Indian Institute of Technology Madras
Present

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS
Lecture 14
Quiz 1
Lyapunov's direct method
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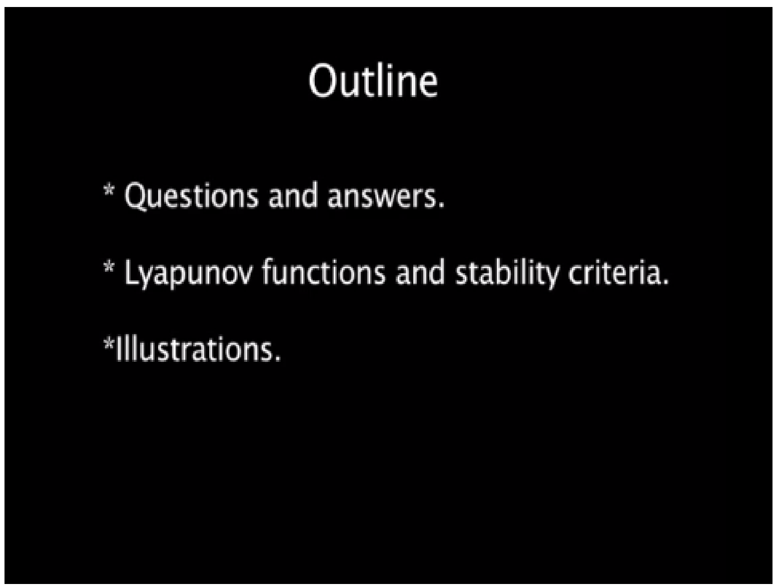

Yeah, so let us begin today with discussion of the quiz that we had last week about through the answers quickly, so you can check it in your answer books. We start with question one which was just true or false. And the first statement was a conservative dynamical system given by an equation of motion of the form x. = F(x).

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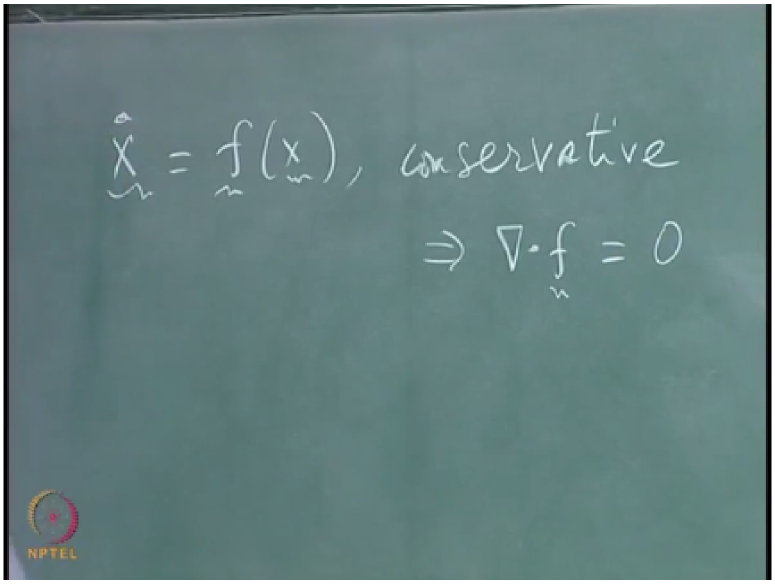

And I told it is a conservative cannot have any attractors. The answer is that it is true, because for conservative system this implies that ∇.f=0 and that is so, then we know that the volume element in phase space is preserved under the flow. So since we have defined a conservative system as one in which the volume elements do not change under the flow, the conservative dynamical system cannot have any attractors.

Because if you have an attractor no matter what the nature of the attractor either a point attractor like a critical point or a limit cycle, a stable limit cycle or most complicated attractors. The fact is that all the points in the basin of attraction of this attractor would eventually asymptotically fall into the attractor. And therefore, this definite reduction in phase space volume. And since conservative system, the flow does not allow for any change in the phase space volume, it cannot have any attractors.

Hamiltonian systems of course are special cases of conservative systems and they would not have any attractors either. The next question asked that how many oscillators the only system holds time period of oscillation is independent of the amplitude of oscillation, and this is false. We have also seen a number of oscillators, isochronous oscillators whose times periods are actually independent of the amplitude of oscillation.

These are called isochronous oscillators, because it means that the amplitude, the time period of oscillation is independent of the energy of the oscillator. Under special circumstances this can happen even in a non-linear problem. Yes, I stated that a basic characteristic of the linear harmonic oscillator is that is isochronous, that it does not have a time period which is dependent on the amplitude or the energy.

But the converse is not necessarily true, that if you have isochronous city it does not necessarily imply that you have a harmonic oscillator, you could have non-linear oscillators which would do the same thing. Here is an example of an oscillator whose time period would actually be independent of the energy of the oscillator.

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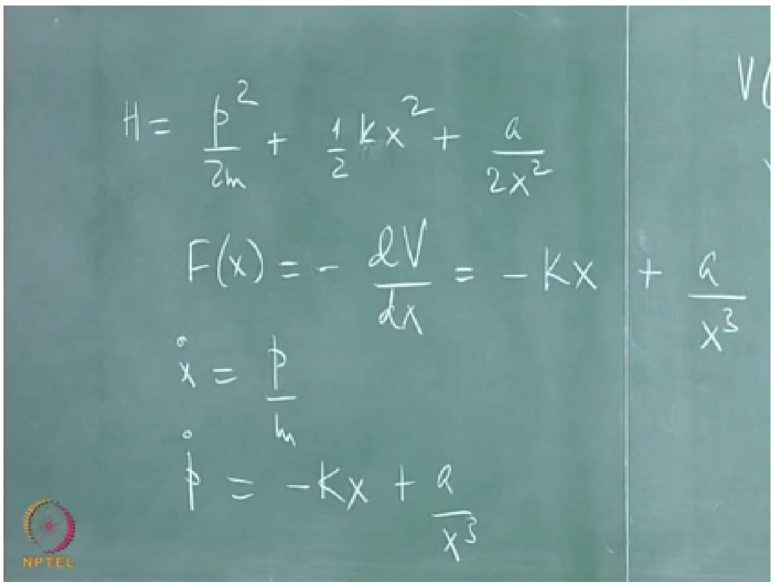

Consider for instance motion in one dimension with Hamiltonian given by P^2/2m+1/2m let us call it kx^2+ a force which goes like some constant a/$2x^2$ potential of this kind. Now what would the force on this particle be, it would minus the derivative of the potential, so the force on the particle is –dv/dx = -kx that is the harmonic oscillator part, and then the derivative of this potential which would go like +a/x^3. And what does this do as x goes to 0, what does the potential look like in this case.

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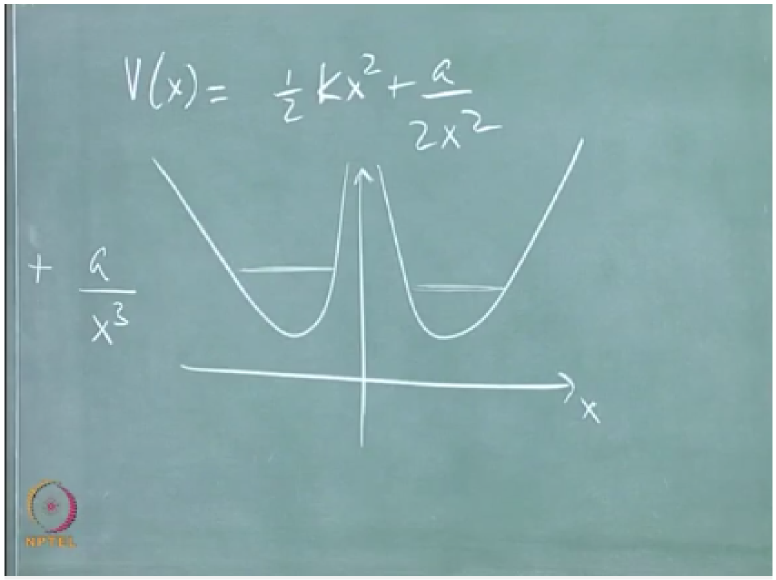

What is the shape of the potential, so let us plot V(x)=1/2 kx^2+a/$2x^2$ where k and a are positive constants. And then if I plot this potential here is x versus V(x) it clearly goes to ∞ as x goes to 0 on both sides. It also goes to ∞ as exposed to + or - ∞, then of course it is clear that in between it comes down an does this and symmetrically so on this side as well. and the particle can oscillate either in this well here or in this well here quite symmetrically.

