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# TOPICS IN NONLINEAR DYNAMICS

## Lecture 13
## Illustrative examples

**Prof. V. Balakrishnan**

**Department of Physics**
**Indian Institute of Technology Madras**

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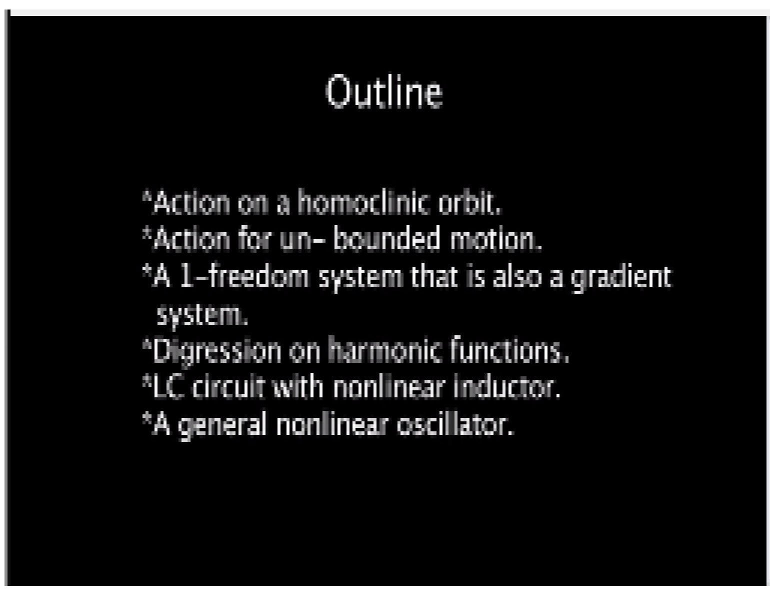


Okay the question asked is the action on a homo clinic orbit I believe first let us go back and look at what we meant by the action.

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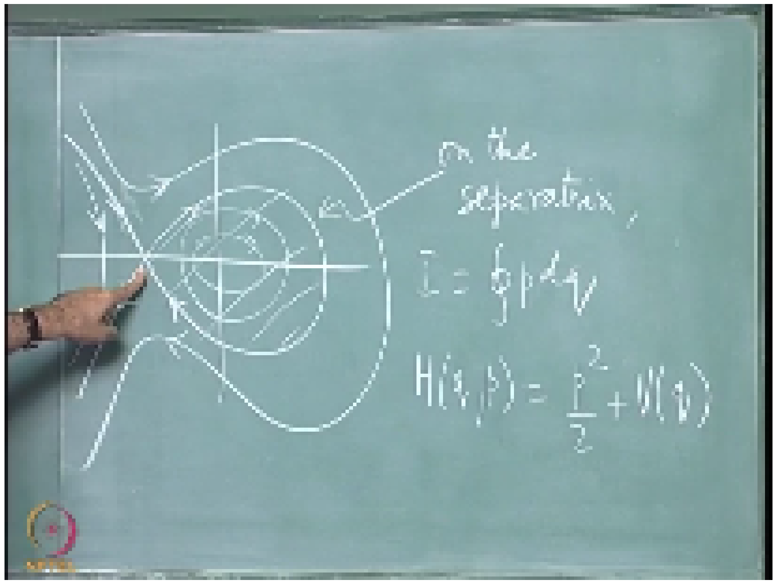


Recall that we started with the Hamiltonian system with variables Qs and PS and then we made a transformation of variables a canonical transformation which took us to a set of variables Thetas and eyes these quantities were called angle variables and these were action variables right now we also discovered that under this transformation the Hamiltonian QP for an integrals system goes over into some function of the eyes alone & Hamilton's equations were preserved in structure.

The action variables turned out to be constants of the motion because you have stated OTT is Δ K over Δ I and I dot was - Δ K over Δ the ta sub that i and this was identically zero which implied that each of the eyes is a constant of the motion therefore this quantity here which is some function of all the action variables also a constant of the motion and the motion as we saw lay on some kind of n dimensional Taurus.

Now the question is can we compute this action for specific instances and in the case of a one-dimensional problem one degree of freedom problem the answer the straight forward the action is in fact integral PDQ for each periodic motion the action is just the integral of P DQ over the full period and what you have asked to compute in that particular problem is a situation where you have in the phase plane some kind of saddle point monoclinic orbit which goes back in this fashion and then which comes out.

And this is what the phase trajectory is looked like and you had open trajectories of this kind and this is the separatrix that separates periodic motion remember there was a center somewhere here and then you had periodic motion here in this fashion and it is separated periodic motion from open an unbounded motion and the question is to compute integral PDQ on this loop so we would like to know on the separatrix the action is integral PDQ which is geometrically nothing but the area enclosed by this loop.

And we know the equation to this loop we know what P is on this loop because if you are call the Hamiltonian q h of q come up e in that particular problem was equal to $P^2$ over 2 plus twice the mass is I + V of Q and on the separate it had a specific value the Hamiltonian had a very specific value which corresponded to an extremism of the potential here the maximum of the potential so if I call that V Max this is also equal to $P^2$ over 2 this thing is also equal to V Max the numerical value of the potential energy at its maximum.

And this is the equation to the separate matrix therefore you can find the equation to this loop and all you have to do is to integrate and find the area of this loop which is twice the area enclosed when going from here to here across this segment so you simply have to compute integral PDQ over that so it is just a definite integral which you have to compute and the answer is straight forward this V of Q itself was some kind of crazy cubic curve and that curve if you plot for this particular value of V Max would actually represent this full curve here.

