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TOPICS IN NONLINEAR DYNAMICS

Lecture 10
Elementary bifurcation

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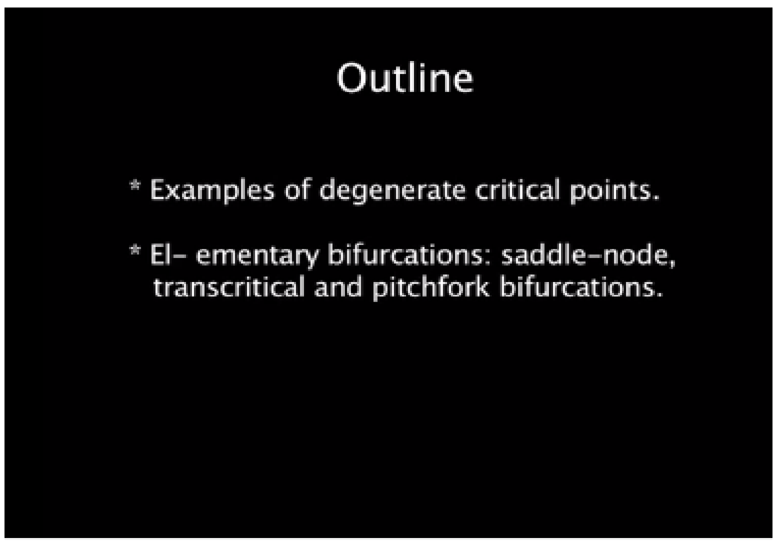

One of the questions that arose in our studies was what happens if you have a degenerate system namely one in which the liniearized matrix is has a zero Eigen value well there are several kinds of degenerate systems but let us look at a specific example the simplest of these.

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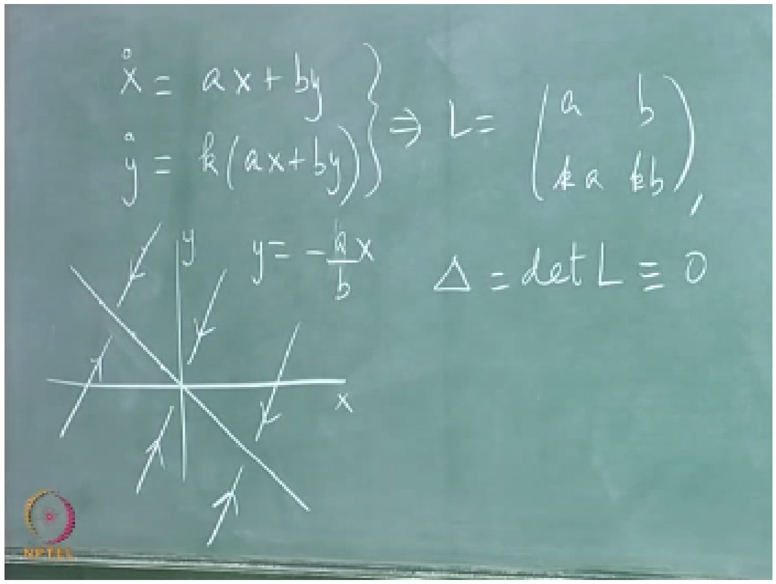

So let us look at one for which X. is perhaps a x + B Y and y. = some constant times the same thing ax + B Y, so of course this immediately implies that L is a b k a. kb so δ equal to determinant L is identically equal to 0 then the question is what kind of critical points you get here.

And the answer is very simple you have a degenerate system these two equations are not linearly independent of each other it is just a multiple of that and therefore in the XY plane if you plot locus of the points on which the right hand sides are 0 you get ax + B y = 0 or X or y = - B a /B X which perhaps is some straight line of this kind and at every point on this line critical line if you like the system is in equilibrium no time change at all, then of course you could ask what is the flow like what happens if you have an initial condition which perhaps starts here.

This would depend on what the signs of these constants are but in general the flow would be either along this line inwards everywhere into this line, or perhaps outwards all the arrows going outwards. So this is a very simple example of a degenerate system it is not of particular interest to us right now, more serious would be what happens if the system is not linearizable intrinsically in the example that we look at is even in a one degree of freedom system for example just a single variable.