For the same energy to neither have a center of oscillation on the left hand side or in the right hand side. And if you come to the time period of oscillation in this potential it turns out that it is actually independent of where you are of what the energy is. I leave that you as an exercise to check out that this oscillator is isochronous and it is not a linear harmonic oscillator, because the force is not just proportional to the negative displacement, but also has an extra term here.

When I say linear harmonic or harmonic, I mean that the potential is a quadratic function of the displacement and that is all that is needed. Anything beyond that is non-linear, and the reason it is non-linear it is because the equation of motion becomes non-linear. We call that the equation of motion here is x. = Δh/Δp, so it is P/m, but P. = - Δh/Δx and that is –kx if you restricted yourself to just this. But if you also included the non-linear term you have a a/x^3.

So this is the reason why we call these oscillators non-linear, because the equation of motion is no longer linear it has non-linear terms in the coordinates. Yeah, now the reason I use, I kept saying linear harmonic oscillators which we are used to calling as a simple harmonic oscillator, because we also looked at the word linear there was a bit of a misnomer. I should have said one dimensional harmonic oscillator.

Because the oscillation was in one direction, the x direction alone, we also looked at combinations of oscillators, oscillator on a plane, oscillator in three dimensions and so on, they are not linear harmonic oscillators, they are just harmonic oscillators. So I use the word linear in the other sense namely a longer line, motion of longer line a given line. So perhaps we should not do that, so just call it a one dimensional harmonic oscillator in that case.

But every harmonic oscillator has a linear equation of motion, linear in the coordinates and momenta. The many want to use non-linearity in the potential, you at once have these extra terms and it is not at all guaranteed that the oscillations would have a time period independent of the energy. But in this particular case it turns out that they do and that is an interesting exercise to look at.

There are profound implications to this, this model here is not polled out of the hat, it has further implications and it is a specific, it is a member of a specific family of potentials and model which has many, many other interesting properties. So even though the potential does not look parabolic at all, the oscillations in this are independent of the energy of the oscillator, this could be very well happen.

We saw however that if you took a potential of the form a power of x here in one dimension just the power of x no extra terms when the only oscillator which had a time period independent of the amplitude was in fact the harmonic oscillator, the quadratic power here, nothing else. But that was restricted to the class of potentials which just had a single power of x and nothing more than that.

More complicated functions could do this, there are in fact an infinite number of potentials which h would lead to oscillations independent of the amplitude of oscillation or the energy in general. Okay, the next question, in what sense. The question is, is there anything which characterizes the potentials. Not so simple to classify, not so simple it is possible to a certain extent, but not so simple.

These models the one with the x^2 and the $1/x^2$ they form, they are part of a much bigger family of potentials where this property would be satisfied not just in one dimension, but in high dimensions as well or not just with one oscillator, but many oscillators on a line. This is not a sufficiency condition, this is not the only one that does it in the sense that, it is not a unique property to this particular oscillator.

There are other potentials, many other potentials which would do this and to some extent they can be classified do we not get into that right now okay. The third question straightforward it said consider a canonical transformation of an autonomous Hamiltonian system. Under such a transformation the form of Hamiltonians equation is preserved although the functional form of the Hamiltonian in the new variables need not remain the same as the original one.

That in fact is what happens in the general canonical transformation, so the statement is quite true. The next one says, every dynamical system given by an equation of the form x.=F(x) can be transformed into a gradient system by a suitable choice of dynamical variables.

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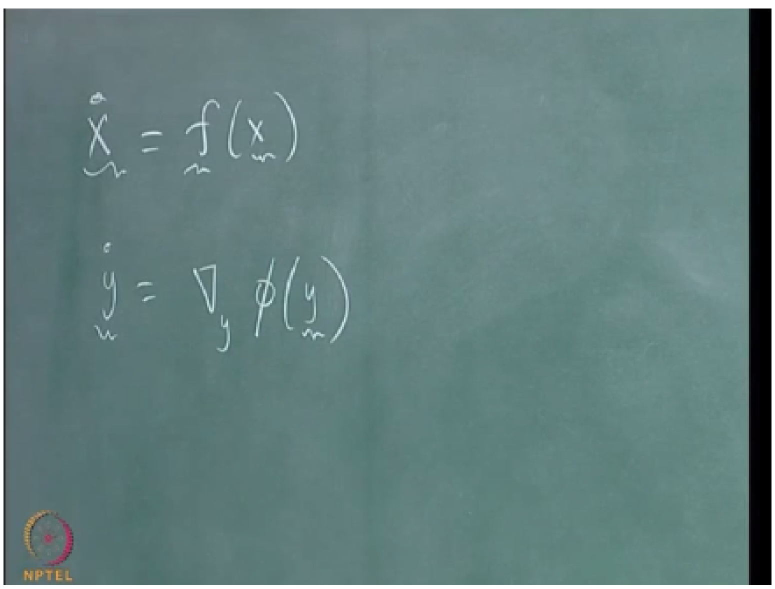

So the proposition is that this could always be transformed by a suitable choice of variables to look like y.= the gradient with respect to y of ϕ(y), no such guarantee at all, because this would imply that every vector field could be writable as the gradient of some kind of scalar, this is not certainly not true at all. So it is not necessary that this should happen, this is not true that every dynamical system can be transformed to a gradient system by suitable choice of variables, change of variables.

We go onto the next, homo clinic orbits can occur in both conservative and dissipative system, and the answer is yes, indeed they can, because all that of homo clinic orbit does is to start at the saddle point, the part of it is unstable manifold goes out and comes back and forms a loop eventually. And the part that comes back is the part of the stable manifold of the saddle point, and there is no restriction on this, it could happen in a dissipative system, it could happen in the conservative system as well.

Pardon me, it does not matter whether the volume elements change or not, because you only talk about a single trajectory here and there is no reason at all why this cannot happen, independent of what else happens anywhere else okay. Linear stability analysis need not reveal the correct nature of the flow in the vicinity of a critical point that has a center manifold, this is the whole point about center manifolds and it is certainly a true statement.

It could, but on the other hand it might let you down. And therefore, you have to go beyond linear stability analysis as we saw with specific examples once you have a center manifold. The level on criterion for integral ability is applicable to any even dimensional dynamical system. And that is false, because a levelonel criterion is specific to Hamiltonian systems which form a special class of dynamical systems of even dimensionality.

So there are many, many other systems innumerable systems which have nothing to do with Hamiltonian systems, and there is no question of anything like this criterion on those cases. The next statement said a bifurcation occurs at some value of a parameter in a dynamical system, if the nature of the flow changes qualitatively as the parameter cross that value and yes indeed that is the very definition that we have used for elementary bifurcations the critical points corresponding to a undammed simple pendulum can only centers or saddle points.

That is also true because this is a Hamiltonian system undammed simple pendulum therefore there is no attracter in the problem it is a Hamiltonian system and the only critical points it could have this simple system are centers are saddle points and we saw that you have centers and saddle points alienating corresponding to the minima and maxima of the cosine potential consider the 2 dimensional dynamical system given by the flowing equations.