But what we are interested in is this lobe here the monoclinic orbit and you simply have to compute the action on that which is twice the area of whatever the area is in going from here to here I do not recall what the exact answer to this is but it is 650 or something like that just the area PDQ which you need to compute what is the action for the simple harmonic oscillator what would that be well recall for the simple harmonic oscillator where you have bounded periodic motion of this kind on any of these trajectories.

The action is simply the area here which is the area of the ellipse and that straightforward to compute and you can in fact related to the Hamiltonian itself in a very simple manner for the oscillator.

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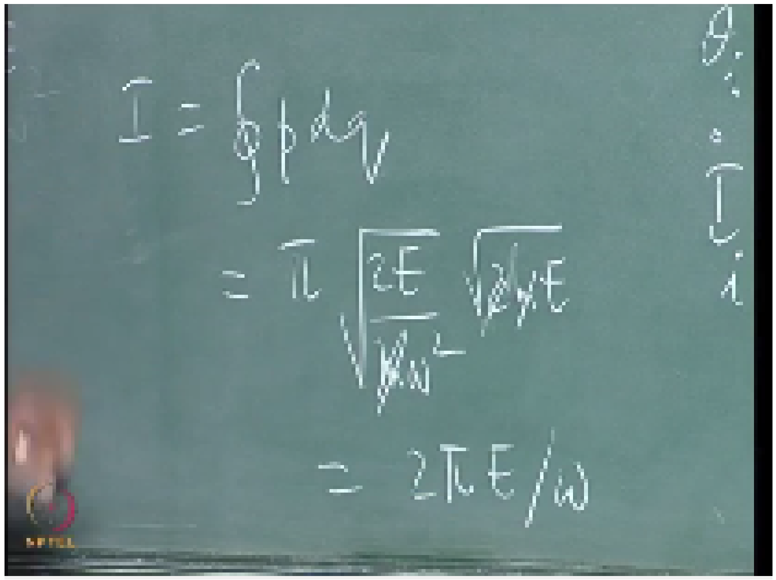


The Hamiltonian is $P^2$ over 2m +1-1/2 M Omega squared $Q^2$ and if you took any typical orbit of energy e then this point here the amplitude is represented by 1/2 M Omega squared if on this orbit H of QP = e that would imply that this particular point is given by 2 e over m Omega squared √that is what would correspond to the amplitude and this point here would be given by square root of 2 m.e therefore the action integral PDQ is simply.

The area of the ellipse which is π times the semi-major axis which is root 2 e over m Omega squared times √2 m to u can sorry the n cancels and this is equal to 2 π e over Omega so it is quite clear that the action is related to the Hamiltonian the Hamiltonian essentially is omega times the action itself in this particular instance it is immediately clear the important thing is that I is linearly related to the Hamiltonian and therefore when you compute.

These quantities ΔK over Δ I you get a constant as you should but in a non linear oscillator this is not true as we have seen the Omega depends on the action which in turn is a function of the Hamiltonian and therefore the time period of oscillation depends on the energy of the oscillator which it does not for simple harmonic motion okay let me also mention that you can find the action for unbounded motion let me give an example of it you could have unbounded motion.

And then ask for what the value of the action is for such motion well for instance in the case of the simple pendulum if you had trajectories of this kind you recall the simple pendulum you had hetero clinic orbits of this kind and then there was open unbounded motion which corresponded to rotation of the pendulum and you could ask what is the action on one of these things the answer is you can define an action in principle it could diverge.

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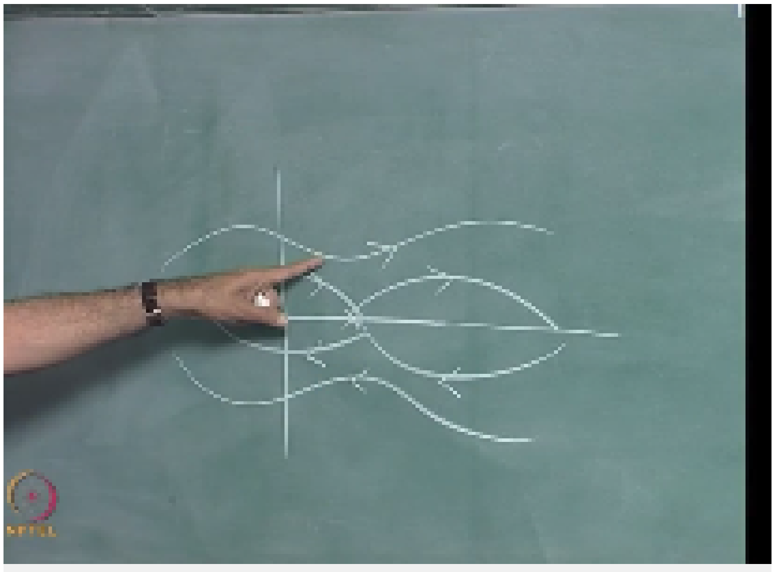

If you simply compute at PDQ it might diverge but you compute PDQ with the potential and without the potential and subtract the second from the first and you get a finite answer so action for unbounded motion would correspond to integral PDQ over the trajectory but you must subtract from it whatever you get without any potential for a free particle so that the divergence formally goes away.

And finally to recall the boards are muffled quantization rule simply said that the action is quantized so it is the action that is quantized always and you could ask what happens if I have motion on an n-dimensional torus what is quantized how do you do that well pictorially for the two-dimensional case two degree of freedom case you have one action here one angle here and you have another angle here in this case.

And you have two independent paths that you can follow for integral PDQ and each of those is equal to an integer times Planck's constant so that is the way quantization has done for systems in the semi-classical quantization is done for systems with more than one degree of field so you would compute for each of those cycles let me call them C sub I you would compute PDQ and of course these quantities will be vectors in general.

