Indian Institute of Technology Madras
Present

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NONLINEAR DYNAMICS

Lecture 12
Poincar'e index

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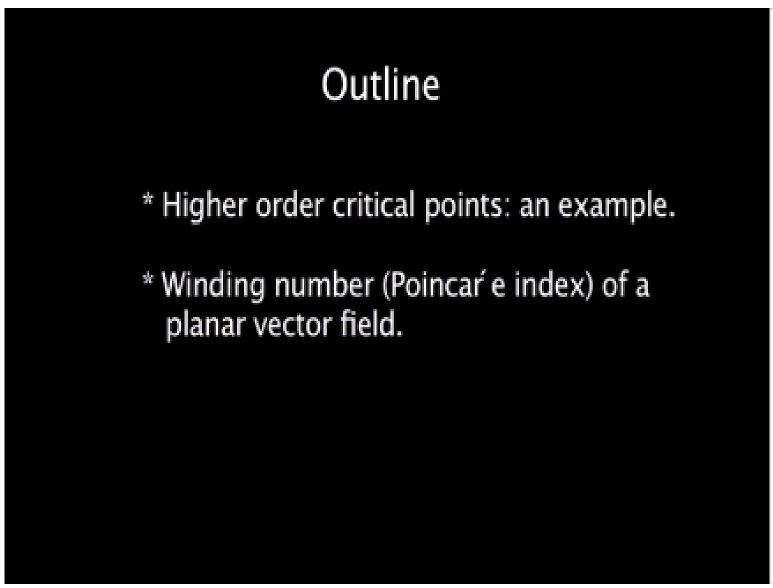

So we begin today with an example of when you have a center manifold in a dynamical system and how linearization could lead to an erroneous conclusion and again I do this with the help of an example so let us consider the following.

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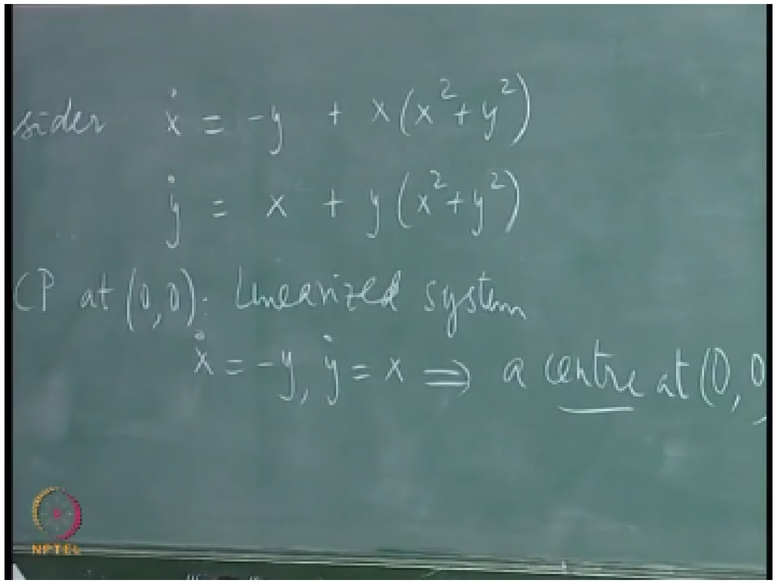

System consider x. =- y and y. =x as we know this linear system has a critical point at the origin which is a center because the eigenvalues of this matrix of + or - I the linearize matrix and you would expect, the center it is like the harmonic oscillator problem but now I add nonlinear terms to it, so suppose I add x + x times this + y times x^2+ y^2so there is clearly a critical point at 0 and the linearize system x. =- y. =X → a center.

Therefore expect the trajectories to go around in small circles or ellipses about the center which is a stable critical point on the other hand we can solve the entire system the full nonlinear system because of the form of the non-linearity it is straightforward to solve this problem completely and what would the solution be all I have to do is to use the fact.

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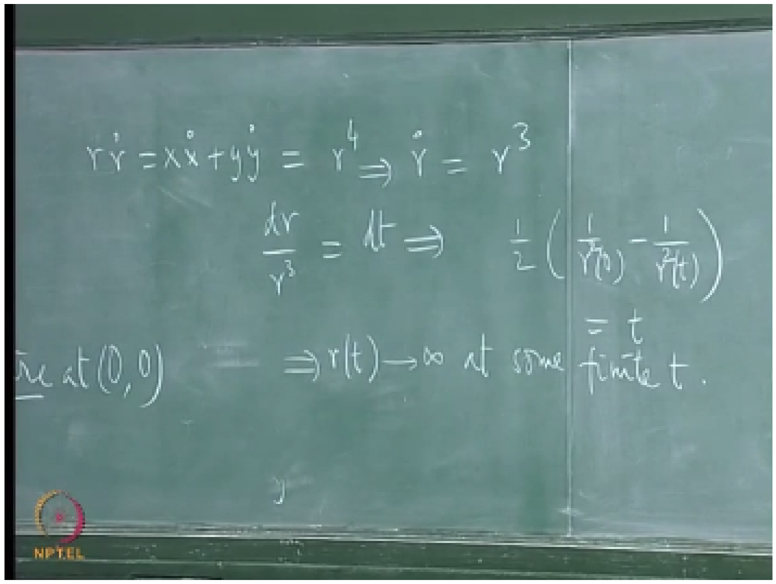

That rr. is xx . + yy . if I do that I multiply this by X and that by Y and add these cancel and I end up with an r^4 which → that r. is r^3. Which immediately says since r can only take on non-negative values as soon as you have a finite r0 which is non0 r increases and keeps increasing indefinitely and the question is you could also find out how it increases as a function of time that is straight forward.

Because this says dr/r^3 =dt and that of course → that 1/2 1 over r of 0 - 1 over r (t) this quantity is = t itself r0 ^{2} and it is easy to see from this if you solve for r(t) so this → that r(t) tends to infinity at some. Moreover if you do this in plane polar coordinates and look at what θdoes.

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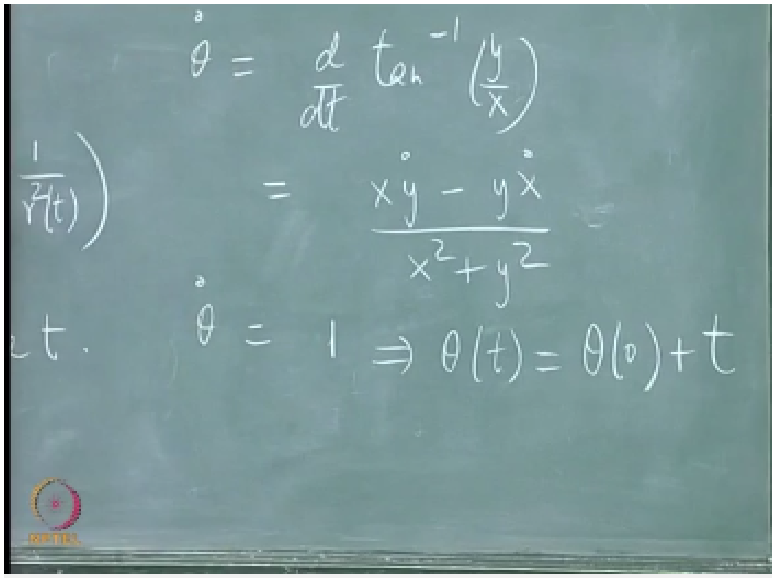

Then see θ is =if you recall this is D /DT of tan inverse y/x and of course we already have a formula for it which is =xy. - y x. / x ^{2}+ y ^{2}and in the present problem θ. works out - so I multiply this by X and this by Y and subtract these terms go out and you end up with an x ^{2}+ y ^{2}divided by the same thing so this is = 1.