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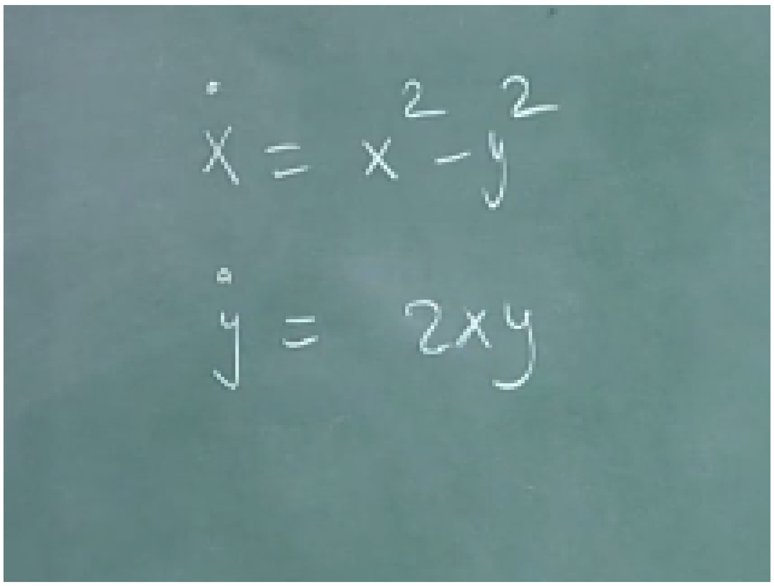

X^2 x dot is $x^2 - y^2$ and y dot = 2xy and the proposition was the critical point at the origin is a saddle point and that is falls because the saddle point a saddle point as a winding number the vector field corresponding to a saddle point as a winding number of -1 but this is a dipole field and the winding number here is 2 and this is topologically quite distending from a saddle point and it is higher order critical point because this is not even linearizable in the vicinity of the origin.

There are no linear teams here at all it is intrinsically non linear and it is happened by the equalizations of 2 singularities 2 simple critical points yeah yes it is not no, no it is a critical point I mean that is it so there is no question of any unfolding or anything like that this is just a critical point as it stands yes it could like a saddle note for example it could yes but I would not call.

No I would not call it the saddle point at all because this is a the statement being made as to do with the singularity of this vector field at the origin so it is a local statement about the singularly at the origin and that is in this case undoubtedly a dipole singularity and therefore as a winding number 2 and it is not a saddle point it cannot be deformed smoothly into a saddle point because the winding number is a topological invariant and it says that no matter how you transform coordinates and shift and bend and so one you cannot change the nurture of the you cannot change the winding number so saddle point remains the saddle.

On the other hand that was not true for nodes you saw for instance that something that looks like a source radial field could become a tangential field so something that looks like a node could transform into spiral point and so on these could be done by smooth distortions but certainly you cannot take a saddle point and convert it and distort it into dipole field or anything like that a hope bifurcation can only occur in a dissipater system.

And it is true because a hope bifurcation is one where a limits cycle is involved and it is a bifurcation where a stable critical point bifurcates into a stable limit cycle and an unstable critical point or if it is a sub critical bifurcation an unstable critical point bifurcates into a unstable limit cycle and a stable critical point neither case attractors are involved and therefore this sort of thing cannot happen in a Hamiltonian system or more generally unit is conservative system.

But it can certainly and does frequently happen dissipate systems finally if the Poisson bracket of A with B vanishes and that of B with C vanishes then the Poisson bracket A with C necessarily vanishes that is falls because all you can say from carboy identity is.

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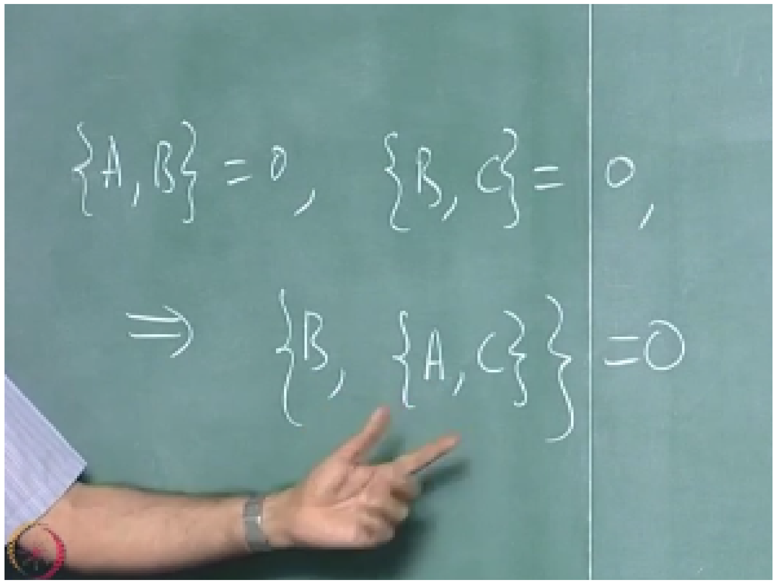

If A with b is 0 and B with C is 0 this would imply then and your asked to find out if the Poisson bracket A with C vanishes what you can say a set is a the Poisson bracket of this 0 by the carboy identity because the other 2 terms drop out and all you can say is that this quantity need not be 0 some function who is Poisson bracket will be happens to be 0 and that is about it you cannot say anything more.

Whoever if you have 1^0 of freedom systems and they are Hamiltonian systems for example a single 1^0 of freedom Hamiltonian systems and then you know that the Hamiltonian in an autonomous case is the only functionally independent constant of the motion in the problem than anything else which you find out which is also a constant of the motion would necessarily have to be a function of the Hamiltonian.

Then of course if A is the Hamiltonian B and C functions of the Hamiltonian then of course all the Poisson brackets of these quantities with the Hamiltonian vanish but that is not true in general in general this is all that you can assert we had an implication to this the implication was that Hamiltonian system.

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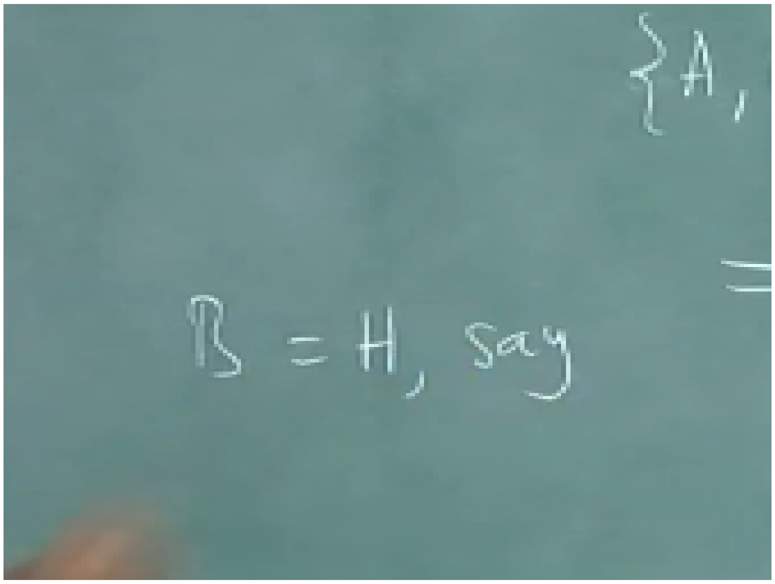

If you set B = to the Hamiltonian for instances then A with B = 0 implies that A is constant of the motion similarly C with the Hamiltonian = 0 implies that C is a constant of the motion and then this statement implies that the combinative the Poisson commutator of A with C the Poisson bracket of A with C is also a constant of the motion so what it implies is that an Hamiltonian system if you find 2 constants of the motion there Poisson bracket if it is not trivial is also a constant of the motion.

So it as a practical use in that instance the next question was multiple choose using we have considered a general Hamiltonian system in the first statement was the Hamiltonian is always a sum of a kinetic energy term and a potential energy term which depends only the generalized coordinates this need not be so at all because pointed out early on the.

That all Hamiltonian system needs is that you have even dimensional phase space with the certain structure the Poisson brackets structure canonical Poisson bracket structure and a Hamiltonian function specified to you which then determines the equations of the motion of all the variables there is no restriction that the Hamiltonian should be of the form of the kinetic energy verses the potential energy.

That is only true for simple mechanical systems not true in general in fact even the statement I made about Hamiltonian system namely that it should even dimensional and it should have this canonical Poisson bracket structure could be generalized there are more general forms of writing Hamiltonian system where you do not even need to have an even dimensional space where the meaning of the Poisson bracket itself could be generalized further. That is the mathematical detail we have not got into.