So you would compute this and that would be your the corresponding action those would be the action variables and those would get quantized however i must say straight away that it is not straight forward to quantize these systems if they are complicated and it is just that once you use used to are things like linear simple harmonic oscillators or systems where you can decouple an N degree of freedom system into n 1 degree of freedom systems.

The Hamiltonian simply adds up and that course the matter becomes simple okay pardon me what is Lyapunov stability I have not discussed this yet I am going to come back to it and do it but let me explain what Lyapunov stability is and that in the problem sheet but I have not yet discussed it I will be coming back to this okay you see we have seen that linear stability analysis works very well when you have the echo be in matrix at a given point in a dynamical system has eigen values.

Such that none of them has a zero real part in which case you have a center manifold and then linearization may not give you are liable picture as to what is really happening in the dynamical system now as long as you have purely hyperbolic points in other words points where the according matrix has no, no I then value with a zero real part linearization is fine but the question you could ask is we have seen examples of this linearization fails spectacularly very often what would you do.

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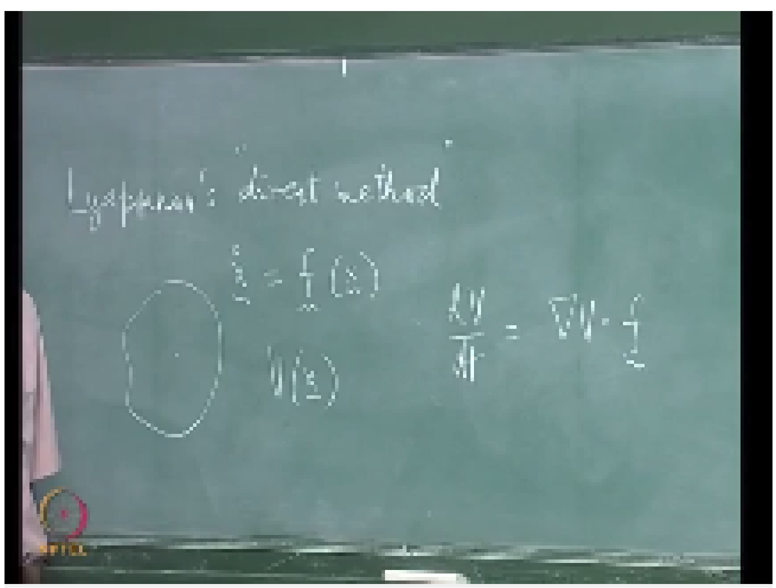

Then well the Lyapunov stability analysis that is also called Lyapunov direct method corresponds to saying that if you give me a flow X.= f of X and I would like to examine the nature of the flow in the neighborhood of some given point then I discover a function V of X with certain properties if this V of X is such that it vanishes at the point you are interested in.

And does not manage anywhere else in the neighborhood but for instance V dot the time derivative V dot if the function has a definite sign everywhere else in this neighborhood and vanishes at that point if it is positive everywhere here and vanishes here I say we is positive definite and if it is negative everywhere else and vanishes here I say V is negative definite then by examining such a function and what its time derivative is because in principle I can always do that and find DV over DT this time derivative would occur.

Because X changes as a function of time so all I have to do is to write its partial derivatives with respect to the X's and multiplied by the corresponding X dots and what do I get what would this be I start differentiating with respect to each of these X's and i multiplied by the corresponding derivative of that particular X what would I get on the right hand side yes exactly i get the gradient of V with respect to these variables dot.

The velocity vector which is f itself on this side and I can make statements based on whether we is positive definite or negative definite I can make statements on the stability based on the sign of this quantity here so I introduce an auxiliary quantity called the Lyapunov function V no statements as to how you discover this function based on experience for instance and then analyze the stability there are stability theorems which I will quote and discuss depending on what t be over DT does in this neighborhood.

Now in the problems that we are looking at simple one-dimensional problems bounded motion the Hamiltonian itself acts like a Lyapunov function for instance a simple harmonic oscillator function which was 1/2 $P^2$ + 1/2 $Q^2$ in suitable units clearly vanishes at the origin and in a neighborhood of the origin never vanishes it is positive definite therefore this function is positive definite and you can make statements about the stability of the center of the origin by analyzing this Lyapunov function.

So we will come back to this, this statement gave a two-dimensional system and asked whether the Hamiltonian is unique and what happens if you interchange the role of x and y and further whether this could even be regarded as a gradient system or not.

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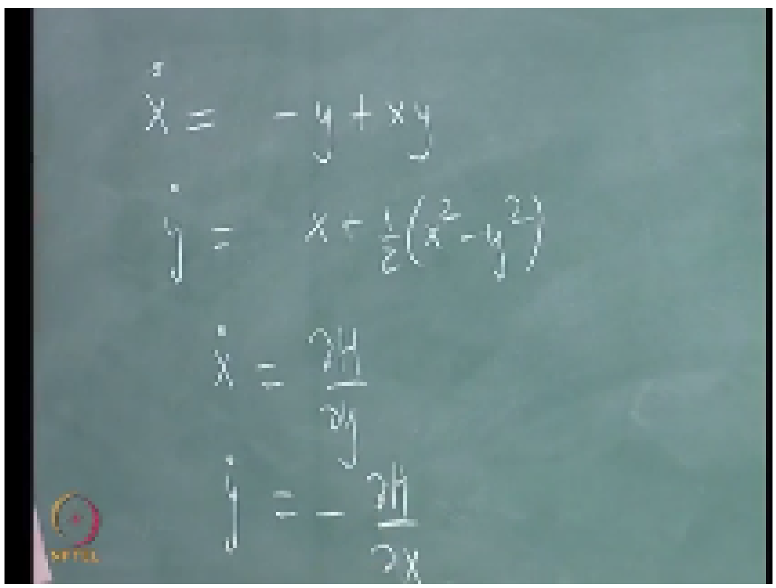


So let us look at that and the problem was X .= y +X Y and Y .= X +one-1/2 $x^2 – y^2$ the question is, is this a Hamiltonian system and if so what is the Hamiltonian is it unique can you interchange the role of x and y well if this is a Hamiltonian system in which explains the role of the generalized coordinate and y plays the role of a generalized momentum then we should require that X . Δ H over Δ Y and Y .-Δ Hover Δ X well.

