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TOPICS IN NONLINEAR DYNAMICS

Lecture 11
Limit cycles

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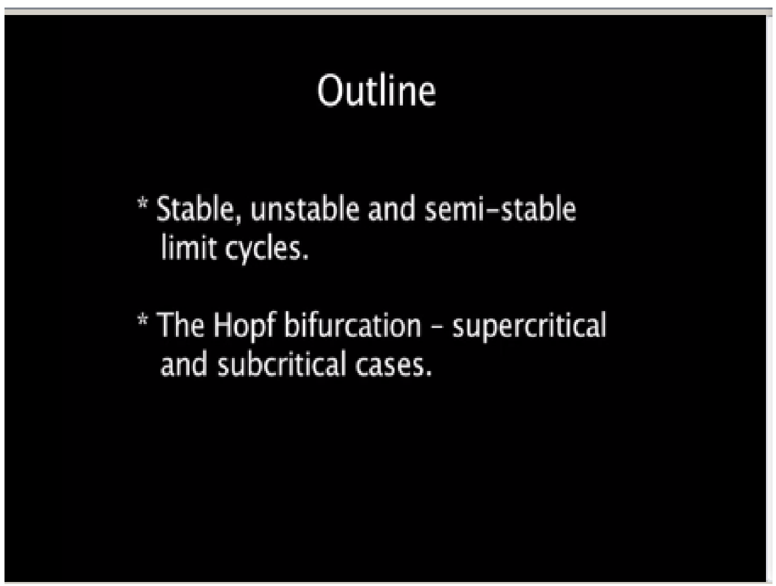

Let us consider the possibility of critical points which are like lines now in some sense and. I am going to do this with the help of an example we look at a simple nonlinear system where this happens again in two degrees of freedom to two dimensions

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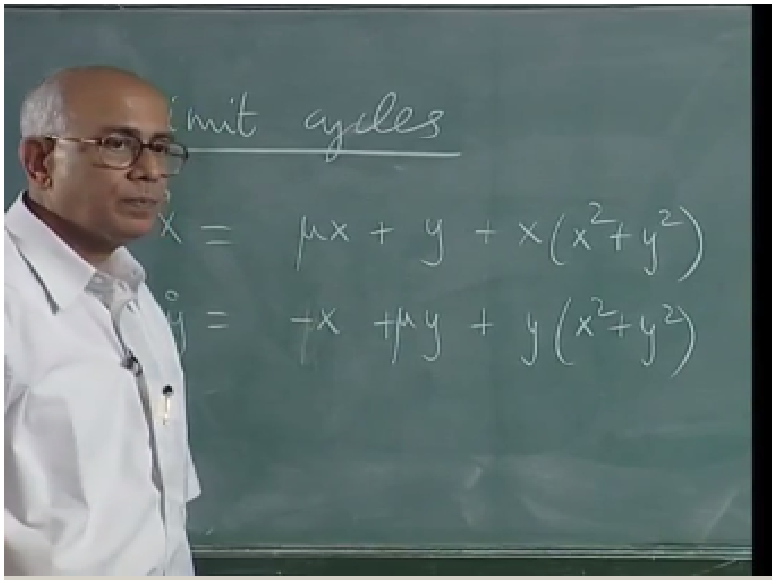

So let us look at X. = μ X + y + x (X ^{2} + y $^{2)}$ y. + μ Y so let us put an X here +Y with a μ / μ is a real parameter so this portion is a linear part of it and that is the nonlinear part of it and it is immediately obvious from here that you have critical points.

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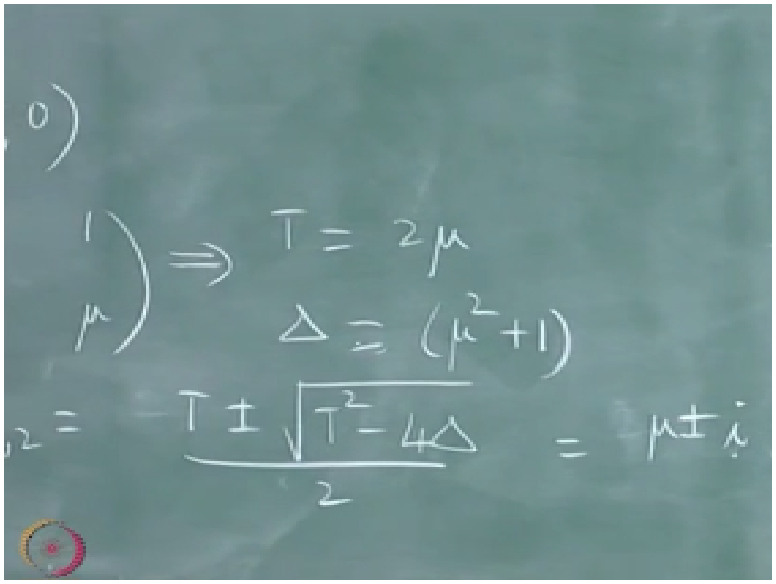