But certainly does not have to be some of a potential in a kinetic energy saddle note bifurcations cannot occur in this system they certainly can saddle note bifurcation we saw with an example of a potential itself that a saddle note bifurcation certainly can occur in a simple potential problem Hamiltonian problem the dynamical symmetry group of transformations need not necessarily be identical to the group of the canonical transformations.

And that is certainly true because the dynamical symmetry group of a Hamiltonian system could be much smaller than the group of canonical transformations if you recall in n degrees of freedom the group of canonical transformations was the simplex group to n over the real's where as the dynamical symmetry group would depend on whether the Hamiltonian had some special symmetries are not and most Hamiltonian do not and when they do they have much smaller symmetry groups.

The example we took was the 2 dimensional harmonic oscillators which had a symmetry group which whose canonical transformations was the simple group 4 Sp 4 on the real's on the other hand the symmetry group of the Hamiltonian itself was the group Hamiltonian itself was the group of the rotation in 4 dimensions so 4 and the intersections of these 2 was much smaller group which was isomorphic to Su 2 so we saw that this need not be true at all.

Action angle variables necessarily for exits this system again no because the Hamiltonian need not be integrally completely at all in fact you could have a few action variables less than n number and that is sufficient to intergraded the system complete so they need not exist at all in this scene remember that once an action angle pair exits then the angle variable does not appear in the Hamiltonian and it becomes a cyclic coordinate.

And this si not always possible in general if it is fully intertribal then the statement is on the suitable conditions you have an action angle transformation which will then lead you to set of variables in which all the angle variables disappear from the Hamiltonian and the Hamiltonian is a function of the action variables alone which are then constants of the motion where that need not be true in general the next statement to attain to a general actiniums dynamical system described by a set of n coupled non linear first order ordinary deferential equations like x0 = f(x) the phase space can be either even dimensional or odd dimensional and that is certainly true automatically.

Is always at least one attracter in the system not necessary for instance if it is conservative system they need not be attractors at all the dynamics is necessarily measured preserving not all it could be a dissipations system so it could very well have measure with shrinks they must exist at least n functionally independent constants of the motion that that do not have any time explicit time dependency know that happens you cannot have nay motion at all in the n dimensional space.

So if you have n constant of the motion which are independent and do not depend on time at all then the system cannot even be there integral there is no motion it is absolutely nothing to do once you specified the initial conditions the system just remains there it cannot move it out after that so that is not valid here we will go on to question 2 it was fairly straight forward and the system was specified.

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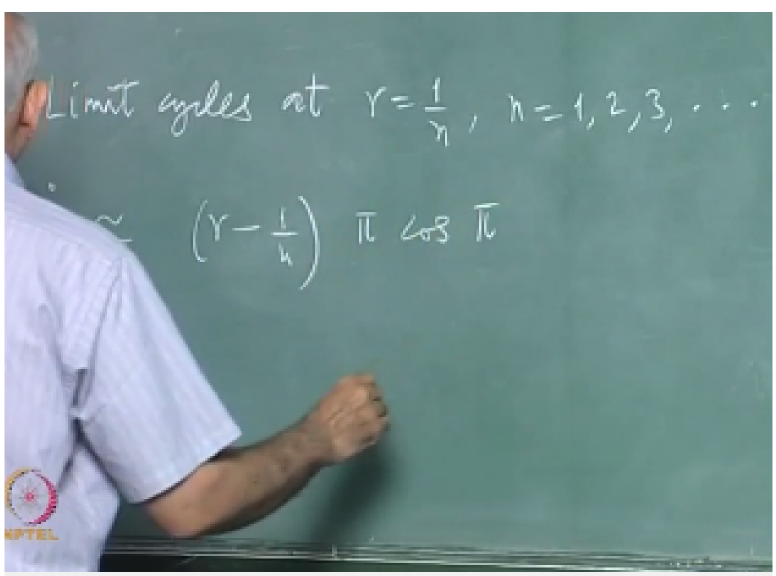

Polar coordinates by sin π/ r and θ dot = r in plane polar coordinates you are asked to find in this case a limit cycles of the systems if any as well as the stability and cumulatively clear from here that limits cycles at r = 1/n where n = 1, 2, 3 and then these becomes sin π and vanishes so R does not change on those points and it is circular limit cycles so you get a family of concerned limits cycle of radials 1 ½ or 3rd so on so forth.

Now if I took the system near of these limits cycles in the resonantly of these one of these limits cycles then r dot is approximately = sin πn which is 0 + r – 1/n the derivative of this at r = 1/n the derivative of this s of course π times cos π/ r but r is 1/n so this becomes cos πn times the derivative of this which is $-1/r^2$ which becomes $–n^2$ θ

But r is 1 / n so this becomes cos π n times the derivative of this which is -1 / r2 which becomes –n2 + higher orders, so this becomes to equal to –π n2 – 1 n r – 1/ n + higher order terms and if I n = 1 this gives your – sin along with this and then it says r . is proportional to r – 1 with the positive coefficient here. Therefore the sufficiently large values of r is flows out because r. is positive it goes away and if r < 1 it floes away inwards.

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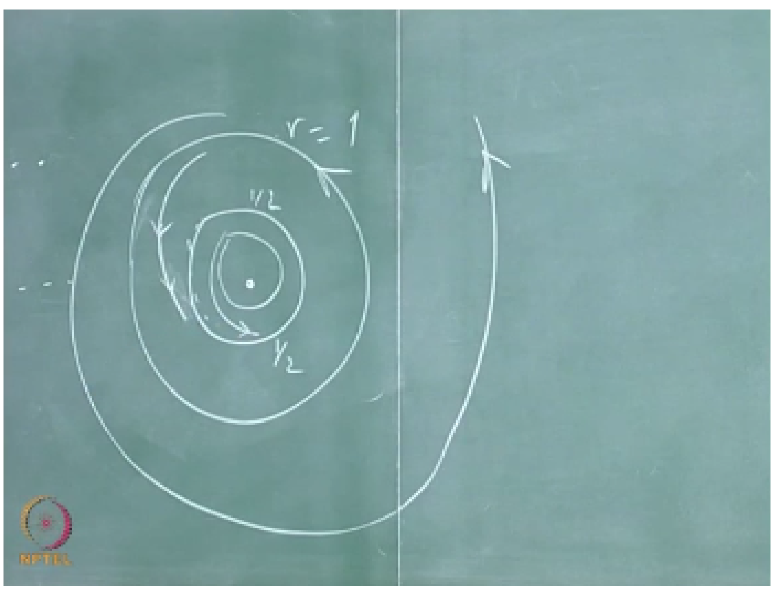

So it is clear immediately that r = 1 this particular limit cycle this limit cycle here whatever was inside here flows out whatever starts here flows out because θ. Is r if r > 1 θ. Is positive number and it increases and therefore you expect it flows up in this fashion, and since θ. Is positive the flow is in the counter clock wise direction on this limit cycle. And anything which starts in is going to flow away from it towards the next limit cycle which is the limit cycle at r = 1/2.

So things flow in towards this and similarly between 1/3 and ½ things flows out towards the ½ so you have an infinite number of limit cycles nested within each other the outer most one r = 1 is unstable the next one is stable to one inside is unstable and so on. So that is the full space portrait, what happens at r = 0 this case, what can you say about r = n, well this function does not have a limit is r goes to 0 here that is quite clear and what you have is an accumulation of limit cycles of alternative stability.

So it is a crazy singular point but it just an accumulation point for limit cycle that is what all you can say that is r goes to 0. So the flow gets more and more intricate as you get inwards towards this, this not a simple critical point by any means, it is like asking as n tends to infinities that d1 are rod when it right so it is a same problem as before. The limit does not exist so the whole point is that the limit has r goes to 0 of sin π / r does not exist there is no definite limit.

In what sense well it is clear that everything we write down here mathematically is modeling some physical system to some degree of accrues. So the question of you know whether it actually is describe does it actually describe a physical system write down to r = 0 is a mode point that sense very unlikely to happen. Let us go on to the next topic and I would like to introduce to you the idea of Lyapanov's direct method.