So in this particular problem we see that linearization has led to completely erroneous conclusion whereas if you restricted yourself to the linear part of the system, you would conclude that the origin was a stable center it turns out that is not.

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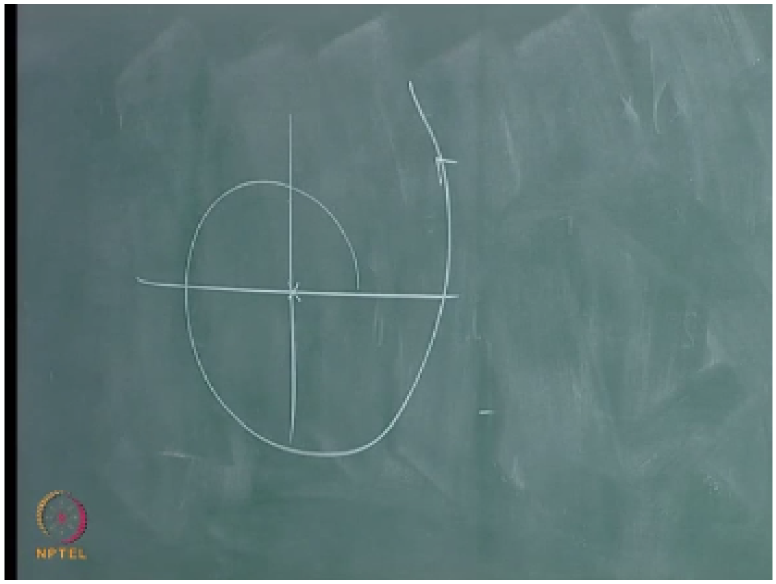

So at all and wherever you start you are actually going to flow outwards in this fashion such that you hit infinity at some finite value of time because of this specific form of non-linearity. So the origin is not a center it is an unstable node in this case and things flow out and this shows you the perils of linearization when you have a center manifold very clearly shows you that the non-linearity has completely changed the behavior r the dynamical system from what the linear one would predict the linearize form would predict.

And this is the reason I said early on that when you have a center manifold there is no guarantee that linearization produces the true flow in the vicinity of the critical point and this is a simple example which illustrates this point. Let us go on to the next topic that we had started namely we were looking at higher order critical points and I made a statement that these higher order critical points are generally formed by the coalescence of simple critical points.

And indeed we saw in the case of a saddle node how a bifurcation occurs and you get a higher order critical point at the point of an exchange of stability bifurcation. Now let us look at some further examples of this let us look at for instance.

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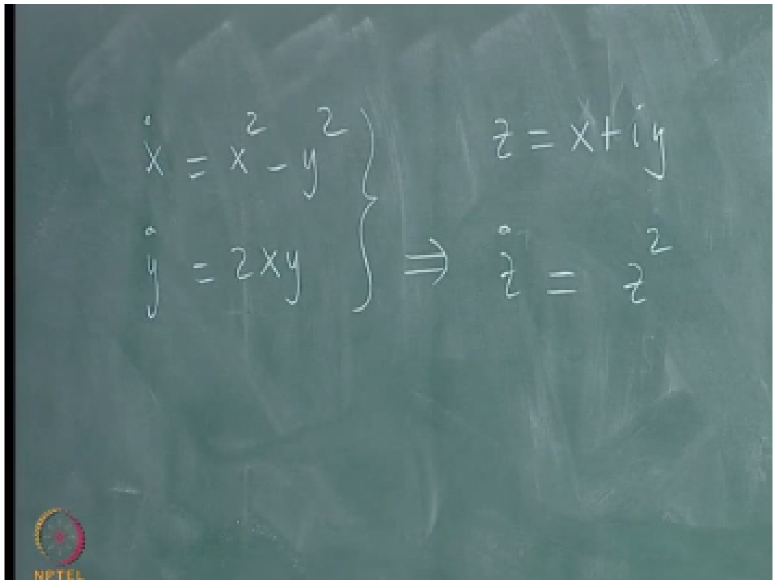

X. = x ^{2}- y ^{2}and y. is twice X Y it is immediately clear that the origin is a critical point but there are no linear terms on the right hand side and therefore this thing is intrinsically a higher order critical point at the origin. The question is what kind of critical point is it what does the flow look like in this instance what would you suggest looking at this function what does it suggest to you?

Let me give you a hint suppose you wrote z =x+ iy and regarded this as a complex variable x + iy as a complex variable what does this suggest to you. So if you set z =x + iy this set of equations → z. = m. So it is = z ^{2}itself so it is immediately clear that the origin the singularity at the origin is such that if you took a circuit once around in the Z plane once around the origin the function on the right-hand side its argument changes by 4 π rather than 2 π because it is z ^{2} .

So this brings us to the concept of the winding number of a singularity of a planar vector field and let me explain what that is.

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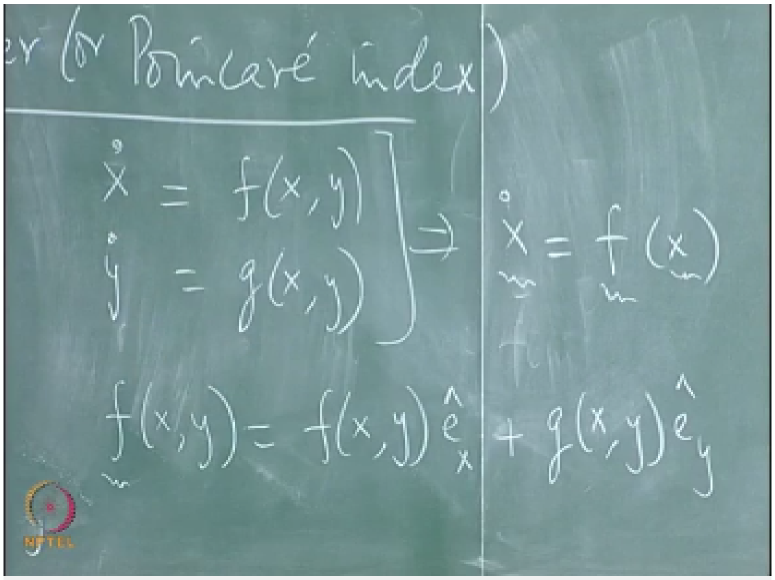

The winding number for correct index of a planar vector field namely a vector field in a plane as a function of x and y, we will come back to this, we will come back to this the statement I made was if in the z plane there is some singularity at z = 0at the origin but it is such that if I move around once in the Z plane and increase the argument of Z by 2 π the argument of this vector field on the right-hand side increases by 4 π because it is the ^{2} and this has a specific implication for what I am about to say the statement is the following.

Suppose you have a singularity of a vector field somewhere in the XY plane say the origin and the field lines around the origin looked in some complicated fashion they perhaps look like this these are perhaps the field lines, this is how it would look for instance near an unstable nodes in this fashion. At every point other than the origin which is taken to be a singularity of this vector field the vector field is unique and by vector field.