If this is Δ H over Δ Y what would this imply for H itself I have to integrate with respect to Y what does it imply - $y^2$ over 2 + X $Y^2$ over 2 + an arbitrary function of X the integration constant here could be a function of X since we are doing partial derivatives plus some function of X let us call it π of X in but this would imply in turn that H of X y = I have to integrate this with respect to X and flip the sign.

So what would this correspond to I believe there was a mistake in the sign there so thisis minus $X^2$ over 2 all right as it stands and then what you get here $-X^3$ over 3 $+y^2$ x over 2 you have to integrate this with respect to X so it is $y^2$ x over 2 plus some function of Y pardon me $x^3$ over 6 plus this so are these compatible can they be simultaneously satisfied yes indeed so it is clear that H of X ,Y is equal to well there is certainly a $-y^2$ over 2 - $x^2$ over 2 could be this function and thus I of y would be this function.

So it is clear that you have a Hamiltonian which is $-1/2\ x^2 + y^2$ that certainly permitted + $XY^2$ over 2 that is a nonlinear term and there is a non-linearity - $X^3$over 6 here so it is evident that this part of it could be π of X and this part of it which depends on Y alone is sy of Y all you have to check is that the term that involves both an x x and y multiplicative lee is the same in both cases is this unique or could I have done something else.

I could have added a constant to it plus any constant and it would  have changed the Hamiltonian at all so it is indeed a Hamiltonian system but not a usual sort of Hamiltonian thing because why it certainly does not look like the kinetic energy of a particle moving in one dimension because it is got a minus sign here but it is a Hamiltonian all the same could I have inverted this could I have done this the other way could.

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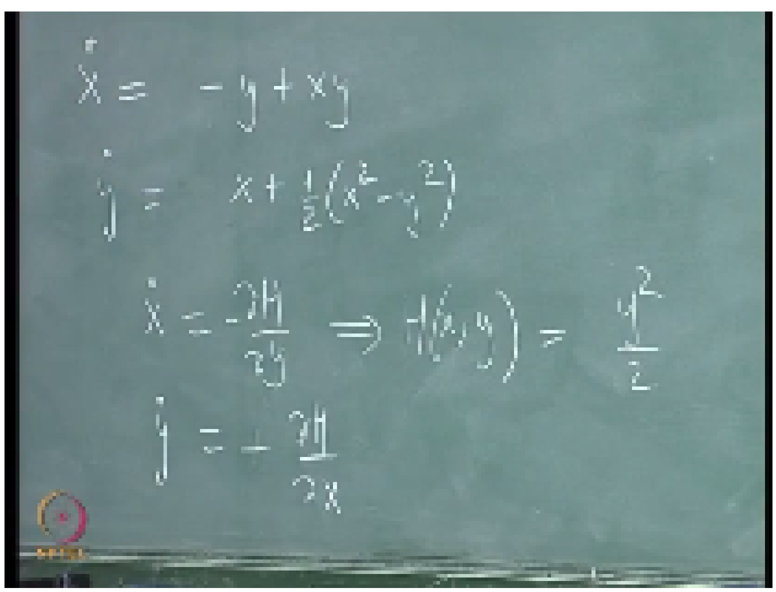

I have written this as - ΔH over Δ Y and made this a +Δ H over Δ X would this be possible what would happen is this possible. Yes it is yes it is you could have taken it this way either but let us see let us work it out so this would imply that H of X ,Y is equal to let us work this out let us see this is really true or not because - Δ H over Y so this is y $^{2}$ over 2 herein this case and then minus X $Y^{2}$ over 2 and this would imply that H of Xy = $x^{2}$ over to hear this case + $X^{3}$ over 6 -X $Y^{2}$ over 2.

So exactly as he said all you have to do is to flip the sign therein the Hamiltonian and it is equally good and this is in fact a little more respectable because the quadratic terms appear with a positive sign so indeed it looks like a particle which is moving in one dimension in which X plays the role of the momentum and why place the role of the coordinate and you have the correct kinetic energy with the right sign but you also have momentum cubed and we have complicated combinations of this kind.

So it is certainly not a particle moving in some static potential but something more complicated than that could this system also have been a gradient system could the same system also have arisen as a gradient system what would happen then in other words the question I asked is could you have the following X . Δ H over Δ Y and this is -Δ H over Δ X but.

Put this also be equal to some gradient of some scalar function can a Hamiltonian system be a gradient system simultaneously is the question if so this is they have to be true h & 5 would have to be two scalar functions such that these conditions are satisfied is this true or not this is what we are trying to examine can I Hamiltonian system also be a gradient system at least in this simple example of a one freedom system could.

This be possible can you have functions π and H not necessarily in this particular case but general functions π and H such that these partial derivative conditions are satisfied what would that imply for each of these functions what would it imply for example if I differentiate this a second time i get d2 π over DX 2 = Δ H over Δ X Δ Y and that would appear with a minus sign here right so this would end up being equal to in other words $\Delta^{2}$ Δ =0 in two dimensions and similarly for H this would also imply that $\Delta^{2}$ H is zero which is certainly possible.