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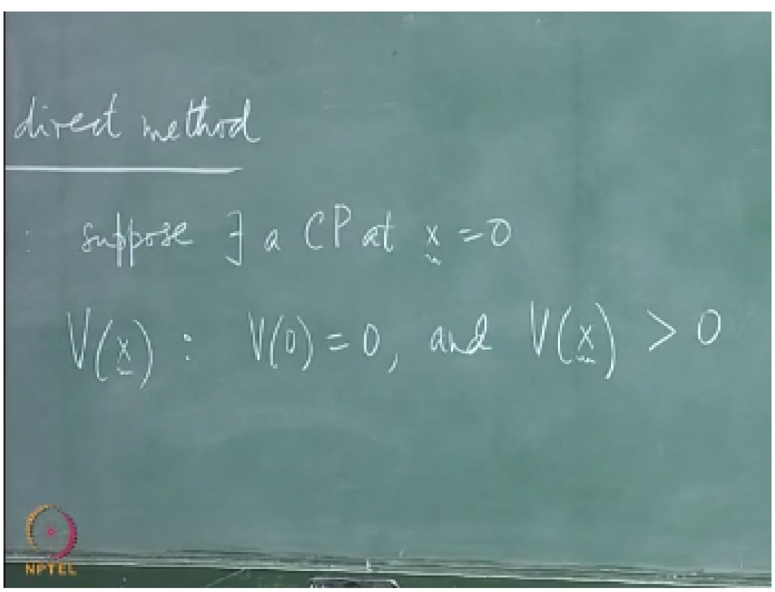

Which I briefly mentioned little earlier and what I intent to do is to work out a little bit of this in terms of examples and maybe give some problems, so that you could work out things and see how this method works. Now this method is useful it is method for analyzing the stability of a critical point and it defers from the linearization method which we learnt about so far, where if you recall we took a particular critical point we liniearized a system about this critical point and then if the Eigen values of the accobean matrix at this point at knows 0 real part and the point was hyperbolic then we identified the stability of otherwise of this critical point based on what the real parts of this Eigen value statement.

If all the real parts if at least one real part is positive you ended up with something there was unstable some direction based things would flows away but if all of them are negative the things flowed in asymptotically you had asymptotic stability. We also saw that if you have a center magnified if there are Eigen values whose real parts a 0 then either you have a center or you have more complicated behavior but the stability is not uniquely decided by linearization about that point.

In those cases Lyapanov's direct method helps you to do this there is a method works as follows, suppose for this dynamical system x. = f (x) suppose you have a critical point at x = 0, so let us consider the origin to be critical point without lots of generality and see what happens in the neighborhood of the origin. So here is the origin and there is some neighborhood of the origin in which if I can find the function let us call it v (x) it is call the Lyapanov function with the following properties.

V (x) V (0) = 0, so it vanish this is the origin at the critical point and v(x) > 0 at all other points in this neighborhood I then say v(x) is positive definite function so if this is true I say V is positive definite, on the other hand if it is also possibly equal to 0 at one or more points in the neighborhood other than the origin then I say it is positive semi definite. And the same similar statement is true for negative definite and negative semi definite if it is < 0 everywhere then it is negative definite less than or equal to 0 it is negative semi definite.

Now we has nothing do with the dynamical system it is just a Lyapanov of function and auxiliary function which is am going to try to find then the statements that has follows Lyapanov stability theorem, so there are many of them but in the simplest form the stability statements are as follows.

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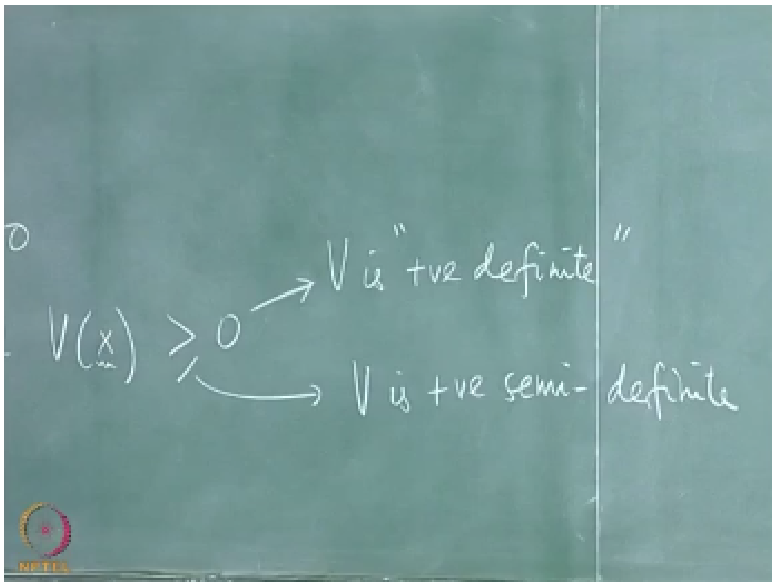

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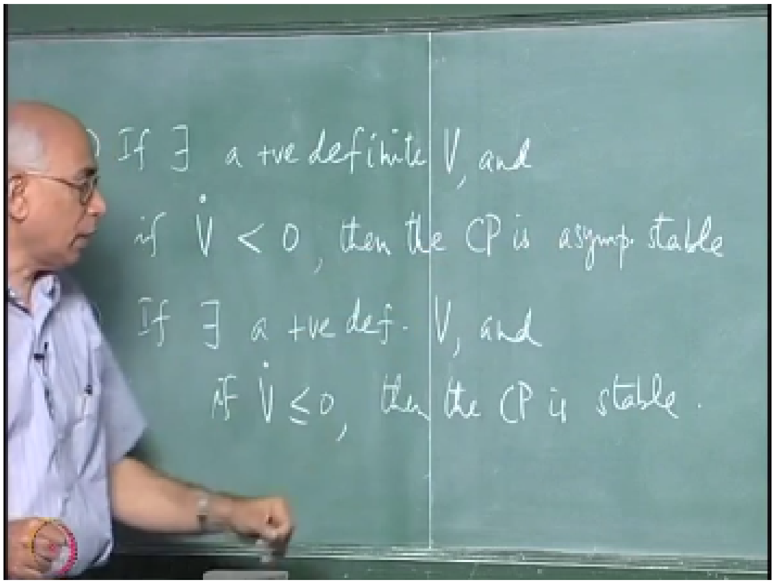

Statements are in brief one if where exist a positive definite V, and if V. < 0 in the neighborhood in which V is positive definite by dot I mean the time derivative of this V then the critical point is asymptotically stable to a little less stringently if there is exist a positive definite V and if V. is less than or equal to 0 could vanish at some points in the neighborhood then the critical point is stable recall again that a center was stable but not asymptotically stable a spiral point in asymptotically stable spiral point definitely things fell in to this spiral point but need not be stable.

So these two do not exclude each other and some sense they are independent concepts, and finally yes not necessarily we will see examples, so things could be stable and asymptotically stable but they could be asymptotically stable without being stable so the statement is caught to do with finding specific Lyapanov functions. Of course as you can see if I can find this then definitely I assert that the critical point is asymptotically stable.

But if I can only prove this and there are point where it vanishes and they cannot prove that it is actually non 0 with at every point that v. is < or at some points equal to 0 then all you can say is at the critical point is stable. We going to see this with examples right away what happens and free you can also have a statement of instability.