I mean the set of equations which we wrote down once again x. is f of x, y y . is G of x , y and you recall I combine these two into reading as a vector equation f of x and this vector field F has two components F and G so let me write that explicitly and write f of x , Y is times the unit vector in the x Direction + times the unit vector in the y direction, this field vanishes at the origin both F and G vanish at the origin and you have a critical point of some kind.

And now I would like to characterize this singularity by the concept of what is by the concept of the winding number of this singularity which is defined as follows if I start at any point here any arbitrary point there is a unique direction to this vector field F and it is evident that if I write this vector field F itself as a modulus times an argument this is equivalent to writing if you like this vector field as a complex number.

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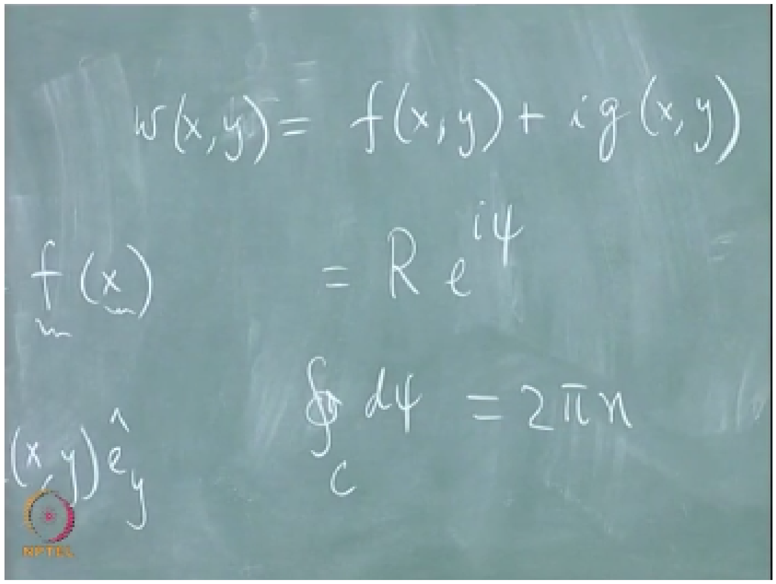

So I could write w = f of x y + i times g of x y and this w is a function of x and y instead of a planar vector field I represent it as a complex number for a 2- dimensional vector field this is only true in two dimensions to start with if I took this W which is a function of x and y and wrote this as some r e to the i where r is the modulus of w and size the argument of w then if I start at any point in the plane and make a circuit of some kind and come back to the same point.

It is evident that should return to its original value because the field at every point is unique and therefore this arguments I must come back to its original value it is quite clear or it must increase by a multiple of two π. So that e to the 2 π is unity and you do not see it at all it is therefore clear that the integral of around any such closed circuit must be an integer times 2 π, if I call this circuit C therefore I assert that integral around the closed circuit C D sine this must be =2 n π where n is some integer. Simply from the single valued Ness of this number of this argument sy which → if I go back here.

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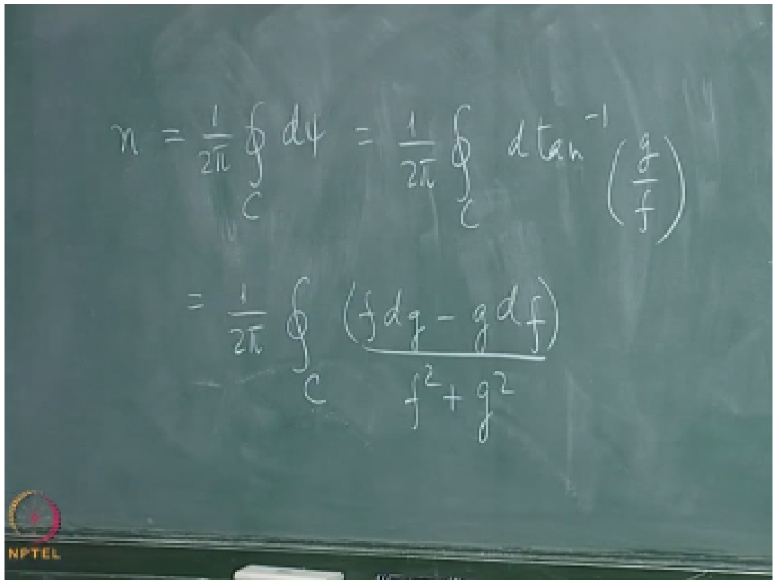

let the combination 1 over 2 π integral d must be an integer but what itself this is 1 over 2 π over C D tan inverse G over F because that is the definition of the argument of a complex number whose real and imaginary parts are given by these two fields scalar fields but this in turn is =1 over 2 π over C and if you simplify this exactly as we did tan inverse Y over X it is immediately clear that you get f DG - GD f / f^2 + G^2 we are F and G are functions of X and Y.

And therefore in principle you could write D G and D f in terms of DX and but if you integrate around any closed path in this vector field you are guaranteed as long as this function is well-defined you are guaranteed to get an integer. Now what would happen and we can see this geometrically what would happen if I took a circuit which does not enclose a singularity of this vector field namely a point where F and D vanish or the vector field is not defined. What would happen then and that is easy to see because if I simply took a region of the space.

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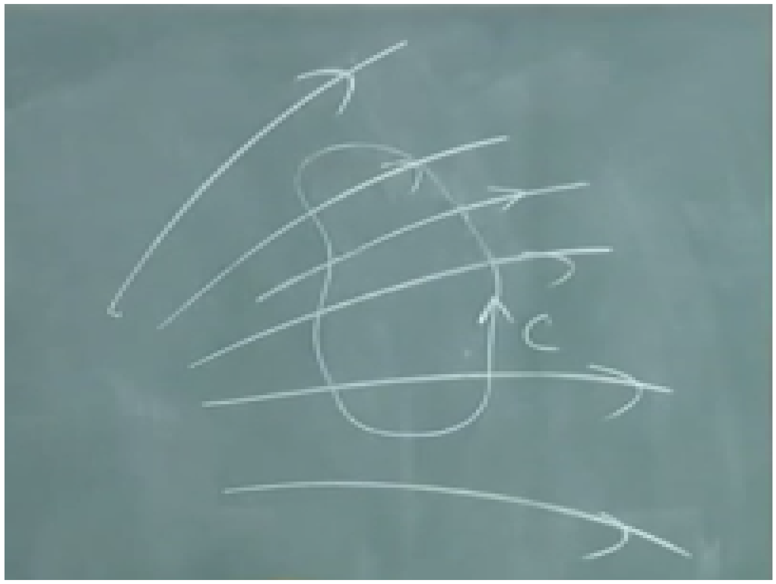

Where the vector field is well behaved and has no singularities of any kind and I start at some point where the vector field points in this direction and took a closed path and came back in this fashion we can track what happens to the argument of this vector field by looking at what direction it points in by pretending there is a little umbrella which you hold as you move along this path and ask what happens to this umbrellas direction.

I start here in this fashion and I move up there so it perhaps tilts in this fashion and then I come down and it perhaps moves this way it comes here it comes back and then it goes out like this and then, when I come back it slowly comes back to its original value. So all it has done is to take a little perambulation of this kind and back to its initial value therefore this n has not increased at all first you had a small increment in the argument and then it decreased back to its original value went back oscillated and came back to 0.

So in this case n was 0 immediately on the other hand if there is a singularity of the vector field and we went around a path which enclose the singularity then this is no longer true.