So you could have functions which satisfy Laplace's equation in two variables and then of course if H is such a function it could also be simultaneously a gradient system what would be the what would be your guess as to what π is suppose H is such a function that it satisfies Laplace's equation a harmonic function something which satisfies Laplace's equation what would that be what would that what would phi b that is a good guess that the level surfaces of π would be orthogonal to the level surfaces of H certainly.

So if these are the surfaces or curves on which H is constant you would have another family of orthogonal curves where five would be constant this case certainly this is true because we know that for a gradient system the flow is along the gradient of π for a Hamiltonian system it is normal to the level curves of H now it turns out you are familiar with this perhaps from complex variables that if you took an analytic function of the complex variable Z where Z is X + iy.

And wrote this as u of x ,y +I times V of X , Y the new and we obey precisely these conditions instead of π index you have u and V and they call the Cauchy -Riemann conditions and we know more ever that the real and imaginary parts of any analytic function of a complex variable Z are harmonic functions related by the Cauchy -Riemann conditions so in case it turns out that H and π are functions of this kind you certainly can have a Hamiltonian system also be a gradient system you could generalize this to n variables.

If you decouple the entire thing if your system is simply kind of additive combination of n freedoms individual couple then this would still be possible so in certain cases this accident does happen you may have this yeah now the statement I made have to do with complex analysis but let me not get into that let us look a little bit at what we mean by a harmonic function it is a good thing to have this digression to see what is meant by harmonic function.

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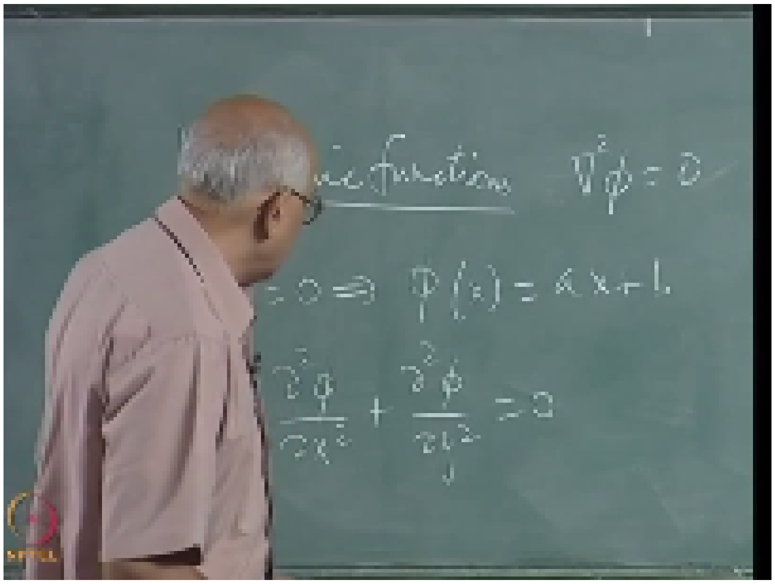

So let us look at solutions of the Laplace equation itself a little digression on harmonic functions their solutions of del squared π =0 in some region any number of dimensions let us look at one dimension what would this function be you have d 2 π over DX 2 = 0and that of course immediately implies that π of X must be at best a linear function in this fashion no curvature at all what would you say is the corresponding property in two dimensions so if I have this what kind of property do you think.

I must have in order to be a harmonic function it is immediately clear that a linear combination of the form ax +B + C see y + D is a solution but that is not necessarily the only solution because the point is each of these terms does not individually have to vanish this could cancer that in general so you have many more complicated functions which are harmonic functions but what is the primary physical property of harmonic functions which leads to this behavior which leads to its being a solution of Laplace's equation.

What is the import of this statement here what is it geometrically in this case if i plot x vs π of x some kind of straight line what property is being exploited by saying that it is second derivative is 0 what is the geometrical meaning of the second derivative the curvature it has no curvature this means of course that if you took any value of x and any value other value of x and you took a value in between.

Here you are guaranteed that this value is the arithmetic mean of this value and this value in other words you guaranteed that π at any point x is fine at any X +α π at - μ /μ u/2 is not it you guaranteed this for any epsilon you guaranteed this as long as there is no curvature the value in the middle is always the arithmetic mean of the values on either side of two points symmetrically distributed about this point that is what it is telling you in fact you could go back from this.

And write this as π of x+ epsilon -2 π of X and let me write that as π of X – π of X - π of X -=0 the just this equation rewritten now what is this if i divide this by epsilon what is this it is the derivative it is the right derivative and this here is a left derivative at the point x and if i take the difference between the two and / another epsilon I actually get the second derivative and that is how you end up with the second derivative being zero.

So it reflects this mean value property the value at any point is the arithmetic mean of the values at two points equidistant from it on either side of it that is the basic property of a function of second derivative vanishes all that happens in higher dimensions is that this is generalized so in two dimensions you took this.

