Slide Time: 59:19)



If you started with something not on the lower half but up here and I leave you to analyze that it is fairly straightforward the only thing is in the constraint equation you have to write z as R + the $\sqrt{R^2 - \rho^2}$ and go through the algebra once again to see what would happen to a particle sitting here when you start rotating but the result we got was completely physical which said that initially you have four small velocities small angular velocities you have a stable oscillations about this origin.

And once you exceed a critical point then this gets flung out and you have oscillations about a new point and the largest value that point can have is in fact when you're right here at this stage which happens when you have an unbounded angular speed and then it gets flung out as far as it can go here we also found out what was the nature of this bifurcation diagram we found out what kind of exponent you had me found that there was a kind of √behavior the function of √of ω - ω C was way the way the new function the new solution took off.

So it is a fairly complete analysis although we did not solve the equation of motion in general that is complicated because the motion in general could be very complicated depending on the initial conditions and you could translate it back to the original Cartesian coordinates but there is not much presentation doing that in this problem it is obviously of interest to stay with the cylindrical coordinates that we had yes pardon me yes you have yes you know we did not draw the phase diagram because we did not really write down we did not really draw the phase diagram at all because we did not write the solutions down we just looked at what happens in the critical points and so on.

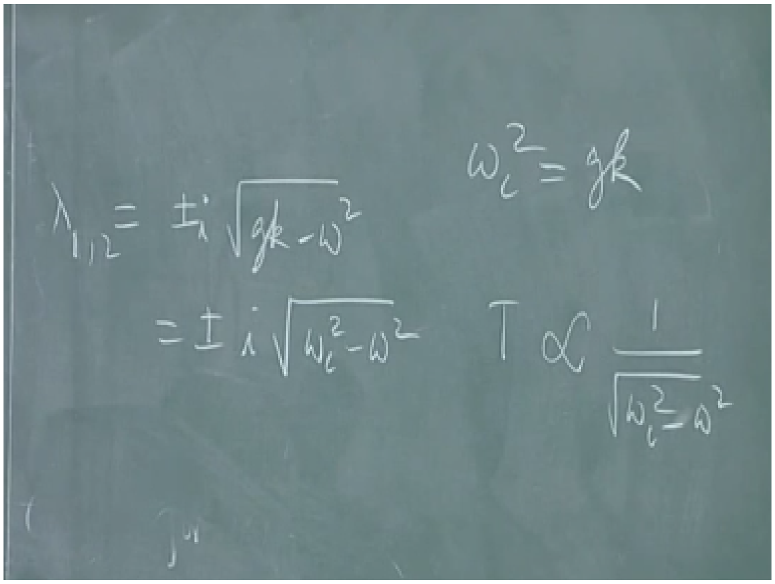
But you are right there is obviously a hyperbolic point at a certain stage and then there would be separate races coming out of it and so on so I leave this as an exercise to try and generate this but in practice although because the expression is fairly complicated it might be messy to do this but you can do this numerically can write these solutions down you can solve this yes it is clear that once the critical point becomes unstable.

And it becomes a saddle point then there is no question of oscillation so absolutely so things were diverged yes we did not look at the time period of small oscillations about this point we did not do that we did not write down what would happen but that is straight forward because for sufficiently close to the origin for small oscillations it was like a simple harmonic oscillator problem for which you can straightaway write down the solution you can write down the time period yes.

What did I word yeah presumably this would end up diverging yes because it becomes unstable beyond that point but one should check this out explicitly we did not do the large oscillations we kept things very close to the origin and that was it but now you make the amplitude larger and larger and then you begin to see what would happen in general your question is what happens to the actual time period of small oscillations itself in the linear problem yes indeed it diverges that you can see directly.

Because let us go back his question is what happens to the time period of oscillations as ω hits ω C in the previous problem the circular hoop if you recall in the parabolic case yes indeed it becomes infinite as you can see immediately because here is our problem here is our miss here is our linear matrix.