You have a critical point certainly at 0 0 both these guys vanish and what sort of critical point is it if you linarite near the origin this L = μ a 1 - 1 here and a μ here which ⇒ that the $T = 2\mu$ and the $\Delta=\mu^2 +1$ in this fashion and remember that λ 1 ,2 =- T + -$\sqrt{}$ T^2 - 4 Δ /2 +T =2 + - $\sqrt{}$ T^2 which is 4 u 2 -μ 2 - 1 so it is 3 μ 2 our four new 2 - 4 Δ so that is 4 i2 – 4/2 so this cancels out here and this just gives you two I so μ + - I so what kind of critical point is this depends on the sin of M but since it is complex conjugate as a pair of complex conjugate critical points its eigenvalues.

It is clear that the critical point is a spiral point so this immediately says that it is an asymptotically stable

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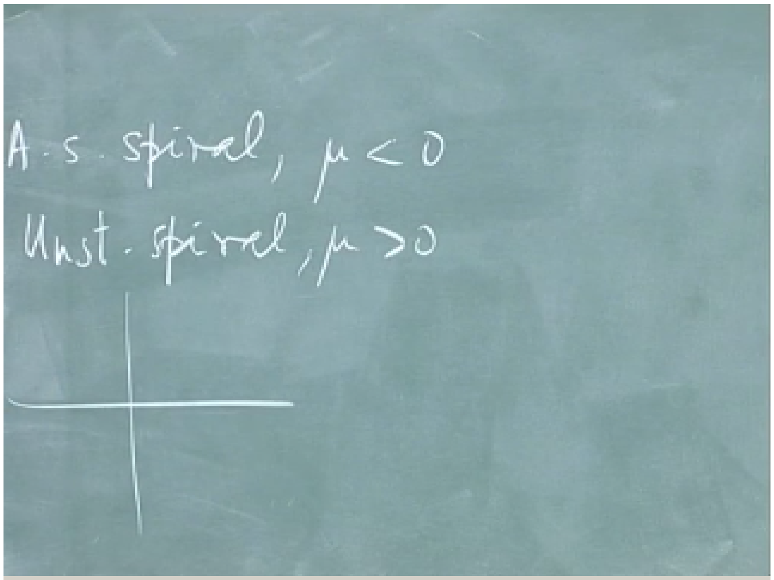

Spiral form $\mu < 0$ and an unstable spiral $\mu > 0$ at the origin and the phase picture as you can imagine is a spiral going inwards into the origin from $\mu<0$ and away from the origin outwards from $\mu > 0$ we should like to know what the direction of the flow is as well for that let us go to polar coordinates plane polar coordinates so I said in the usual way $X = R \cos \theta$ $y = R \sin \theta$ which immediately ⇒ that.

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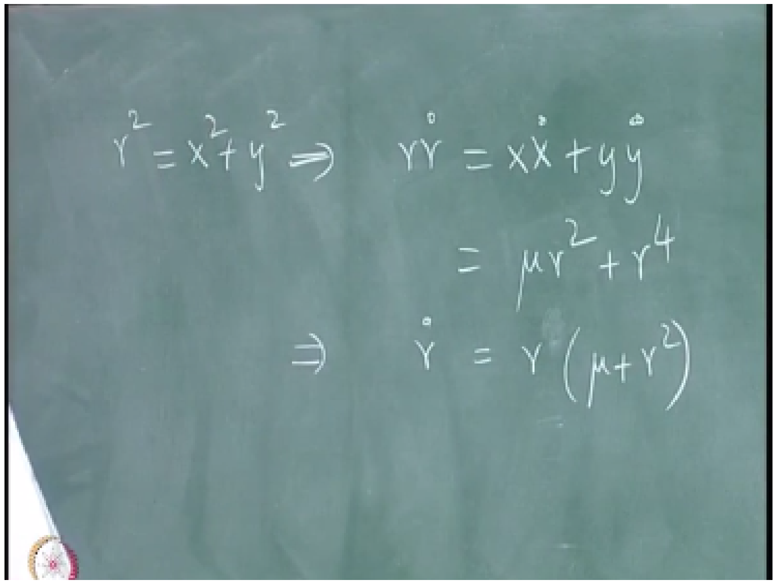

R^2 is $x^2 + y^2$ which ⇒ differentiate both sides are r . = X X . + YY .and that becomes in the present instance I multiply this by X and this by Y add the 2 and I get μ times $x^2 + y^2$ so μ times $r^2 + r^4$ which comes from these two terms and then this gets multiplied by X and this by Y when you add the two it vanishes so this gives me r^2 (μ + $r^{2)}$ in this fashion which ⇒ that r . Is r times μ + r2 therefore the picture which says that from you <0.