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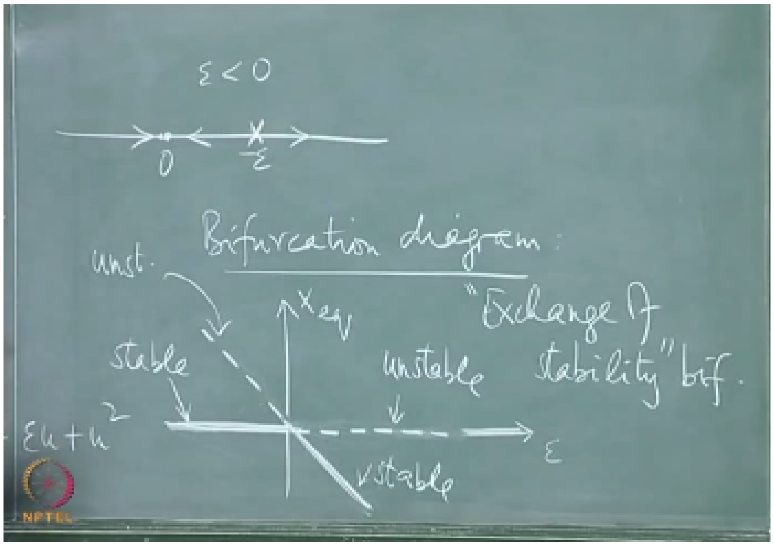

If you had something like X. = x^2 and you ask what does this do what kind of critical points do you have in this situation. Well we have a phase line there is a critical point at zero and we can't tell whether it is stable or unstable in the conventional sense because for positive x the flow is outwards but for negative x also x squared is positive and therefore the flow is inwards in this direction.

Therefore this is as far as the left-hand side is concerned it is an attractor but as far as the right-hand side is concerned it is a repeller it is a higher order critical point. Now the question is can we say something else about it and how does this higher order critical point arrive? Well it occurs because x^2 is not a generic polynomial on the right hand side if I ask you to write down a polynomial on the right hand side or something which is expandable in powers of X about the origin you'd start by writing $a_0 + a_1 X + a_2 x^2$ and so on and so forth and unless you have a special accident.

So that $a_0$and a_1 are both 0 you never start with the x^2 term on the other hand if I put a constant here just a single constant you can always absorb that constant by shifting X to that point, so the constant is irrelevant but certainly the linear term is missing so the sensible way to do this would be to ask instead of looking at this system directly unfold this second-order 0 if you like this double 0 by writing this as equal to some $\varepsilon X + x^2$ and examine what happens for various kinds of ε small positive small negative or 0 ε equal to 0 corresponds to the case you have, but ε positive or negative would be a generic system and you could linearize it in the origin of these points.

Well in this instance for example there is a critical point at the origin and there is also one at - ε so there is one at - ε out here, in the instance in which ε is positive then of course - ε is located here and you could ask what kind of flow do you have once again near the origin near the origin here in the vicinity of the origin I could write x dot is approximately equal to ε X which means if ε is positive the flow is outwards along the direction and outwards along this direction here by continuity therefore it remains in this direction all the way up to this critical point and now you ask what kind of critical point is this for that you have to liniearized about this point so the sensible thing to do is to put in this neighborhood is to put u equal to X + ε.

So that the origin in U is the point X equal to - ε and then ask what happens to this system well that system now becomes u dot equal to u x X because this is U X + ε times X but X is U - ε - ε u + u squared once again if you liniearized in the neighborhood of this point then the liniearized equation is liniearized it is u dot is approximately equal to - ε u and since ε is taken to be positive it is flowing in into this attracting fixed point here and it is flowing in this way in the other direction.