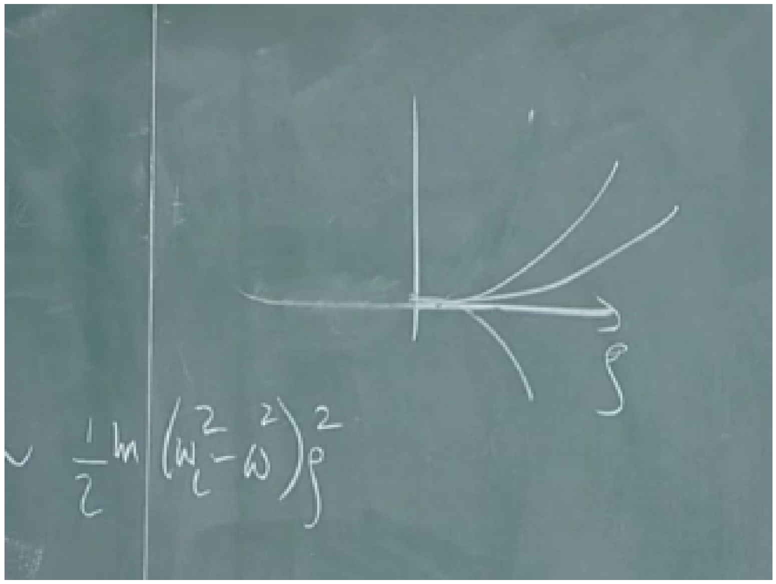
(Refer Slide Time: 01:03: 02)



So what are the eigen values λ 1 , 2 is = what are the eigen values of this matrix right yes okay now we are going to look at small values of ω to start with and therefore ω is less than the $\sqrt{}$ this in which case let us write it as I times $\sqrt{}$ gK - ω 2 and let us call ωC^2 = gK therefore this is = + or - I times $\sqrt{\omega C^2 - \omega^2}$ those are the eigen values which means that the small oscillations will have e to the power + or - these eigen values multiplied by T and the time period of oscillation T is therefore proportional to 1 /$\sqrt{}$ ω C 2 - ω^2 which diverges as ω hits ωC it is exactly his conjecture so it is clear that as you hit this critical point the time period diverges it is about to take off and beyond that it becomes unstable you could also have deduced this by looking at the shape of the potential.

What does this correspond to what kind of thing does this correspond to when I put this in here and you have an equation which says ρ dot is P / m and P dot approximately = and let us write this as ω C^2 and keep ω less than ω C^2 so - M ω C2 - ω ^{2} times ρ what kind of potential are we talking about it is like a simple harmonic oscillator with a potential this is an equation of motion so you have to integrate this.

(Refer Slide Time: 01:05:07)



So you get a potential effective potential we effective which is proportional to ½ m ω C^2 - ω ^{2} ρ^2 is not it if I integrate this and change the sign right this is like the force f of ρ so V is = integral F of ρdρ with a - since F is - dv / dρ this is certainly true what kind of shape is this as a function of ρ because we are looking at positive ρ this is a parabola upwards provided ω smaller than ω C but it is getting flatter and flatter as ω approaches ω C and therefore the potential is starting to look like this and once ω exceeds ω C the potential inwards and inverted parabola is unstable at the origin.

So this is exactly what happens the stiffness is going down as ω approaches ω C it is becoming like a spring with a smaller and smaller spring constant it gets more and more flabby and therefore the time period is longer and longer and it diverges at ω = ω C beyond that it is not going to sustain oscillations because it is not around the other way okay yeah but we have not really written the phase portrait down we have just done things near the origin but this is good enough it tells us all that we need to know okay.

Online Video Editing / Post Production

K. R. Mahendra Babu
Soju Francis
S. Pradeepa
S. Subash

Camera

Selvam
Robert Joseph
Karthikeyan
Ramkumar
Ramganesh
Sathiaraj

Studio Assistance

Krishnakumar
Linuselvan
Saranraj

Animations

Anushree Santhosh
Pradeep Valan. S. L

NPTEL Web & Faculty Assistance Team

Allen Jacob Dinesh
Bharathi Balaji
Deepa Venkatraman
Dianis Bertin
Gayathri
Gurumoorthi
Jason Prasad
Jayanthi
Kamala Ramakrishanan
Lakshmi Priya
Malarvizhi
Manikandasivam
Mohana Sundari
Muthu Kumaran
Naveen Kumar
Palani
Salomi
Senthil
Sridharan
Suriyakumari

Administrative Assistant

Janakiraman. K. S
Video Producers

K. R. Ravindranath
Kannan Krishnamurty

IIT Madras Production

Funded by
Department of Higher Education
Ministry of Human Resource Development
Government of India

www.nptel.ac.in