This is an asymptotically stable spiral also shows that from μ <0 there exists a trajectory corresponding to r√- μ on which there is no change of are at all and since r .=0 the trajectory is a circle of some kind on the other hand if μ is positive this is an unstable spiral nothing happens everything flows off to infinity we would like to know the direction in which things flow and it is evident immediately that we have to draw two different phase portraits one for μ <0 and the other from μ >0 .

So let us do that let us plot this for μ positive on this side and on this side from μ negative what kind of behavior do you have from μ negative from μ+ this is an unstable fixed point a critical point there on the other hand its stable so μ < 0 but it is also clear for new less than 0 r = r times R 2 - modulus μ which is = 0 for r = √of models therefore from μ < 0 there exists a circle of this kind on which you do not change our at all therefore this is a trajectory by itself on the other hand you have asymptotically stable spiral here.

And outside here it is evident from this picture that if r > √ μ r. is positive therefore the flow is outwards off to infinity on the other hand if r <√ of mod μ the flow is inwards R decreases towards the origin towards the stable fixed point at the origin the picture is very different for μ+ there when μ is positive this can never banish so you do not have such a root at all so it is a solution at all and it is clear that are always increases the only question is in what direction do things happen and to find that we need to find out

What θ does so let me write that down retaining this equation θ = tan inverse Y over X which ⇒ that θ .= 1 over 1 + y^2 over x^2 multiplied by the derivative of this which is y dot over X - y X dot over x 2 and that simplifies to XY .- y X ./ X^2+ y^2 if I go back to these equations of motion I multiply this by X and this by Y and subtract if I multiply this by X I get - x^2 so let us write this down.

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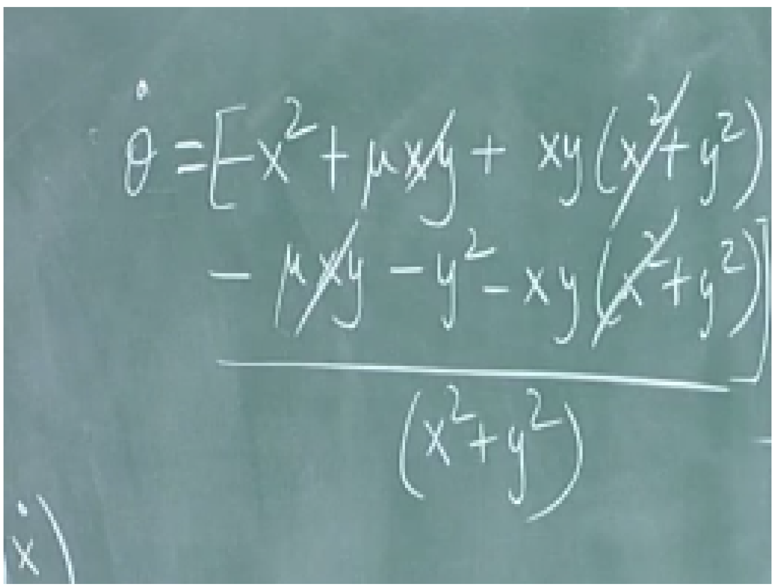

So in the present problem θ .= - x 2 - + μ XY + XY times x^2 + y^2 - this is going to be multiplied by Y and subtracted so - μ XY - y^2 - XY times x^2 + y^2 that is the numerator and the whole thing is divided by(x^2 + $y^{2)}$ and it is immediately clear that these terms cancel out and you end up with is = - 1 which means on that circle θ decreases at angular velocity =1 in the clockwise direction because it is decreasing so this means it does this on the server therefore a point starting in the neighborhood of R=√ of μ this circle had a radius R =√ of - μ what happens to a point in here it is clear that it is going to spiral in and fall into this fixed point as similarly.

It is evident that if I start a little bit outside I am going to spiral away towards infinity outwards that isolated closed orbit is called a limit cycle so let us define it this limit cycle and isolated limit cycle is an isolated closed trajectory it is important to notice that it should be isolated in other words in its neighborhood there are no other closed trajectories there exists a neighborhood of this circle in which there are no other closed trajectories in this problem there is not one in any case anywhere else except here notice.

On contrast that near a center if this were a center you would have a whole family of closed trajectories of this kind arbitrarily close to this trajectory closed orbit there are other closed orbits and this is not a limit cycle on the other hand in this situation it is clear that this is a very special trajectory everything inside falls away from it everything outside falls outwards away from it therefore you would intuitively see that this is an unstable limit cycle you could have a situation where everything falls.

Into it then it would be a stable limit cycle you could have a situation where things fall into it from one side but fall away from it on the other side of and then it would be an a semi stable limit cycle but the general situation is that the limit cycle is defined as an isolated closed trajectory it should be obvious immediately that you cannot have a limit cycle in a conservative dynamical system because this means that whole sets of points are falling into this point here whole sets of points are moving away there.