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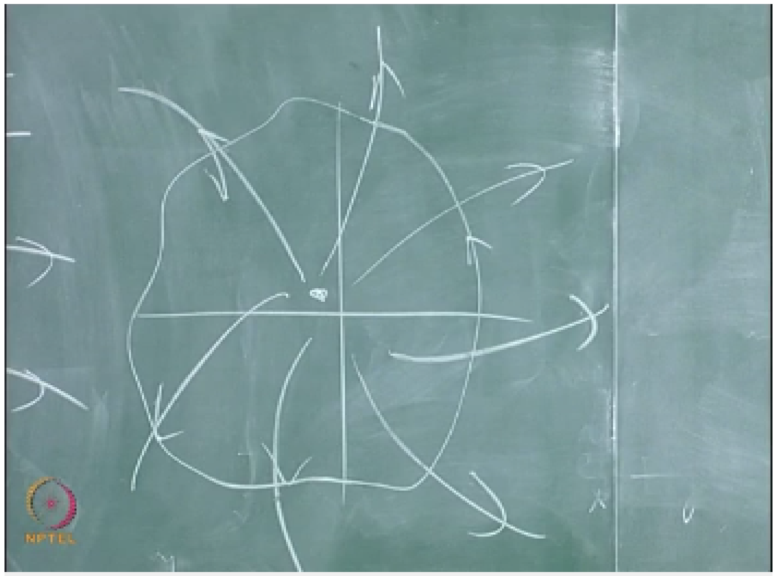

Let us take a simple example let us look at a vector field which perhaps is in this fashion it kind of radial field in this fashion and they will say there is a singularity somewhere there what would happen if I started and enclosed this path once in this fashion. Now let us follow it once again I start here in the vector field points like this in the tangential direction and let me just look at instead of moving along with this curve let me look at simply keep it fixed here and turn this arrow around to reflect what it does at various points in its path.

So it starts in this fashion by the time it comes here it is pointed like this by the time it comes here it is pointing in some direction like this like this out here it is definitely in this fashion and here it is like this here it is back there and it is back here and back to this. So in this path as I go around the singularity once in the counterclockwise sense this arrow this umbrella has also gone around once completely in the same sense counterclockwise.

Since and the N is =1 in this case because the total angle is 2π by which this vector field is rotated and I say that the singularity of this vector field at this point is the winding number is 1 + 1 this is the definition of the Poincare index or winding number of a vector field around a singularity, if it does not enclose the singularity if the singularity is somewhere here and this path does not have the singularity inside then of course the winding number is 0.

So in a region where a vector field is completely well-defined on a contour on which the vector field is well-defined if that contour does not enclose a singularity of the vector field then the winding number corresponding to that contour is 0 because the argument does not change at all it might oscillate but never completes a complete two π, yes if I start here and I do not enclose this singularity it is evident that all this vector field can do is go up like this go down like this again and come back.

It does not turn once upon around its center completely so the argument starts at some value θ_0 increases to some $\theta_0 + \alpha$ comes back to θ_0 goes to θ_0 - some θ and comes back to θ_0 it does not complete a circuit and therefore the algebraic sum of all these increments is 0 and the winding number is 0.

So you play around with this and you convince yourself that the only way in which the winding number is going to be non0 is if this contour C encloses at least one singularity of the vector field and for this radial pattern we discovered that the winding number is + 1 and now we can begin to ask what is it for other kinds of singularities what is it if you had a saddle point or a center or a node or a spiral point of various kinds.

Those were the kinds of critical points we had for two-dimensional flows and we could ask what is the singularity of the vector field looked like but before that I would like to point out to you that this numbers which we got as one for this kind of field is independent of this contour see I could have started here or I could have started there and gone around. The fact is independent of this contour see as long as it encircles this singularity once in the counterclockwise sense the increment inside is also guaranteed to be 2π.

It is therefore a topological property which is not a property of the specific contour C but rather a property of the singularity itself it is very similar to Cauchy theorem in the calculus of residues it simply says if you applause contour which encircles a pole a simple pole of a function and the function of a complex variable Z, then the line integral f of Z DZ of this function around the singularity is reduced to two π i times the residue at this point.

And it is independent of the actual contour as long as the contour encircles the singularity once in the positive sense. Now this statement is independent of the contour also the direction of the contour because if i started here at this point you could ask what's to stop me from doing this i go around once but then i do all kinds of meanderings here and come back and hit this once again it still does not matter all these increments would cancel out and the net result would again be 2 π as long as you enclose this once in the counterclockwise sense.

What happens if instead I started with this kind of radial field moving outwards and I decided to traverse the contour in the counterclockwise sense I did this instead what would happen now? Well it is clear that again if I go around in the counterclockwise sense in the clockwise sense sorry in the negative sense then this vector would also rotate in exactly the same sense as the contour and the winding number is again + 1.

So it is a property not of the specific contour but rather of the singularity that is being enclosed and that is why it is so important so it is remarkable that if you have some functions of x and y which are well behaved in a certain region, except for a singularity at some point where F and D vanish then this combination of algebraic quantities is guaranteed to be independent of this contour C provided it encloses the singularity and is an integer.

Yes it does not make it itself so the statement is if I go around once in the positive sense in what direction does I increase it increases by + 2π, if I go around once in the negative sense it increases by - 2π therefore the statement is if increases by the same amount as what happens when I go around once then the winding number is + 1. It should be independent of this C so I change by going around once like this if this is a complex Z plane the complex plane in which X and y are the real.

And imaginary parts the argument of Z increases by 2π once in going around this way and the arguments I also increases by + 2π, so the winding number is + 1 if I went around like this the argument of sight decreases it changes by - 2π and so does therefore the winding number is again + 1 it is independent of the sense in which I describe this contour. What happens if I took other kinds of singularities well let us look at some of them.

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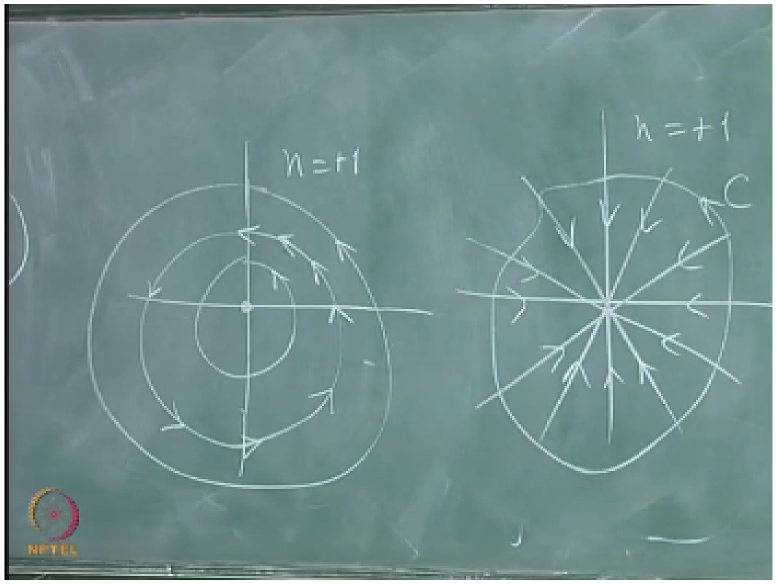

What if you had a singularity which is like what happens in the case of a Center, so we have field lines which go round like this is what the field lines are for this vector field and let us put a sense on it what would be the winding number corresponding to this singularity at the origin well again you do exactly the same thing start at some point the vector field looks like this and go around and follow this trajectory all the time.