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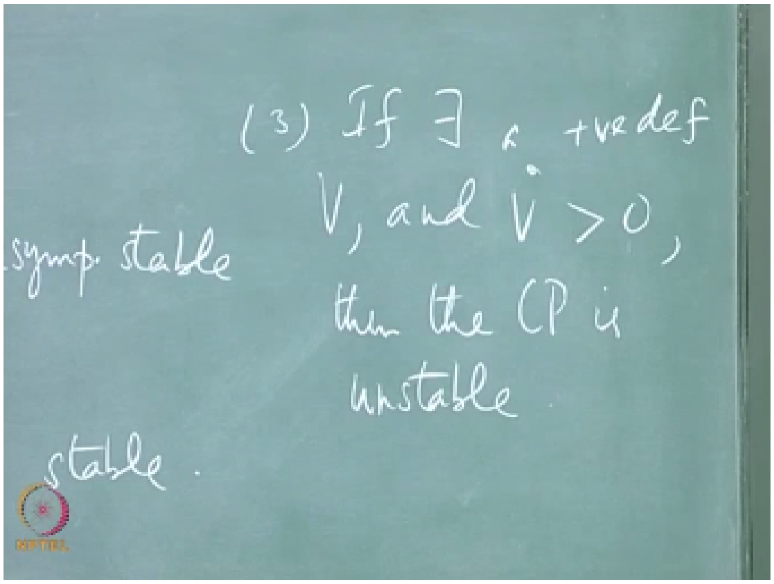

If there is exist once again positive definite V and v. > 0 then the CP is unstable when a refinements of the statements are possible but I am just giving the simplest version here I am going to look at examples. Of course you can turn this around and instead of V you consider the –V then of course if v is positive definite and v. > 0 it would translate in to saying if V is negative definite and V. < 0.

So instead of V you could always chose –V as your Lyapanov function and then whatever you say about positive definiteness becomes a statement about negative definiteness. Now let us look at a examples and see how to apply this and I am going to go through a series of examples but you will see how powerful this theorem is but we need to know what V. has to do with things and the reason is the explanation is very simple if I consider v as a function of x and I consider dv/dt.

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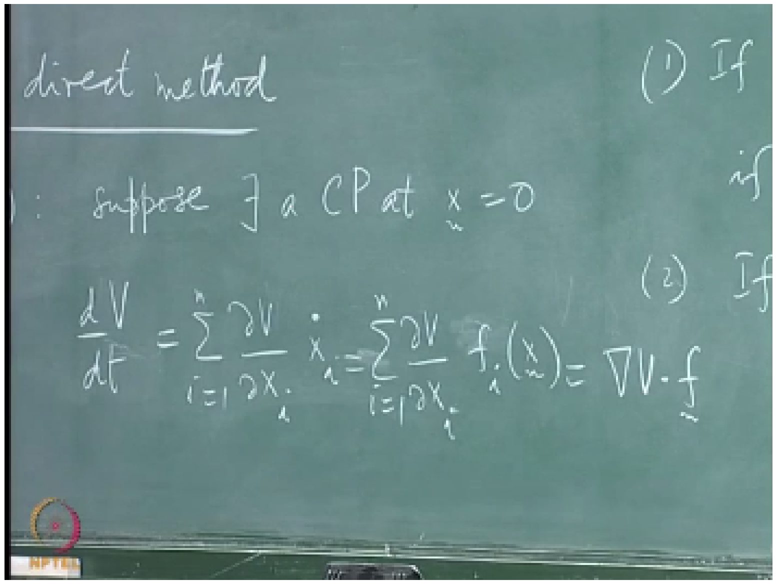

Which is V. what is this equal to? Well it is the function of x and therefore this is equal to δ v / δ xi. But from the flow equations on the solution trajectories this is also equal to δv / δxi fi of x the ith component of the vector field f, and this is summed over I the summation of repeated and this is imply I have not written it down but this is I = 1 to n and this is of course = ∇v . f.

Now what is the direction of gradient of v, v is the function and what direction is gradient of v? It is normal to ht level surfaces of v and the flow specifies the direction and which the trajectory moves in phase space. So you can see that this quantity here is telling you something about the relative directions of the gradient of v and the direction of the flow. So it is like having a level surface and finding out of the flow is going inwards in to the surface art all time and which case it has goes and hit some point or it is flowing out. So in very erotic terms that is the way in which the stability theorems emerge from consideration of dv/dt but let us look at an example right way.

So let us look at this simple 2 dimensional system and it is not restricted to 2 dimensional at all so this is true and n dimensional system and that is what make it interesting but let us look at simple 2 dimensional example.

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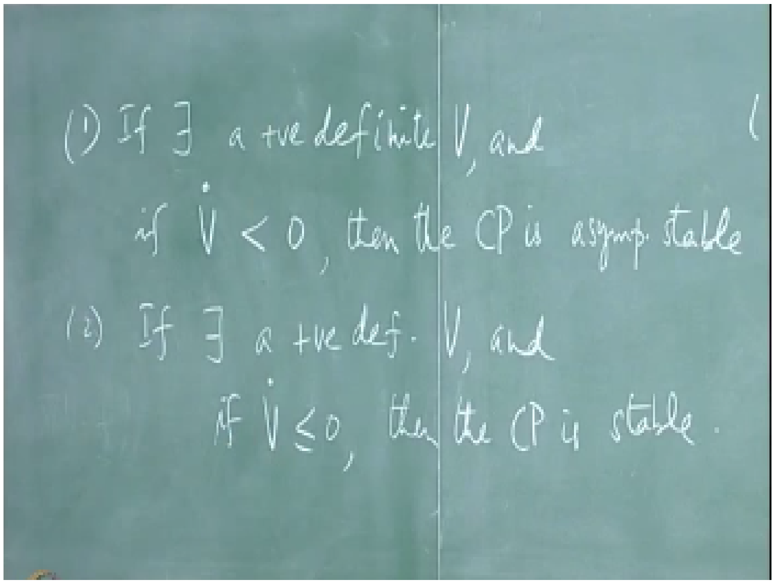

So the 1st example I am going to look at is x. = y, y. = -x which is of course the harmonic oscillator all over again in suitable units and now what you can tell me about this, what should I choose to be the Lyapunov function? Here is the way there are no simple guidelines available. What would one choose as the Lyapunov function in this insistence. Well a good choice will be we know already that this is going to be centre of origin.

And we know that it is stable but what would you choose as a Lyapunov function? I would like to choose something that is positive definite something that is vanishes at origin and in the neighborhood of the origin does not go to 0. Well I would like to choose something that is got a definite sign, so it must be at least quadratic pardon me, I could chose $x^2 + y^2$ so I suddenly choose that, v (x,y) = ½ $x^2 + y^2$, which you recognize as energy of the oscillator and suitable units as set the mass and the frequency =1.

The this function is positive definite in a neighborhood of the origin in the sense we have defined it today, it vanishes it to the origin and it is not 0 anywhere else in the neighborhood of the origin. So what is the gradient of the v it is component suggest x and y. what do you get if you combined it with this?

So what you get from gradient of v.f this is the vector field f so it says take this component multiply with this and add the 2, you get 0. Therefore we go back here and ask which of these apply? We have a positive definite p and we have v. < 0 strictly but = 0. So certainly this criterion applies and with the guarantee that this critical point is stable. Now of course if you could find some other v some other functions all together.

Where this was valid then the critical point will also have been proved as stable and we know that is not the case in the harmonic oscillator. So it implies that you cannot find a positive definite function v such that v.u because if you take then it will contradict what we already know about the harmonic oscillator. No it does not soya that at all why should it say that? If all you can prove is that v.≤ 0 then all you have established is that the critical point is stable okay.

Now it is quite yes, no why would you say that? Okay now I see the confusion no, if you can show that v. is strictly < 0 not 0 then you have shown that it is stable and in such cases, yes indeed, you prove stability but you proved a much stronger statement as well. But you do not go the other way as in the example. All I have succeeded in showing with this Lyapunov function is that v. is 0.

So it hits this case here, I am unable to show that it is <0, so the statement I am making is that I have been able to find a Lyapunov function which was positive definite and for which v. strictly negative then I would have lead to a conclusion that this critical point is infact as stable. In addition to being stable no it need not be but the Lyapunov function criterion is telling you, see this is the question of the choice of the Lyapunov function.

The same function does not satisfy this as well as that is quite clear, so let us go with this again. I guess and pulled out of the top of my hat a certain Lyapunov function and for this Lyapunov function I show this is the positive definite that is trivial and I see that v. is strictly 0 and then I looked down here and ask which of these applies and this is the case which applies. I conclude that this critical pint is stable.