So there is no question of conservation of phase space volume here at all a limit cycle therefore a stable limit cycle is an attractor which occurs only in dissipative systems no possibility of limit cycles in a Hamiltonian system in particular and in conservative dynamical systems in general we have defined conservative dynamical systems as those for which the phase space volume is preserved under the flow limit cycles therefore are a feature of dissipative systems in this instance the limit cycle was actually unstable but let us look at a slight modification we tweaked this model a little bit till we get a stable limit cycle what would you need for that we would like to have things go into it from both directions what would you require for that what should we do.

To this model we change things a little bit here so that we end up with stability there what would you suggest well had the situation been reversed namely had this become unstable and this becomes stable then the job would be done right all we need to do is to ensure that this becomes stable and this becomes unstable what should I do therefore upon me - me okay we put a - μ X then what for a - here - here this do the trick well let us try write so what do you think happens well the liberalized matrix near the origin is now - μ a one here a - 1 here and a - μ here.

And the trace is = - 2 μ the determinant is still μ 2 + 1 so λ1 , 2 = - to M + or - $\sqrt{}$of 4 μ 2 - 4 μ 2 - 4 so this becomes a 2 I so it is just - μ + or - I what next and what happens to the equation 4 rr .t what happens here 4 rr. pardon me where should I put a - in the nonlinear term I should put a - sign here this is not going to work we will try let us write exactly I am just replacing number - μ exactly simply replacing μ by - μ so do you think anything is going to happen maybe not okay so what is his suggestion what do you suggest

What do you think I should do all of them should have negative signs this should remain as it is the non linear term should have a negative sign okay we put a negative sign here and we put a - sign there the linear the Eigen values are not changed so let us draw this picture all possibilities can occur so let us draw what see what happens now it is evident here that these are the eigenvalues the way I have written it here because this is not affected does not affect the linear terms and it says that when μ is positive.

You end up with a spiral which is stable and when u is negative you end up with a spiral which is unstable right yes okay, so let us go back to our original way of looking at it we had this and we change the sign here in which case this is μ and that is μ so we go back we do not change anything now it is this and it is this so this does not change it is an asymptotically stable spiral form you negative and an unstable spiral from you positive but the equation for rr. That definitely changes.

Because this immediately says that rr .= multiply this by X multiply this by Y and add the two and you get μ times $r^2 - r^4$ this fashion which $\Rightarrow$ that r .= to Is take out an $r^2$and calculus of the R so that R times $\mu - r^2$ in this fashion so once again we see there is a limit cycle at $r = \sqrt{\mu}$ but this time for positive values of μ so what does the picture look like this for negative values of μ and this for positive values of μ now I have X as well as Y.

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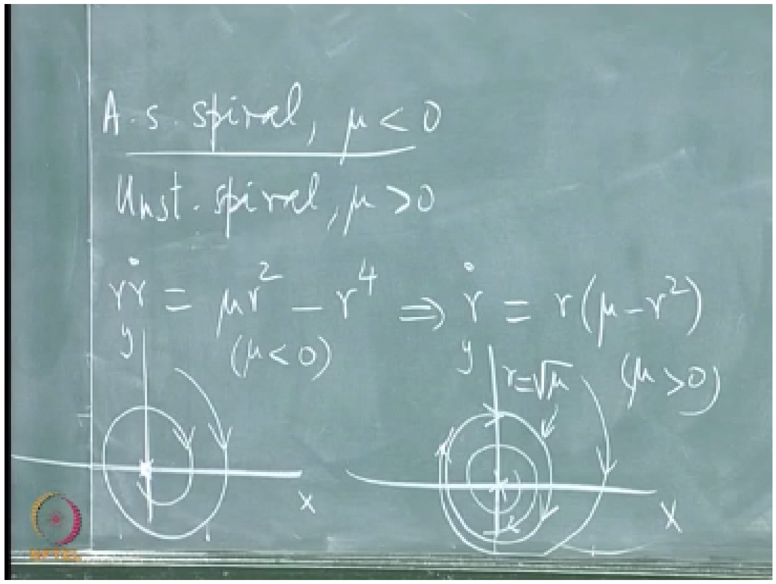

So I really need to write down a proper bifurcation diagram but let us first draw what happens in the phase plane here is X here is Y here is X and here is y and this corresponds to μ positive this corresponds to μ negative and for negative values of μ there is no root here our dot does not vanish anywhere other than or =0 and this is an asymptotically stable spiral and we should also check what θ .does.