I start here in this fashion when I come here I am like this when I come here I am like this when I go here I am like this like this back to this back to this therefore, the winding number is again + 1 n =1 what if I had a radials outward field well n was =+ 1 we saw but what if I had a radials inward field what if I had something like this these, are the field lines what happens if I did this what would happen?

Now again we do the same thing I start here at this point the field points, so then I am going to go around in this fashion so it is when I am here it is like this when I am here it is like this when I am here it is like this like this and then when I am here I am back here and when I come back to this point I am back here. So as I go around once in the positive sense this field also goes around once in the positive sense and the winding number is again + 1, so we see that at a node the winding number is + 1.

Regardless of whether this node is stable or unstable therefore the idea of this winding number does not say much about the stability of the critical point but it says something about the local geometry of the vector field at this point for this as well it is + 1. So is it for that and it is so even if the arrows are pointing outwards does not matter at all so for a spiral point a node and a center it is easy to check regardless of the stability of the center or this of the nodes or the spiral points the winding number is + 1 always.

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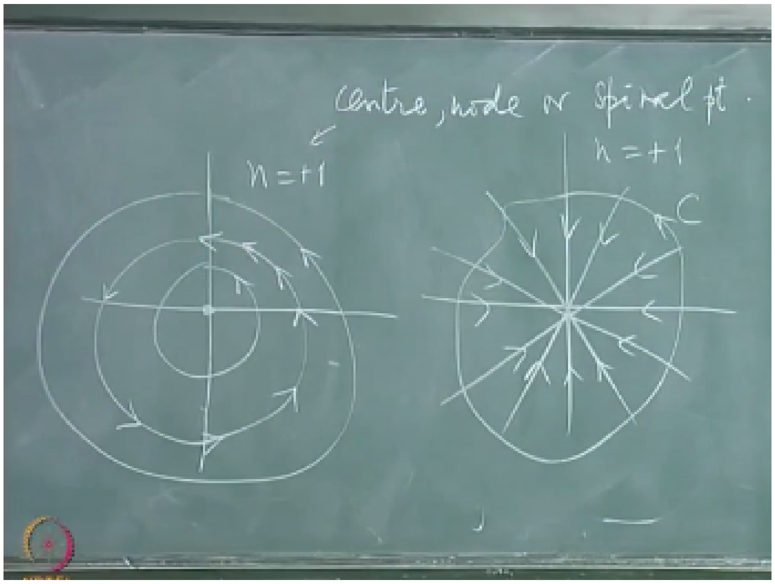

What if you had a saddle point at a saddle the vector field let us say this is a saddle point two lines in two lines out and let us say these guys go out in this fashion, in this fashion what would be the winding number around the saddle point well I start with the same trick as before I make a contour of this kind now look at what happens? When I am here the field points, so when I come to this point what does the field do?

So it starts here and by the time I come here to this point the field does this, so please notice I am moving in the counterclockwise sense along this contour but the field on the other hand is moving in the clockwise sense, so it does this does this my time it comes here it is done that and by the time it comes here back again it is done this. Therefore it goes in the other direction so while the argument of Z increases by 2 π the arguments I decreases by 2 π it becomes - 2 π is the change.

And therefore the winding number in this case and =- 1 for a saddle but it is + 1 for a center node or spiral point regardless again of the stability of these points, what that suggests is that what looks like a radial flow in one region can actually be uniformly deformed to look like a tangential flow as you move out, this is certainly possible because this is what topology is all about you could start with a flow.

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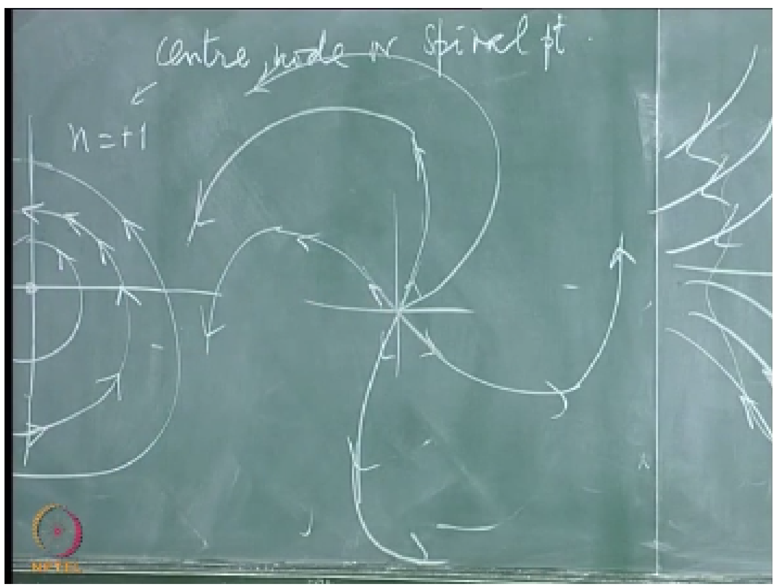

Which perhaps near the origin is radically outwards let us say and as you go up it starts curving and as you go further out it starts getting more and more curved completely smoothly, so that it is clear that the flow eventually could even look tangential if you are far enough away from the origin. So this suggests that flows which look like sources or sinks could be made to look tangential by smooth changes in a smooth manner without actually go crossing any singularities and that is the reason why the concept of a winding number did not distinguish between centers nodes or spiral points.

On the other hand a hyperbolic point the saddle point is very different and there is no way in which one of these flows which correspond to any of these singularities can be deformed by a smooth change of variables to look like this not possible. So it is of limited use but it gives us some hints as to how vector fields behave what would happen if you had more than one similarity.

Inside I made a statement that the winding number if you enclose a singularity you are guaranteed that the winding number corresponding to the singularity is not 0, okay on the other hand if I have a closed contour and I discovered the change in this argument as I come back is 0I cannot conclude that there are no singularities in sight for exactly the same reason, that I cannot do that in complex variables because you might have two poles whose residues cancel each other just as you might have two charges in Gauss's theorem.

It simply says the total flux across a closed surface of the electrostatic field is equal it is proportional to the sum of the algebraic sum of the charges inside and you could have two charges whose fluxes could actually cancel as you took the full integral in, exactly the same way it is possible that if you had a closed contour in which you had both for instance a node as well as a saddle point then the - one of the saddle point and the + 1 of the node could add up to give you a 0 and this is not difficult to see let me give you an instance right away where this happens.

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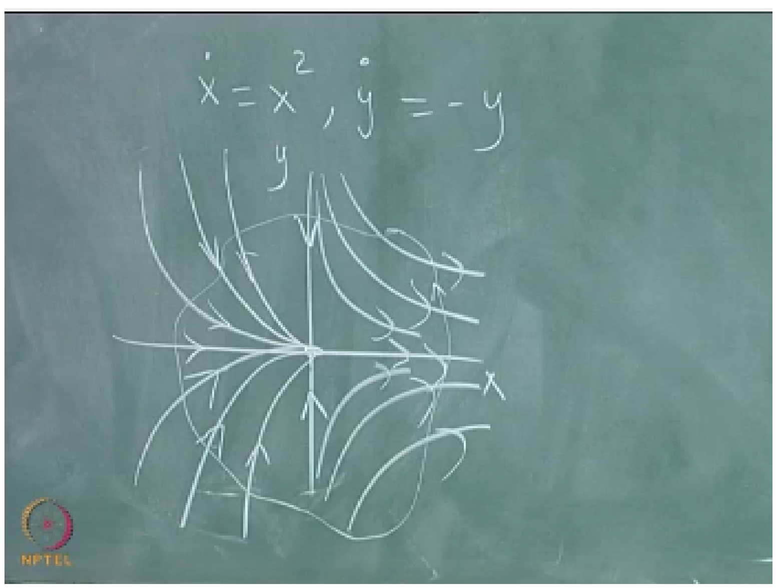

We looked at the example of a saddle node which was x. as x ^{2}y . =- y and if you recall the flow here was along the positive x axis, here and along the y axis it was inwards in this fashion and this side looked very much like a saddle point, so the flow was like this and out here the flow rode into this point in this fashion. And the question is what kind of winding number this field has around this point.