Now if I did not anything more about this system this is all that I could conclude about and then I might wonder perhaps this critical point is not only stable, I would like to examine if so I need to find another the way of Lyapunov function some other cleverer choice of function where I could actually establish this and I am unable to do so. So the point of Lyapunov functions as to do with the clever choice of a Lyapunov function.

And the remarkable statement is if you can find even one Lyapunov function which satisfies the conditions of the theorem, then you can conclude whatever theorem states. But that may not be the best this is certainly true but this is the statement about stability here. So we have seen stability for this particular problem but we also know this problem can be explicitly and it is a Hamiltonian.

So the conclusion would be that you cannot find such a v no matter how hard you will try but you have to understand that this not that you can do this in all cases, you are trying to find out if in the absence of any information some statement can be made about stability and that is what all the Lyapunov function does. So let us see a few more examples and we will come back and answer some of these questions.

Yes you will not be able to find success Lyapunov function which would then satisfy this criterion as well, yes indeed just as in this case, we are not able to find out the Lyapunov function I am just ascertaining this so, that you cannot find Lyapunov function. so let us look at the next instances we added something to the oscillator and we did the following. We put –x times if you recall $x^2 - y^2$ but let us be general and put some π (x,y) – φ(x,y).

Where φ is a continuous function it has a derivatives and so on, what can one say now? I still choose this v and then what do I get, I get the gradient of v = x, y and therefore v. is the gradient of the v. with this and what does that give you. So these two terms cancel but I get a $- x^2 \varphi$ and out here I get $- y^2 - r^2 \varphi$ (x,y). Now the Lyapunov function I have chosen is certainly is just the energy of the oscillator it is a positive definite function vanishes at a origin non 0 everywhere outside the origin.

And I happen to choose this function and decide and see that v. in these cases $- r^2 \varphi$, now what one can say? Suppose φ is positive definitely then what would you say, if this is > 0 $\varphi > 0$ in the neighborhood of the origin is positive definite then v. is negative definitely and therefore I would say the critical point is stable. On the other hand if I know that φ is < 0 if that is the way this function is then v. becomes positive definite v is positive definite and I can certainly assert that the critical point is unstable.

So without further analysis depending on what this function does and the fact that I know Lyapunov function exist I am able to make the statement about the stability of the critical point. So this is the power of Lyapunov method it is called the Lyapunov 2^{nd} method or direct method because it depends on your clever choice of these functions. Yes the question is can I choose let us do the next example and see if you can choose $x^2 + y^2$ as a universal Lyapunov function see what happens.

So the other problem I do not want to erase this, the other problem we looked at was the simple pendulum which to had linear harmonic oscillation for the separation of the small amplitudes, so let us look at that example and see what happens.

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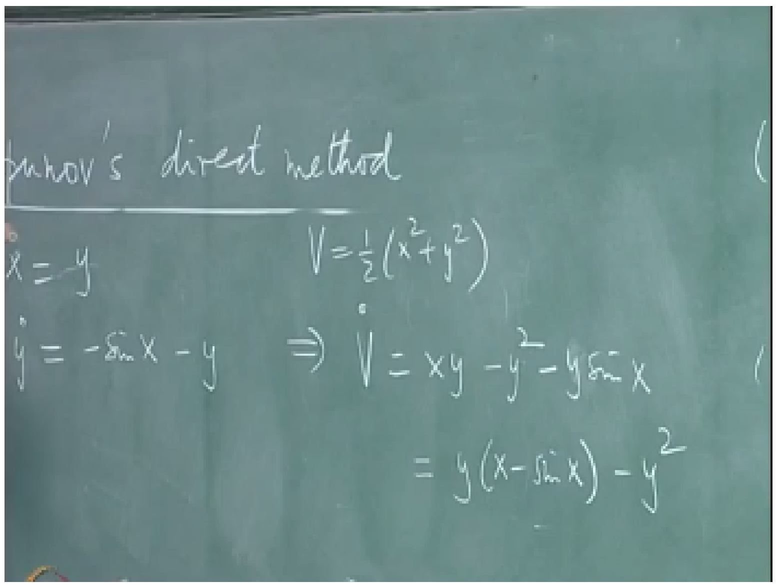

There we had x. = y and y. = on the right hand side – sin x and then if you put damping you also had a – y in this fashion in suitable units this was the damn simple harmonic oscillator with some special choice of units for the frequency and the damping coefficient. What happens now I am going to choose v as he suggested I choose v = ½ $x^2 + y^2$ so this would imply that v. = x so this is xy + y x this is – y^2 – y sin x.

I am interested in seeing what happens near the origin that I know there going to be small oscillation I have included damping here so what happens now? This becomes = y times x I sin x – y^2 and even if I say the neighborhood that I am interested in restricted to a small neighborhood of the origin and y is approximately 0 so I neglect the quadratic term as you can see immediately that x-sinx could either sin therefore we finished you cannot make a statement about whether it is positive define or negative definite not able to prove that in this case.

So this is not a very good choice what would you suggests that this is not that energy of the simple pendulum at all they are pendulum so the next choice would be to say shall I choose the energy of the simple pendulum itself and dam simple pendulum is of that too is positive definite function let us see if we can choose that as function so instead of this.

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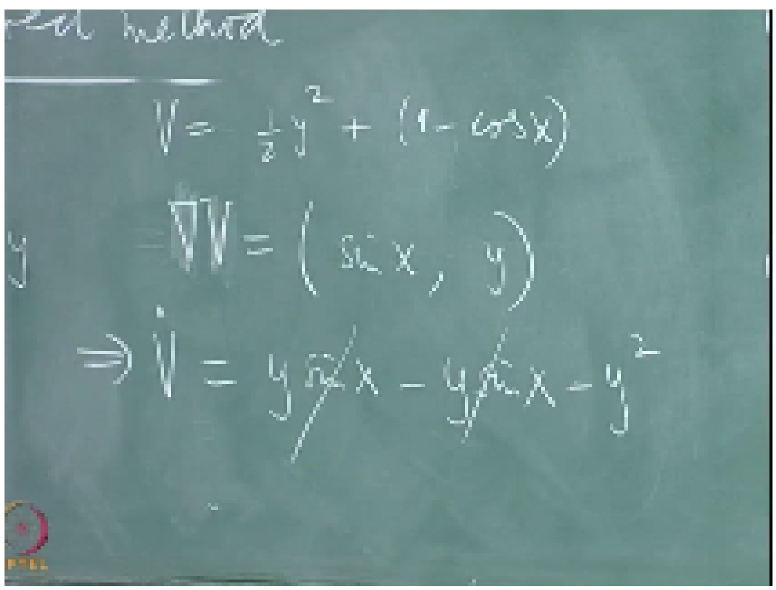

I replace it with $1/2y^2$+1-cosx which was the potential energy and that is the positive definite function also because the least value it has is 0 at the origin and then there is a neighborhood of the origin in which it is not only the positive values so what happens now the gradient of v= I differential this with respect to x and what do I get here is a vector with the following compounds derivative with respect to x this becomes sin x and y so this implies immediately that v.is y sin x-ysinx-vsin and this cancel off and you get v.-y^2 what can you conclude.

Now this is the dam simple pendulum remember it is a dam simple pendulum and we are looking at what happens in the neighborhood of the origin but we have already know what is the origin is what kind of critical point is the origin in the dam simple pendulum it is definitely as asymmetrically stable if it is under dam pendulum it is a spiral point it falls in and symmetrically stable by the point but what are we achieved here we have got an lyapanovs function which is positive definite we have got a v. which is $-y^2$.

And what can we say about that can I apply the first one is v.<0 or can I only apply the second one this function if it is negative definite then certainly I can assert it asymmetrically stable but unfortunately this vanishes not only at the origin but all along the x axis so in the neighborhood of the origin it vanishes everywhere here and therefore it is negative semi definite not negative definite this is all one can assert therefore the conclusion is this critical point is stable but we know it is stable and also it is asymmetrically stable we proved its stability now because the existence of the this lyapanovs function guarantees that this critical point is at the very least stable but it is also unstable.