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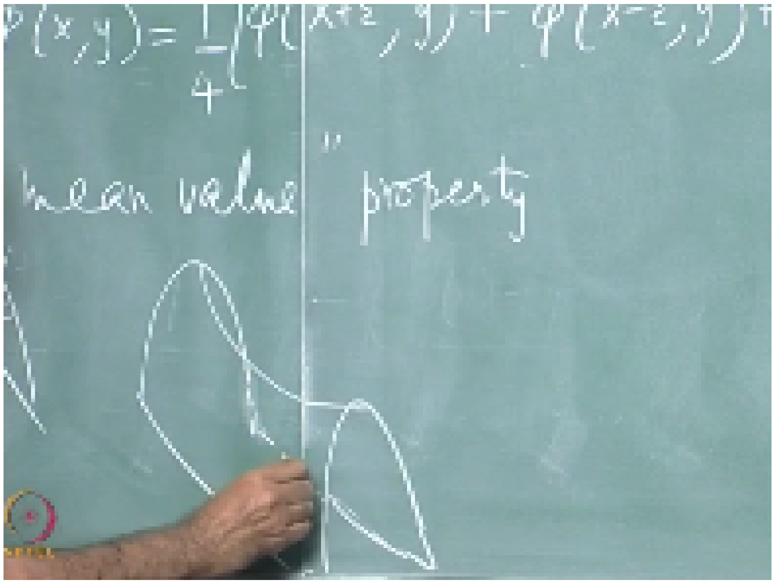


For example and you ask for its value in the XY plane at any particular point the statement is if I wrote these as finite derivatives finite differences from infinite decimal points infinite simply apart then this difference between the value here and the value here minus the difference between the value here and the value here divided by epsilon squared gives you the second derivative in the limit and similarly on this side so it is not hard to see that a function which satisfies.

This essentially satisfies the property π of X , Y = π of X + epsilon y + π of X - epsilon Y with a quarter here because all I have to do is to move this for here move it across to the other side and start taking differences split this π the four times this into one difference here another difference here another here another the head and then take the limit epsilon goes to zero appropriately after dividing by epsilon and you end up with this you could go backwards this it is this mean value of property that distinguishes the harmonic function.

This is the actual meaning of what a harmonic function is and of course this operator here this derivative second derivative operator is a scalar operator because recall this is the same as saying $\Delta^2$ π but $\Delta^2$ is $\Delta . \Delta$ on 5 and that is a scalar operator therefore it is actually invariant under rotations of the coordinate system so it would look exactly the same if for example you took these four points and rotated them.

So they look like that if you want the value here or it took any other points in between I take this point this, this and this and take the mean value of any of these point and that gives you the value here in fact you could generalize this and say let me take the values on this entire circle here integrate over all these values and divide by the circumference of the circle and you get the value at the center what it implies is that you cannot in such a situation have an absolute maximum on an absolute minimum at any point for a harmonic function.

Because if this point where an absolute maximum it is larger than the values at all these points but then it turned be the average of those values and similarly for an absolute minimum so harmonic function cannot have apps maxima or absolute minima in the region in which its harmonic so it satisfies a mean value theorem and has no absolute maximum minima it could have saddle points so it could have a local maximum with respect to the X direction in which case the second derivative would be negative.

So this quantity would be negative but then it could simultaneously have a local minimum with respect to Y so that this derivative would be positive and they would cancel each other so the shape would be it cannot have this kind of behavior it is an absolute minimum on an absolute maximum it cannot have this kind of behavior so this is not allowed this is not allowed but it is permitted to have a saddle point so it is possible to have something which looks like this so it is possible to have a shape like this, this point.

So at this point you have a local maximum in this direction but a minimum in the other direction and you can have a saddle point such points are permitted the same thing is true in three dimensions a harmonic function is one which is whose value at any point is the arithmetic mean of the points of the values that all the points of a sphere surrounding this point divided averaged over the area of the sphere.

And that is the reason why in electrostatics you know that in a charge free region $\Delta^2\pi$ must be zero the electrostatic potential must be zero and therefore it cannot have any absolute maxima or minima in particular it cannot have any stable equilibrium points because that would correspond to an absolute minimum of the potential this is impossible simply because of the harmonic property of the electrostatic potential there many, many other profound properties which arise because of these functions.

But it is useful to know that the generalization of the statement that the function does not have any curvature in one dimension which implies that it has a mean value theorem is satisfied is generalized to higher dimensions in this fashion to harmonic functions and of course because you have more than one dimension to play with you have a very large number of possible functions which satisfy this equation Laplace's equation on the set of solutions of Laplace's equation is highly non-trivial in two.

And more dimensions this has to do with an LC circuit with a non-linear inductor and we need to write down the equation of motion for the charge and we have to show that a certain quantity is a constant of the motion so you have a lossless LC circuit of this kind with a core through this circuit here so this is l it is an LC circuit and this is the capacitor C and you are told that the flux through the core is $\pi$ of I anon linear function of the instantaneous current I in the circuit now the back EMF as you know is d $\pi$ over DT this is the back EMF so the normal equation would be this plus potential across the capacitor.

And what would that be q / see this would be equal to zero if this inductor were a linear inductor when we know that the flux across this inductor $\pi$ is equal to is proportional to the current through it and it is equal to L times I wear this thing here is the self inductance of the inductor and therefore this equation would immediately become ld2 q / d d 2 + q / c +to 0which is nothing but simple harmonic oscillations with a natural frequency omega naught is 1 over root LC that is a linear conductor.

Now we have a nonlinear inductor in which you have an equation of motion which looks like this in this fashion and the question is having written this equation of motion down what does the Hamiltonian look like what is what is a constant of the motion in this case there is no dissipation in this problem at all no resistive element so the question is what is the constant of motion and how would you find that what we need to do is to integrate. This equation in a conventional way to integrate this equation is utterly trivial you multiplied by Q dot on both sides then you observe that.