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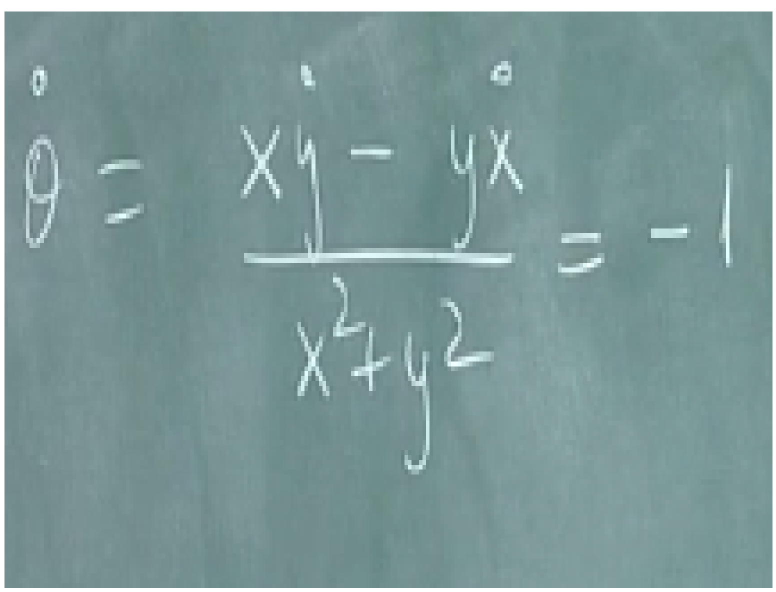

So let us do that see the. = XY. - Y X. / x 2 + y 2 and I think it is not going to change. So I multiply this by X I get - x 2 + XY they multiply this by Y I get a μ XY and I put a - sign so it is – r^2 - 1 the nonlinear terms cancel out in any case so once again when r μ is positive there exists a circle of radius √ r =√μ this time on which the flow is in the counterclockwise direction and inside you have an unstable spiral point so it is evident that wherever you start other than r=0 you are going to flow into this limit cycle if you start with a value of r bigger than √of μ r is negative and therefore r shrinks inwards and therefore if you start here you are going to flow.

In into this point from outside on the other hand when μ is negative you just have an unstable spiral so this point is unstable sorry you have negative you have an asymptotically stable spiral and therefore wherever you start you are going to flow in this direction into the asymptotically stable spiral point it is quite clear that what has happened is that if you start from negative values of view and move towards positive values what started off as a asymptotically stable critical point has bifurcated it is become unstable.

But it is also given birth to a stable limit cycle something which is asymptotically stable and so the bifurcation diagram in this case looks like this I need to plot x and y equilibrium values.
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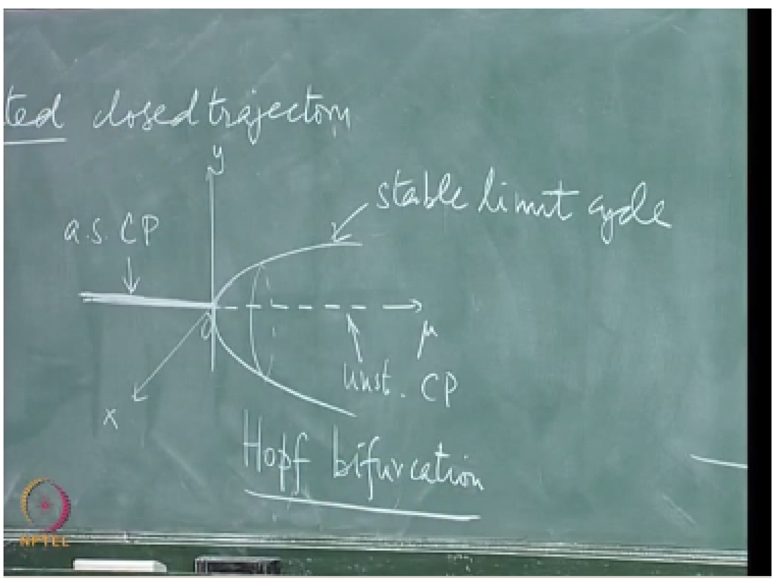

So let us do that here and let us put this to be the μ axis and this is X and this is y I mean the steady state values the values on which things do not change at all then the only equilibrium point for μ < 0 is an asymptotically stable spiral point at X = 0 y = 0 so it is on this axis as soon as μ crosses over to positive values this point becomes unstable therefore in keeping with our usual notation we should really draw this the dotted line like this and in its place you have a limit cycle which is stable of radius $r = \sqrt{\mu}$.

But on the limit cycle x and y change without changing r and this radius of this limit cycle increases like the $\sqrt{\mu}$ therefore the picture would be something like this really is a kind of parabolic Bowl which comes out whose size this radius here increases like the $\sqrt{\mu}$ and it is in both x and y you can see so it is not a critical point it is not an equilibrium point there is dynamics going on there but the moment you hit $r = \sqrt{\mu}$ it remains at that $\sqrt{\mu}$ and this is a stable limit cycle this is an unstable critical point.

And this was a stable asymptotically stable critical point in the bifurcation which happens at $\mu = 0$ of a stable critical point into an unstable critical point and a stable limit cycle is called a hope bifurcation it is one of the most important bifurcations in nonlinear dynamics happens all the time we will see examples of this when we study chemical and biological systems and oscillations in these systems this is a standard mechanism by which periodic behavior suddenly emerges from nowhere as you change a bifurcation parameter.