And that takes much hard work to do so it is not enough to do this you need a much better lyapanovs function in this and there exist one where you can actually show that this as critical point is also asymmetrically stable so I hope this goes a little way answered some of the questions that you raised namely do this imply that or thus this imply this and so on as you can see if you choose a bad lyapanovs function you can make no conclusion at all you choose a reasonable lyapanovs function you get some conclusions.

But it could be even stronger most stronger results could exist but we are not been able to find it because we do not have a suitable choice of lyapanovs function so the whole thing rest with finding a suitable lyapanovs function trying to see the best possible one is found in any case what would happen here if I took the under oscillator but I wrote an equation of motion which was our generalized oscillator stop with that example today.

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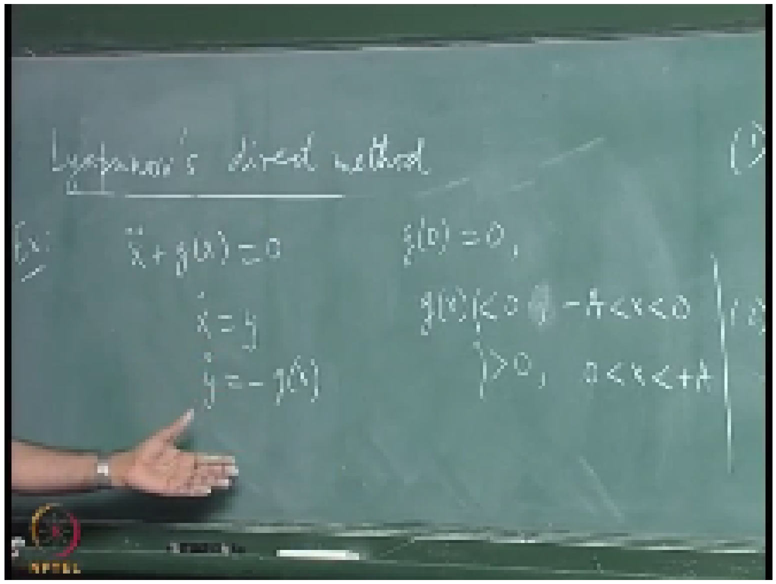

So if I had x.+gx=0 this is no damping in this problem and now I have given to you the following properties this is 0 gx<0 –a<x<0>0 so it is an under oscillator with some possibly non linear function of this kind what can we say about this what would you except happens at the origin what kind of critical point you have at the origin it is a center you completed it is center there is no damping I except stable oscillations about the center.

Therefore I except to be able to find the lyapanovs function where I have this property here what could you say this suitable function of lyapanovs function here that the energy of oscillator could be able to lyapanovs function now it is Hamilton's system in this case what is the energy of this oscillator this is force here this is like the momentum so what would the energy be I choose the lyapanovs function.

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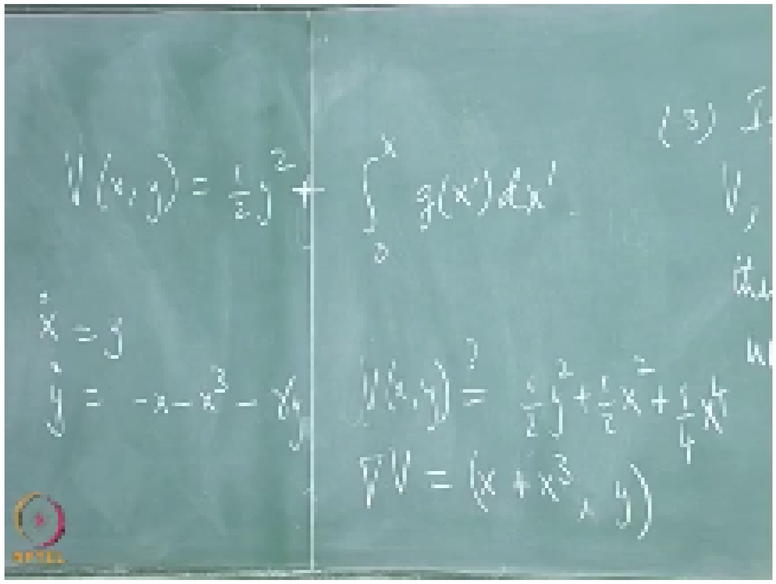

V x,y to be =1/2 y^2 that is the kinetic energy + the potential energy which is the integral of the energy with the force with change of sine so this thing here is equal to 0 to x there is the positive definite function as you can easily check with these properties that is a positive definite function and now you can write down v. and find out whether it is stable or symmetrically stable and the simple exercise show you that the second of the criteria would apply and this would indeed the stable critical point that just like this one more example.

If I do not finish this we will look at it next time x.=y and y.= let us look at the non linear oscillator cubic oscillator that we looked at earlier doffing oscillator or variant operator $-x-x^3 -\gamma y$ so this is the friction term with the positive coefficient γ this corresponds to motion in the potential which is x^2 part and x4 part and this is just the first statement velocity x. is the moment what would be the lyapanovs function in this case the energy one second.

So let us try that so see the best possible choice so ½ y^2+1/2x^2+1/4x4 the corresponding integrate this potential this potential is just a quadratic potential it is not the double well potential of oscillator that would happen if I put a plus sign here in this case you get a inverted parabola near the origin and then the potential that goes like x up there but it does not matter conclusion both these potentials independent of the what the sign we choose in this.

So I leave you to figure out what happens in this case what can you say about this potential what can you say about this cr8itical point then it is very hard to see all you have to do is to take radiant of this v compute it and multiply this and see what happens so this imply at the radiant of x+x3 the first components and the other components is just y so I have an xy so this is v.=$xy+x^3y$ so that $–xy-xy^3-\gamma y^2$ off course everything cancels out what can we say about v. now is it negative definite or it is negative semi definite.

So once again we see that this is not good enough all it says is that this critical point is stable does not get established if it is asymmetrically stable or not it is possible to find and I will give you next time a better lyapanovs function you can actually establish that is also symmetrically stable not surprisingly it will involve γ you need to involve the constant γ and then you can show that you have a lyapanovs function to do the trick similarly for the simple pendulum problem choose the following lyapanovs exponent and show that the system is actually also asymmetrically stable for the pendulum choose the lyapanovs exponent.

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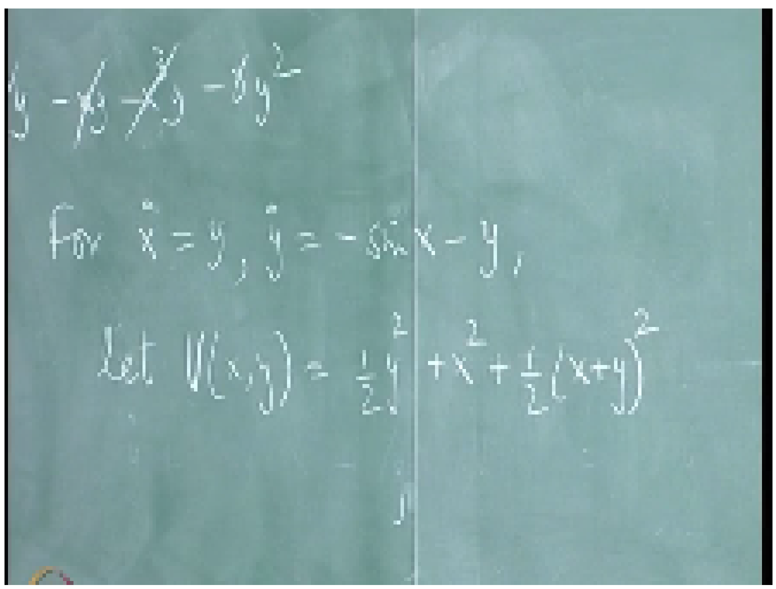

Choose this way of lyapanovs function and then we can see there it is positive definite function vanishes only in the origin in its neighborhood and then show to negative definite and therefore the critical point is indeed and asymmetrically stable so let me stop here today.

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