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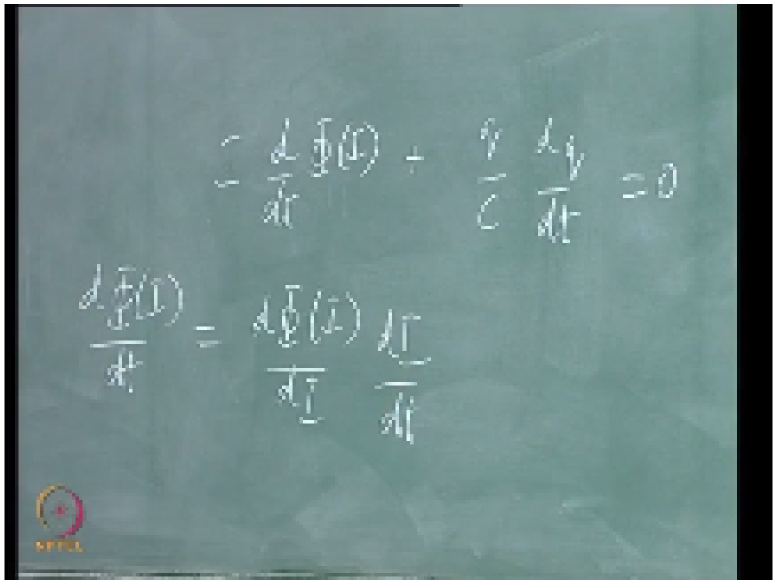

This is d over DT of Q $.^2$ apart from a factor l and once you multiplied by Q dot this is d over DT of $Q^2$ and therefore you get Q $.^2+Q^2$ equal to constant which is the harmonic oscillator now what happens in this case when you do not have a π but you are told that π is some non linear function of the current and is an odd function in general so what is this thing how would I integrate this equation like to integrate this equation so this is d over dt by of I well the obvious thing to do is to multiply both sides by Q dot once again by I.

So this becomes an eye here and multiply this by I which is DQ over DT =0 I can integrate this directly right so what is this equal to if I find the indefinite integral I so can I rewrite this in some fashion how do you integrate this yes indeed so all you have to write is d π I / DT = D π over d I di over DT and this is equal to some I prime of AI di DT so let us put that in here so this is equal to π prime I with an eye here i Ds = 20 so what is the constant of the motion what is the constant of the motion the primitive of this.

So if i do integral I try prime of AI di and I call this something else SCI FI for example the indefinite integral it is evident that sigh of i $+Q^2$ over 2 C is a constant of the motion all I have to do is to differentiate this right here you can easily see that in the linear case when π 0 π is proportional to I it is l times I then π prime is just Land you get indeed get 1/2 Li $^2+$ Q $^2$ over 2 C equal to constant in the linear case.

But in the nonlinear case this is the constant of motion I would generally expect that this flux this π here is some kind of odd function of I as a function of I if I plot fire I it would be linear for sufficiently small currents but as the current increases I would actually expect some kind of saturation behavior in this fashion much more realistic model of what happens then a pure linear function all the way through if you have a ferromagnetic code for example this is precisely what happens.

If you tell me the functional form of Y if I could write down explicitly for you what the constant of the motion is this does have the import of the energy of the system this is the magnetic energy so this term comes here from the inductor this comes from the capacitor both are reactive parts and this corresponds to the total energy of the system there is no dissipation in this problem but it is not a linear system simply because this function here could be very complicated so the differential equation that you write down does not have to be linear in particular.

We have got five I is alpha I +β tan inverse γI but αβγ are positive constants and you asked to find the phase trajectories of the system well for sufficiently small values near the origin it is clear that it is exactly like the simple harmonic oscillator but as you go outwards they would become nonlinear and you would have some crazy ovals some kind so it is just generalizes from the harmonic oscillator.

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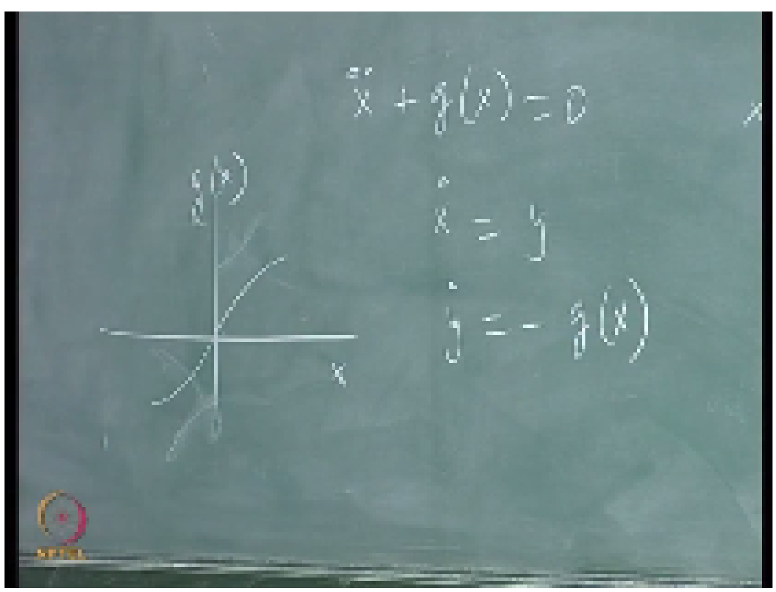

Similarly the next problem also says you have zero and you are told X is an element of - ay ay and you further toll is some positive number and g of x is a given continuous differentiable odd function with G prime of X >0 it is an odd function it is continuous so what can G of X possibly be as a function of X G prime of X is zero what is G of 0 it must be zero because it is an odd function and it is continuous at the origin if not.

There is no true this is not true for instance 1 over X goes like this it is discontinuous infinite discontinuity the origin it could be odd and have a finite discontinuity at the origin in which case it could go like this but here you are told it is continuous so it has no choice but to pass through the origin in this fashion may not pass through it linearly however you are told g prime of x is > 0 also but as if it is a cube.

For example g prime of 0 would be 0 do flat so in this case it is clear that is a function which perhaps does this may or may not saturate we do not care but it passes through the origin in this fashion and it is between minus a and plus a so you can easily write this down as x .= y y =- G of X this is a Hamiltonian system if so what is H equal to yes what is the Hamiltonian in this case if i write this for instance as Δ h over Δ y what is the Hamiltonian equal to in this case yeah indeed it is $y^2$ plus an integral this fashion up to some X.