So on this thing you can see that it is a periodic orbit it is a periodic it is a closed trajectory but an isolated closed trajectory because I have tailored this dynamical system to illustrate this point these were very simple functions here very specific functions and cancellations occurred therefore the limit cycle had the shape of a circle that is not necessarily true it could have a very complicated shape it could change there could be families of limit cycles but the basic definition of a limit cycle is that it is an isolated periodic trajectory in a dissipative system and discovering a limit cycles is not anywhere near as easy as finding

The fixed points or the critical points of the system for which all you have to do is to equate the right-hand sides to zero in some sense but this is not so for limit cycles especially if you do not if you are not able to reduce things to very simple things simple algebra using say polar coordinates or anything like that so the shape of the limit cycle is complicated you can only discover this numerically but there are theorems and criteria which will tell you when these limit cycles could exist and whether such limit cycles could exist incidentally from this picture.

It is also clear that inside this stable limit cycle they must exist at least one unstable critical point otherwise there is no way these things would get thrown out and move into this limit cycle here and that is measured by something called the winding number or the pawn carrier index which I will come to a little later yes can open trajectories we are not considering here at all because they are not of much interest to us at the moment but closed trajectories would mean periodic motion and we would like to start with things which are periodic.

And maybe apply perturbations and go to things which become quasi periodic or a periodic this is going to be the thrust of what we do it open trajectory is unbounded motion in general is not of direct interest so there are of course many physical situations where it is important but they are not directly of concern to us here specifically we are always looking for periodic motion or things close to periodic motion in some fashion the earlier example what happened in the earlier example if I put a + sign here what happens.

Now what kind of behavior do you have once again since we've chosen this new to be exactly as it is this is certainly true you have an asymptotically stable spiral from $\mu<0$ and an unstable spiral $\mu>0$ but because we change the sign of the nonlinear term this becomes a + here it becomes a + here in this case and then

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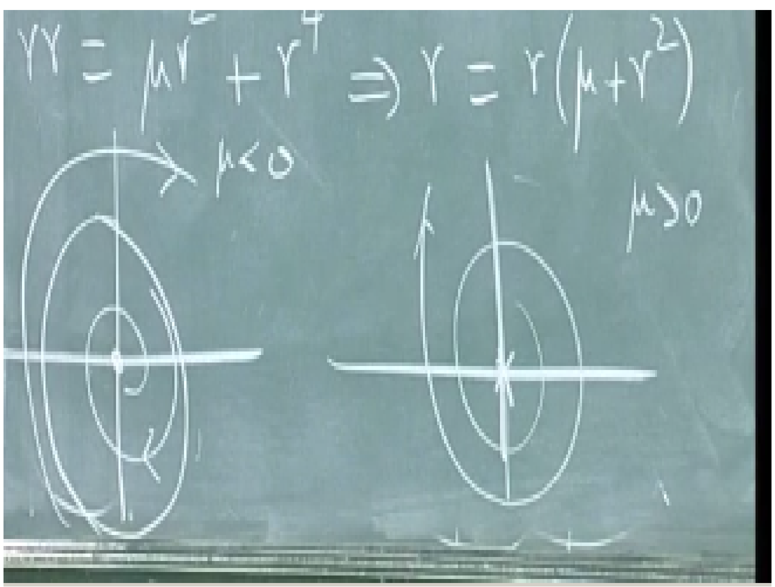

The picture gets reversed this is from $\mu< 0$ and this is for $\mu >0$ from $\mu >0$ this was an unstable spiral here so this is unstable and things are going to go flow away from it certainly from $\mu <0$ this is a stable spiral point asymptotically stable spiral point things are going to flow into it but this trajectory here at $r^2 =\sqrt{-\mu}$ is certainly going to be an isolated trajectory here.

But I should really draw it as a dotted line because it is unstable and it is evident that here things are flowing away in this fashion on the other hand here things are going to flow into this origin this is stable so it is going to flow in here and things are going to flow starting here flow outwards that side this is therefore an unstable limit cycle and it contains within it a stable critical point so here what is happening is as you move from positive move towards negative μ what was an unstable critical point has bifurcated into a stable critical point.

And an unstable limit cycle so if you like it is the image it is the complement of what happened earlier and let us draw the bifurcation diagram in this case and you have a picture.