Of course we unfolded the singularity and we discovered it really came about by coalescence by the coalescence of a saddle point with a node but you can unfold we will put this back together in this fashion and it is not hard to see that if you took a console and went around this contour see. The net change in size is in fact 0and what is happened here is that the + 1 of the known and the - 1 of the saddle have added up to give you a net change in the arguments I = 0.

You cannot conclude based on that that there is no singularity of the vector field there very much is a singularity but the some of these winding numbers has added up to 0 in this case. So the statement is if you discover that the winding number around a closed circuit is non 0 there exists at least one singularity inside on the other hand if you discover that the winding number as you do this integral around the closed circuit is 0 you cannot conclude there are knows necessarily conclude there are no singularities.

Inside there could be a set of singularities whose net winding number is 0 and now let us go back to the example we started with which was essentially that . =Z ^{2} and now this will make sense right away I do not even have to draw a picture because the flow.

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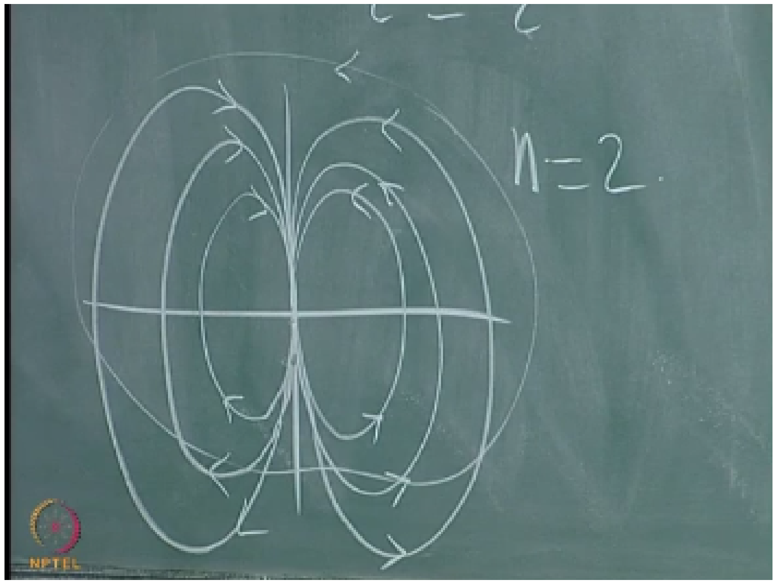

X . is $x^2 - y^2$ y . is 2x y could be combined into Z . $=Z^2$ and the statement I made was simply but if you went around once in the Z plane in the counterclockwise sense then the argument of Z increased by 2 π but the argument of Z^2 obviously increases by 4 π which means that this vector field which was my W has a winding number at the origin of 2, because the angle increases by 4 π and what is the field itself look like.

What does this field look like there's a complicated singularity here and it is a dipole field in this case so the field lines look like this it is exactly, what a point dipole would do when I leave you to verify that if you took a closed circuit around this origin here then the net change in the arguments I would be 4π provided you traverse this circuit once in the positive sense. This is what a point dipole does this is what the magnetic field lines of a point dipole the electrostatic field due to an electric dipole look like.

And what it is and why is it - physically why is this can you tell me this from your experience with electrostatics why is this - exactly there are two charges in there. One of them acts like a source and the other acts like a sink if you take a single point charge if it is positive then the field lines are radially outwards that looks like an unstable node and if you took a negative charge the field lines go directly in words and that looks like a stable node asymptotically stable node.

You put the two together arbitrarily close to each other such that the distance between the two vanishes and the product of the distance multiplied by the charge is finite the charge becomes infinite such that the product is finite you get a point dipole, which looks exactly like this, so this is characteristic of a dipole field and the winding number n =2. In this case well your ceiling fan acts like a dipole field for the velocity field.

If you assume this fan the just the central portion of this fan is like a point source then it is sucking in air from above and pushing out air from below and there is circulation in this pattern so that is a simple example of a dipole source for the velocity field but the charge example is more familiar to you from electrostatics and this is exactly what it looks like. So this is one way in which you get some handle on higher order singularities by examining what the Poincare index of the vector field looks like gives you some hint.

As to what is going on let us go on, now to another model it is very useful and very common and this helps illustrate a little theorem I want to talk to you about regarding limit cycles. I pointed out that limit cycles do not exist in conservative systems but only in dissipative systems the reason is if this limit cycle is for instance stable then it says a whole lot of points in its basin of Attraction get attracted to it exactly, as in the case of a critical point which is an asymptotically stable critical point.

And such attractors do not conserve phase space volume because the whole area falls in into a line or a point and therefore it cannot occur in conservative systems which we have defined as those systems for which the divergence of F is 0 everywhere measure preserving flows. Now let us look at a famous example of a nonlinear oscillator and ask whether in the presence of non-linearity and dissipation you might perhaps have limit cycles and this model is called the Duffing oscillator.

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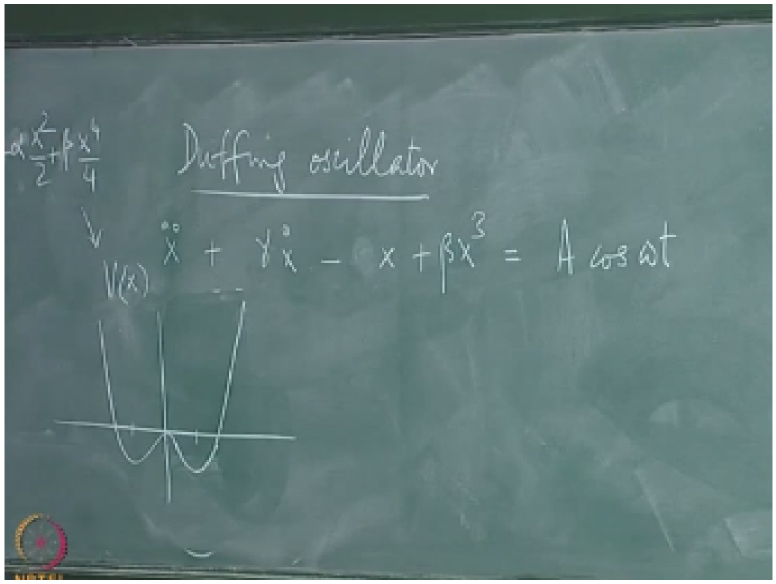

And it goes as follows it is a second-order differential equation that specifies the Duffing oscillator but we will interpret this in physical terms and the equation of motion is X double. + Some friction term gamma X. and then terms which would correspond to non-linear oscillations and this corresponds to oscillations, in a double well potential the potential looks like this versus X and as we know this potential is described by a fourth order potential function which has a maximum at this point at the origin say and to equal minima on either side.