And if you choose the constant of integration appropriately you could make this 0 onwards if this corresponds to motion between -a 2+ a then what would this be so if I start with this what would the largest value of gb of the potential energy be this represents the potential energy what would this be so go back to this guy here and ask it is anyway positive so it would be an integral from zero to a and you could write that down.

So that would correspond to motion the Pretender perhaps it looks like this, this point at this point the integral of this since G of X is an odd function what does it imply for the integral of G of X this integral what kind of function of X is it, it is an even function absolutely it is like a potential it would be some even function there and the largest value it could have is zero to a G of X prime DX prime.

So that would correspond to motion right at that amplitude a that would be the numerical value of this H for this particular motion and once you have that the time period is straight forward to find since you know the amplitude you know the total energy so the time period is easy to find all you have to do is to do integral you have to integrate this place the role of a momentum so all you have to do is to find integral DX over x dot which is why and that gives you the time period.

So let us do that in the XY plane the motion is something like this we need to find the time period of this motion this is the amplitude a here the time period is equal to four times the time taken from 0 to a 4 DX over x dot which is why this case and what is why on this trajectory we know that $y^2$ over 2plus integral 0 to X G of X prime DX prime is equal to integral 0 to a G of X prime DX prime that is the total per ton of the potential energy.

So this implies that $y^2$ over 2 equal to integral X to a G of X prime DX prime with a 2there and on this segment y dot is positive Y is positive sorry x dot is positive so this is equal to the square root of this whole thing therefore this is equal to 4 times integral 0 to a DX over square root of twice integral X to a that is the time period of oscillation you give me G of X and I calculate this in principle.

So this generalizes the oscillator to a more general non-linear oscillator because of this function G of X where is the only critical point in this problem it is a center at the origin it is a center at the origin we know that this function here is an odd function so it must vanish at the origin that is the only place it vanishes so the center is at X = 0 y = 0 this critical point is at X = 0 y = 0 and it is obvious from here that it is a center you need to use this property the G prime of x > 0 and then it turns out to be a center.

What is the necessary and sufficient condition for a liberalized two-dimensional system to have a center at the origin what would be the necessary and sufficient condition for a system which is of the form X dot is ax+ B Y y dot is CX + dy what would be the necessary and sufficient condition for the origin to be a center exactly if you recall what we did in the feed sorry in the t Δ plane we had this parabola which was $t^2 = 4\Delta$ and centers lay here at this point.

So we need to have t = a + d = 0 and Δ = ad - BC > 0Δ here we need this value right Δ =0 is here those are degenerate critical points so these were the center's there were saddles here so this whole thing was saddles and then out here you had unstable nodes and you had unstable spirals you had a asymptotically stable spirals here asymptotically stable nodes here in saddle points here and the center was very fragile it was perched right on this line.

So you must have T =0 you must have a + T =0 and you must have ad –BC >0 to have a center turns out this is necessary and sufficient conditions to have sent a center moreover when you have a situation of this kind the phase trajectories are not hard to find because there is some kind of closed curves around the center here and what sort of curves can be is possibly be you have a linear system what could be the only curve you have that is closed there are lips asst are ellipses.

Because this is a linear system it turns out that you have a quadratic function here this is a quadratic function of x and y equal to constant and they are actually ellipses so I leave you to figure out that given these two conditions that the phase trajectories of the system are in fact ellipses this is not too difficult to prove you can easily integrate this to find out what the phase trajectories are because you can eliminate one of these variables to write a second-order equation in either X or NY and their linear equations.

And you can integrate these equations and you can directly find out what the constant of motion is in this case I am going to do that is not hard to do because dy over DX is CX + dy / X + B Y and you can in fact integrate the system out in the case when T is 0 and Δ is positive and to discover applying the usual rule for conic sections that you indeed have ellipses which is what all you expect which is what you expect what is the condition on an equation of the second degree will describe a conic section which is an ellipse.

If you recall from coordinate geometry the general equation is of the form plus linear terms what notation do you use here something of this kind and what conditions is this an ellipse yes but what is the condition on these coefficients in order this business describes a conic section it is an equation of the second degree so wonder what conditions do you have an ellipse or a parabola or a hyperbola when you have a B - $H^2 > 0$.

This is an ellipse when it is equal to 0 you have a parabola and when it is less than zero you have a hyperbola so this is a nips that is not hard to figure out that the level curves here representing the constant of motion they actually are ellipses in the instance when these conditions are satisfied which is all you could have you.

**Online Video Editing / Post Production**

K. R. Mahendra Babu
Soju Francis
S. Pradeepa

S. Subash

**Camera**

Selvam
Robert Joseph
Karthikeyan
Ramkumar
Ramganesh
Sathiaraj

**Studio Assistance**

Krishnakumar
Linuselvan
Saranraj

**Animations**

Anushree Santhosh
Pradeep Valan. S. L

**NPTEL Web & Faculty Assistance Team**

Allen Jacob Dinesh
Bharathi Balaji
Deepa Venkatraman
Dianis Bertin
Gayathri
Gurumoorthi
Jason Prasad
Jayanthi
Kamala Ramakrishanan
Lakshmi Priya
Malarvizhi
Manikandasivam
Mohana Sundari
Muthu Kumaran
Naveen Kumar
Palani
Salomi
Senthil
Sridharan
Suriyakumari

**Administrative Assistant**

Janakiraman. K. S

**Video Producers**

K. R. Ravindranath

Kannan Krishnamurty

**IIT Madras Production**

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