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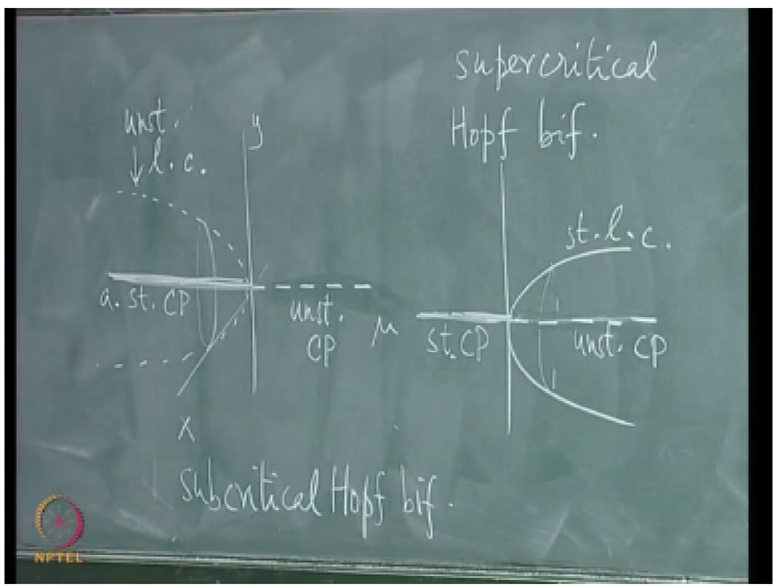

Where as a function of μ so here is X here is Y and here is μ from $\mu > 0$ you had an unstable critical point therefore it was like this and it becomes stable form $\mu < 0$ so this is unstable and this is a stable critical point asymptotically stable critical however you have now a limit cycle that is unstable and therefore. I should really draw it with a dotted line like this and these are the trajectories on it and this is an unstable limit cycle compare this with the earlier case where you had it does not matter which way this parabola looks the way we have written drawn.

It this is moved to the right here but it is not necessary this was stable and inside it you had an unstable critical point and this was a stable critical point so a stable critical point came along bifurcated into an unstable critical point and a stable limit cycle and this was a stable critical point and this was an unstable and this is what we call a Hopf bifurcation this too is a Hopf bifurcation this is called a subcritical and this is called a supercritical of bifurcation in either case it is the bifurcation by which a critical point bifurcates into a critical point.

And a limit cycle and the stability is in the two cases or as I have shown here the fact that this parabola looks to the right and that looks to the left is an artifact of the way we have drawn this with the way we have chosen coefficients but the essential point is that a stable one a stable critical point bifurcates to a stable limit cycle here and loses its stability in a sub critical bifurcation on the other hand an unstable critical point gain stability but also gives rise to an unstable limit cycle in the process both these processes happen very often in nature.

But as I repeat I said earlier they happen only in dissipative systems you have limit cycles only in dissipative systems here is a simple example of a system with limit cycles.

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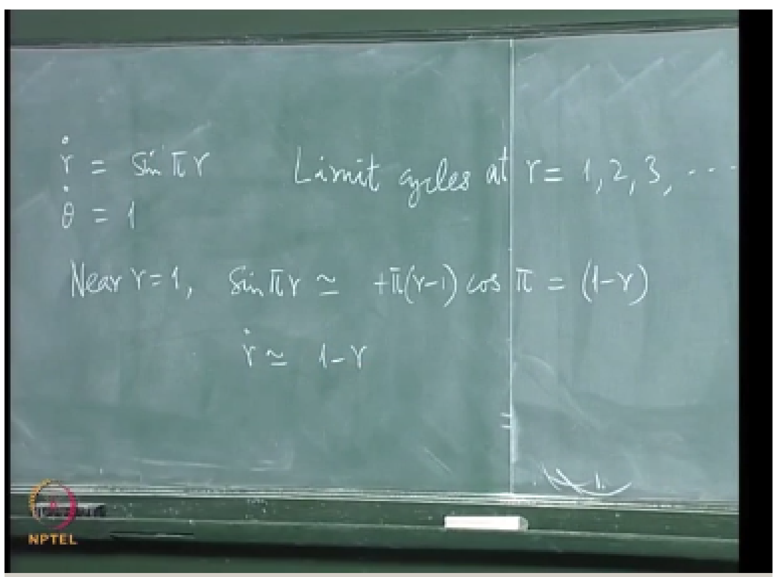

So let us consider r. = sine π r and let us say θ. = 1 in polar coordinates so I have already gone to polar coordinates and done this what would you say is the behavior of this system what does it do in polar coordinates if it does this and we could go back and write the equations in X and X. if you like but we can analyze it directly as it stands here what would you say is the behavior of the system on the right-hand side when does the right-hand side vanish in the first equation.

So it is evident that limit cycles at r= 1 ,2, 3 and so on we do not yet know what is happening at r= 0 which is the origin and you have presumably a critical point at that place we do not know the nature of this critical point what kind of limit cycles do you have at various places would it be stable or unstable or what how would you decide this yes you quite right they would alternate instability one would be stable the next would be unstable and so on what would be the one at r =1 for instance sine π r near r=1 you have to do a Taylor expansion about r=1 and of course sin π=0.