So it is like having an inverted parabola near the origin and then moves out on either side so a model for V of X would perhaps be - x^2 over 2 + x 4 over 4 with some constants multiplying these two cases and with that kind of choice of origin the picture actually looks like this, so that the potential vanishes at the origin and is symmetric has two minima on either side of it. So little inverted parabola and then a fourth order term which takes you up in this fashion.

What would the phase trajectories here look like we will come back to this in a second they would correspond to oscillations here or here or oscillations in D over both across both potentials both wells so this thing here has a term which is - some constant times X which would come by differentiating this point I put an α there. So let us put an α here + perhaps a β X cubed which would come by differentiating this term and putting a - sign and this in general could also be driven.

You could also take the system and put an external force upon, it which could instance be sinusoidal. So perhaps on a cos ω t and you could now ask what about the dynamical behavior of the system and let us interpret these terms once again I have divided through by the mass of this oscillator, so this is just X double this term represents linear π proportional to the velocity instantaneous velocity of the particle these two terms come from the restoring force represented by the potential.

The fact that you have a - sign here comes from the fact that this is an unstable point the origin is actually unstable and this beta X cubed with positive β comes from the fact that you have two stable minima on either side it comes from this potential and that is driven by some external force with frequency ω with amplitude. How many parameters are there in this problem well there is one here, two three four the amplitude of the driving force and five.

With five parameters you have an extremely rich set of possibilities for the dynamical behavior of the system but we could choose the scale of time, so as to get rid of one of these constants and let us choose the scale of time in such a way that this thing here is unity you still have four parameters λ β a and ω and in the parameter space of these four constants these four parameters you actually have many possibilities and the full set of possibilities of the Duffing oscillator can only be understood numerically and it includes chaotic behavior of various kinds it is extremely complicated.

Is this an autonomous system or non autonomous it is non autonomous you are right because there's this forcing term here, so there is explicit time dependence so very complicated things could happen in this system. Let us look at the simplest version of this in some form in which I do not have a forcing at all but let us look at the case X double. + λ X . - X + let us just put this beta =one this =0 and ask what about the unforced dressing Duffing oscillator.

That corresponds to motion in this potential but in the presence of π, now I could go ahead and ask alright suppose you did not have this π at all what does the what do the phase trajectories look like and that is very straightforward because here's what it does.

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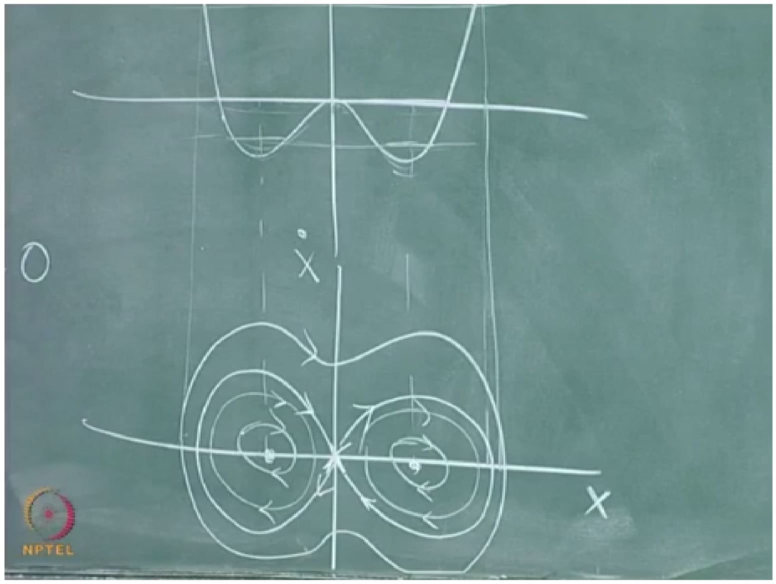

Here is the potential and if I plot X here versus X. for instance on this side then the phase trajectories are very straight forward, you have a little center here you have a center here and you have a hyperbolic or saddle point out there. For small enough energy just above the minimum of this potential you could have stable oscillations about either of these centers and therefore you would have trajectories of this kind for.

Slightly larger amplitude oscillations you would have non harmonic oscillations you would still have this kind of behavior. On the other hand if you had enough energy and you oscillated with this total energy it is clear that the particle could cross this barrier. Go to the other cycle come back and oscillate with this amplitude and it would do so it is something which looks like this amplitude here.
\
So it would certainly do something like this it encloses both the words with the minimum and what is the critical value or separately value of the energy in this case it is 0 itself at this value it is evident that if is start here and give it a little perspiration to the right it would start here go down like this and come back and form a homo clinic orbit which is a separate R ix on which the time period would diverge.

Similarly on this side you would have a symmetric thing which looks like this and you have your saddle point at this point and these are the separate races we can easily find the tangents to these separate races in this model but the actual stable and unstable manifolds are like this it is in a really homo clinic orbits an outside you have oscillation across both wells and inside you have oscillation around this well for that matter.

This is what would happen if you had this system without the γ without any damping at all but now the question is interesting question is I switch on the γ I have a finite positive value of γ what do the phase trajectories look like it is not easy to draw this immediately because I could start at some point in the space and then it would start by doing this but because of the damping the amplitude would keep reducing and eventually it would find itself stuck either at this equilibrium point or at this equilibrium point.

This separatrix is no longer there once you have friction that is the whole point this is only two if you have no friction but once you have friction this is no longer true so I start with this energy perhaps it goes around the first time and then it comes back it slows down it is below this and then it stuck here and oscillates here on the other hand depending on what it does whether it crosses this for the last time to the right or to the left it could any starting point would fall in the basin of attraction of either this attractor or that attractor.

And these would actually be spiral points as an automatically stable spiral points so unlike the non dissipative system in the absence of γ where you have two centers and a saddle point in between in the dissipative system you have two asymptotically stable four spiral points and where a given initial point Falls finally is dependent very crucially on where you start and it turns out actually that in this model and subsequently I will try to show you some pictures of this the basins of attraction of this point and that point they riddle each other the kind of interleave with each other.

So you could perhaps start here and end up there but you start here and you could end up here in this fashion they actually fold around each other in a very intricate fashion and I can only do this numerically at some level but you also can ask the interesting question can this system with the damping switched on have a limit cycle somewhere what is to stop me from doing that after all I start by saying look there is a non-linearity in the problem due to X^3 and there is dissipation so is it not possible that there exists actually a stable limit cycle there is some isolated a trajectory on which you have periodic motion and everything within it falls into some asymptotically stable spiral point.

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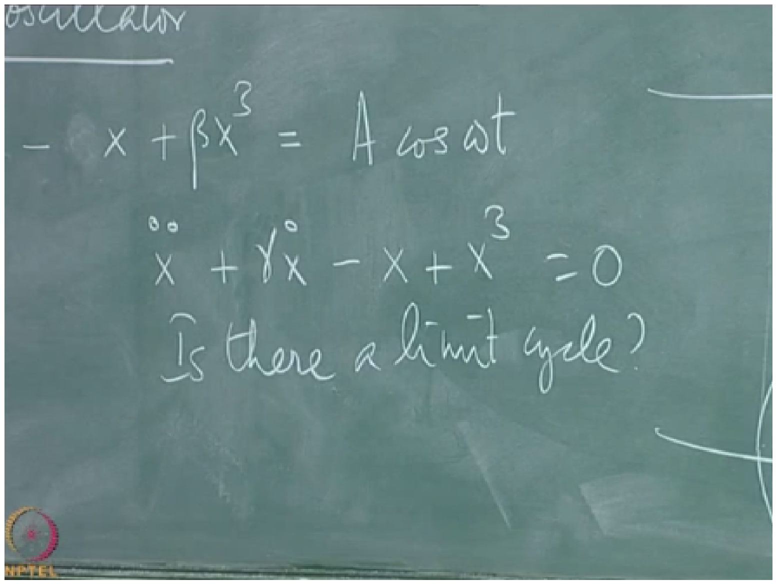

This could happen and we would like to find out if this is true or not so the question is there a limit cycle in this system is there a limit cycles and there is a criterion called the bendixson criterion which says there is not in this problem and it goes as follows let me stop with that.