So the first term is 0+ the next term is r - one times the derivative of sin π revaluated at r= 1 and what is that it is π cos π r right so there is a π cos π r = so that is a cos π and this is =1 1 - R in the vicinity of r= 1 + higher order terms so it is really telling you that r. is of the order is of the form 1 - r is this therefore a stable or an unstable limit cycle at r= 1 if r < 1 it is growing and if r> 1 is shrinking therefore it is a stable limit cycle so we immediately realize that at r = 1 you have a stable limit cycle it is not hard to see it r= 2.

You get a cos 2 π here and that would become + 1 and therefore it would become unstable therefore the picture in this case would look like this at our equal to 1 you have a stable limit cycle at r = 2 you have an unstable limit cycle

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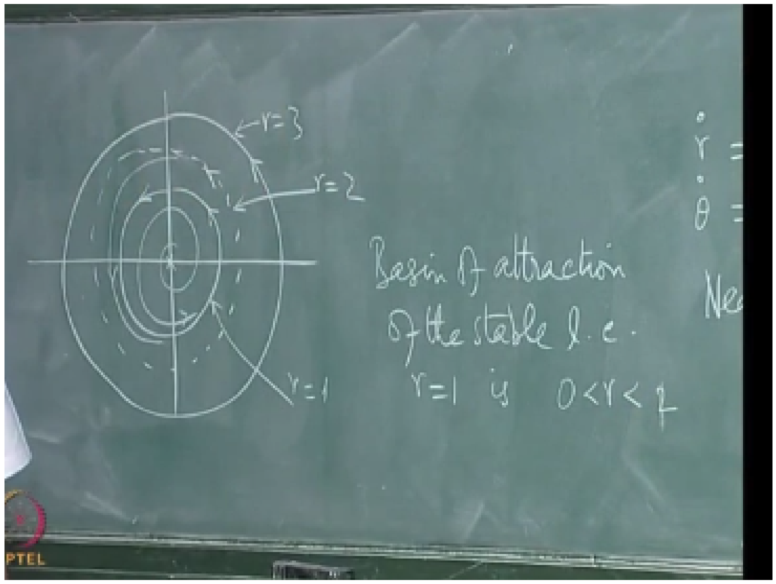

At r= 3 you have a stable limit cycle once again this corresponds to what would you therefore expect is the behavior at the origin which is a critical point should be unstable so that things would flow into it and I would therefore expect that in this case everything goes round counterclockwise direction therefore I did expect trajectories to do this so all points between r = 2 and r= 1/1 would eventually fall into this trajectory they would fall into this stable limit cycle what is the basin of attraction of the limit cycle at r = 1.

So we have the concept of the basin of attraction of the stable limit cycle or equal to 1 is all points which lie inside they are going to move outwards except the origin which is a critical point by itself so you really have an unstable critical point there so this is 0 <r < 1 and what happens to points between 2 and this way to fall in so in fact < 2 points on r= 1 are already there and similarly for the basin of attraction for this is everything between two and < 4 would fall into this and so on.

So this is the second kind of attractor other than a critical point attractors we now have closed orbits as attractors and you can easily see that one could generalize this and have in higher dimensions you could have a torus attractor you could have an n-dimensional torus as an attractive but the interesting thing and we will see this little later is that for high dimensions >2 two you have another kind of attractor possible called a strange attractor and those could be very complicated fractal objects and they would be the ones of central interest.

To us as we go along in drill nonlinear systems so, I stop here today and then we take it up from this point we have seen some of the elementary bifurcations here and we will see what other possibilities can exist for dynamical systems of this kind II know they need the question is can limit cycles do limit cycles have to be circles the answer is no of course not it happened to be they happen to be circles in the examples

I showed because, I have contrived it in that fashion I have written this already in polar coordinates in a simple way and the earlier example I had very simple functions of r^2 there is no reason why this should be so at all the very first example of a limit cycle which occurred was in triode oscillations and it is called the wonder oscillator and it has a term which goes like it is again a two dimensional dynamical system there is an X.. term which would be the inertia term + a term which is proportional to X

So some natural frequency 2 times X + a damping term and then if it is a simple harmonic damped simple harmonic oscillator linear oscillator this would be a constant times X .but you have here a term which looks like 1 - x^2 times X .t times some coefficient and = 0 is a nonlinear equation because of presence of this term x^2 X . and you can see that this damping for x 2 < 1 acts like normal damping but x 2 >1 it feeds back into the system or if you like you put a - sign here this would reverse.

The situation and say for sufficiently small X things would tend to grow outwards and for larger X they would tend to come back this direction so the damping acts in the state dependent manner it is dependent on X so either you have for positive feedback or you have negative feedback depending on what X itself is and the limit cycles of this system but the first limit cycles should be discovered and this is the wonder poll oscillator.

I am not too sure about the signs here but this is essentially what it is you could change this constant it does not matter but this is the way this system behaves and as, I said earlier this occurred for the first time in the consideration of triode oscillations in a triode the old vacuum tube we will see further examples of limit cycles you.

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