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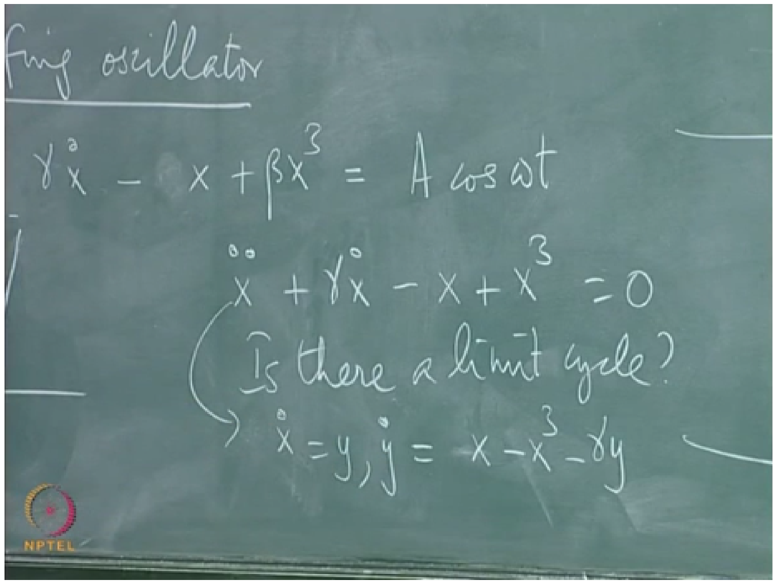

So you have the bendixson criterion which reads as follows I start with x dot is f of x ,y y dot is =g of x, y which is exactly what the system is like because recall that this implies x dot =y and y dot =- =x - x^3 - γ y so y double dot is why dot the same as y dot and that is = this and the criterion says the following it says if you have a vector field of this kind a dynamical system of this kind and you consider this in some domain in the xy plane.

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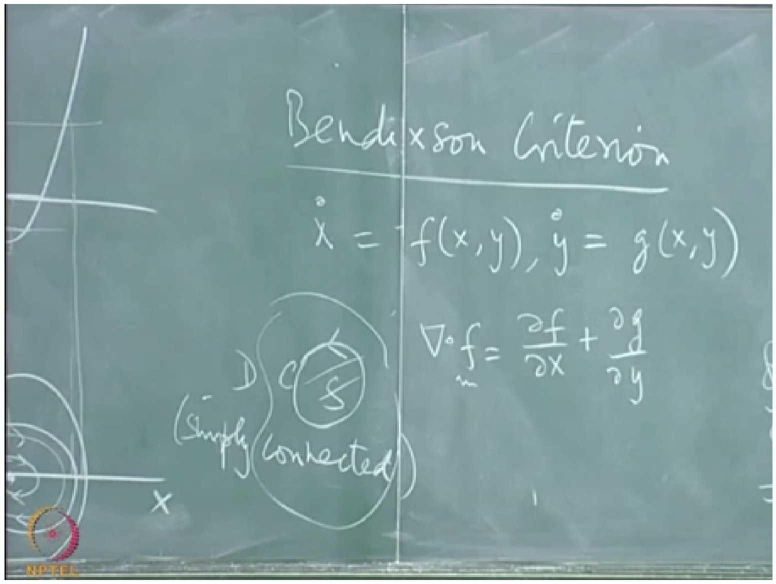

Some domain D which is simply connected no holes there are no holes in this no singularities in this domain and in that domain D F and G are continuous and have continuous partial derivatives first partial derivatives and moreover if $\delta F/\delta x + \delta g/\delta y$ has a definite sign at every point in other words is always positive or always negative in this region and this remember is just the gradient of our vector field F in this case so if this has a definite sign always positive or always negative then there can be no closed trajectories lying in this region in D.

Therefore it would say that can be no limit cycles either which is a closed isolated periodic orbit the proof is very simple if there is such a trajectory then on that trajectory y dot is g of x,y and x0 is F therefore dy which could be written as dy/dx times dx would be =well dy is g/f dx and on this trajectory it immediately follows that f dy - g dx =0 on the trajectory and if I integrate/this trajectory this must be =0 so if you had a closed trajectory of this kind inside this domain D.

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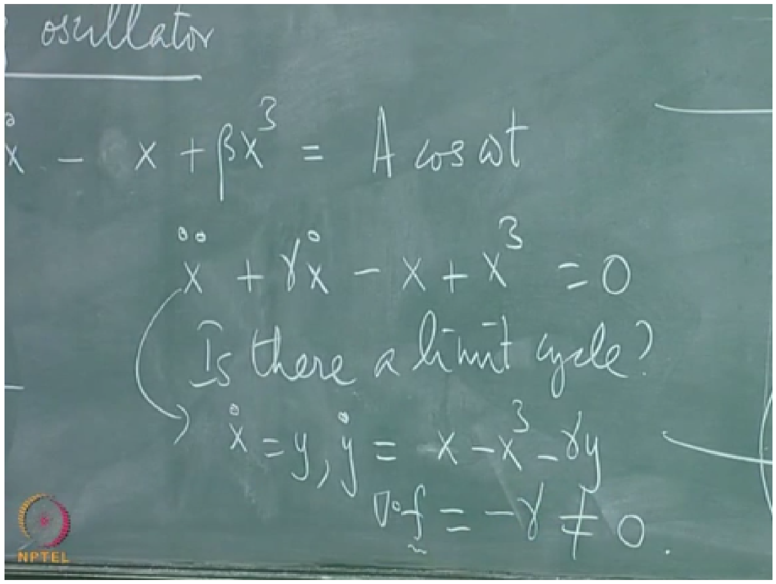

That must be true but my greens theorem on the plane this quantity is also equal to/the surface s bounded by this C it is also equal to/s δ f/δ x + δ g/δ Y dx dy but that contradicts our initial assumption if this function never vanishes has the same sign throughout s then this cannot be zero on the other hand it must be = 0 if you had such a closed trajectory so this forbids you from having a limit cycle if the bendixson criteria is satisfied what is the gradient what is Del dot F in this problem it is the derivative of this with respect to X which is 0 + the derivative of this with respect to Y which is - γ and that is not 0.

Therefore we conclude that by the bendixson criteria the unforced Duffing oscillator linearly damped Duffing oscillator cannot have a limit cycle it is an illustration of a fairly powerful theorem which works for these planar vector fields otherwise you would have to examine numerically.

And we are never sure whether there could be such a case or not a limit cycle or not some isolated periodic orbit but this tells you no matter what your initial conditions are you are not going to have a limit cycle you are always going to flow either into this or this attract attractor point attractor eventually a very useful criterion and we will see more properties of this next time.

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