So you have an attractor at - ε and a repeller at the origin and the coalescence of these two when ε is 0 reduces for you this figure this higher order critical point and this picture is valid for ε > zero this picture therefore corresponds to ε = 0 and one could ask what happens if I chose an ε to be negative to start with that is easily taken care of because if ε is negative the fixed points are at the critical points are at zero once again and at the point - ε which is now on the positive site since ε is negative and it is easy to see that linearization about the origin is always X dot as ε X.
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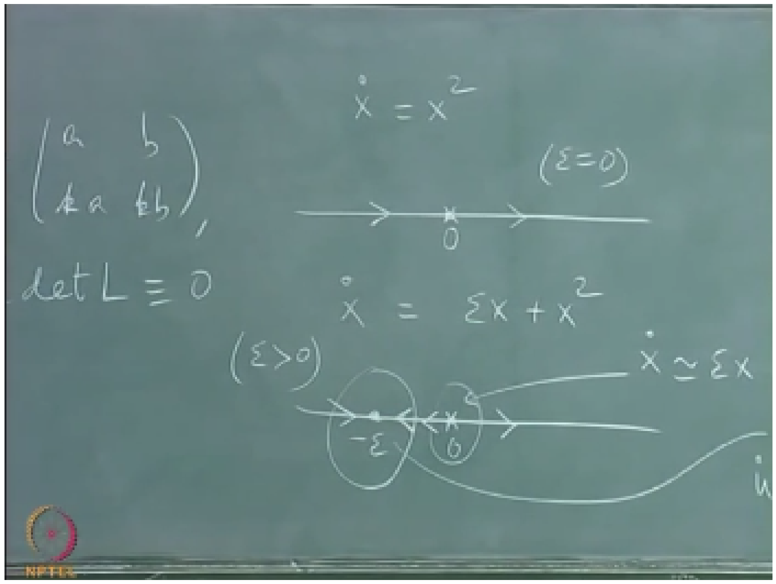

And since ε is now negative the flow is inwards into the origin on both sides in its neighborhood and it is a trivial matter to check once again that the flow is outwards from this point the point you equal to zero or X = - ε because ε is negative this number becomes positive here - E and it is a repeller, so what happens effectively is that this point becomes a repeller and this becomes an attractor.

So now we get a much better picture it says that the system X dot is equal to x squared which is intrinsically nonlinear really arises by an accident it arises by the coalescence of two critical points one at the origin and one at - ε as the second critical point - ε crosses zero as it crosses the origin as it crosses the value zero you get what was initially to start with starting with positive values of ε an attractor at - E and a repeller at the origin becomes a higher-order critical point win of two of them coincide and then as you move over to negative values of ε you have an repeller at - E and an attractor at the origin so there is been an exchange of stability and this is called an exchange of stability bifurcation it is an honest bifurcation one of the simplest.

When can think of and we draw what is called a bifurcation diagram in which I would plot as a function of the parameter ε and plot the equilibrium or stationary value or steady value of X in this case the value at the critical point but there are two critical points one of them is always at the origin and therefore you have a line lying on this axis and the other one is at the point is at the value - E and that is a straight line which is tilted at 45 ^{0} and of this kind and this if you like is

X equilibrium, on the other hand I know that for negative values of ε this critical point is unstable.

And therefore I denote that in standard notation by a dotted line as a dotted line here for negative values of E it is unstable, and therefore I have a dotted line of this kind and a solid line here to indicate that this is a stable critical point. So this is unstable and this fixed point is stable when you cross over two positive values of E what was stable becomes unstable in this fashion and what was unstable continues on unstable.

So this is stable and this is unstable and we have an example of a bifurcation which in this case is called an exchange of stability bifurcation we would like to know ask a slightly more general question what kind of bifurcations can we expect in such systems this was a simple one-dimensional system but it can be generalized to higher dimensions and the question is what kind of elementary bifurcations or distinct kinds of bifurcations do we have this problem too has been analyzed in great detail and bifurcations have been classified at least the elementary ones have been classified in simple dynamical systems.

I should mention that bifurcations which involve a single parameter which you tune in this case just ε they call bifurcations of core dimension one and this is distinct this dimensionality is distinct from the actual dimensionality of the phase space that we are dealing with this is in the parameter space and there is just a single parameter here. So it is a bifurcation of co dimension one in this instance we will see later that we have bifurcations of higher core dimensions and classifying them is a non-trivial task.

Classifying bifurcations in general in higher dimensional dynamical systems it is a fairly non-trivial task. Now I would like to put this in put this in the framework of a slightly more general setting and that is as follows, so I would like to look at it as two dimensional phase space but just to get a physical feel for what is meant by tuning this ε let us cast it in the language of a mechanical example.

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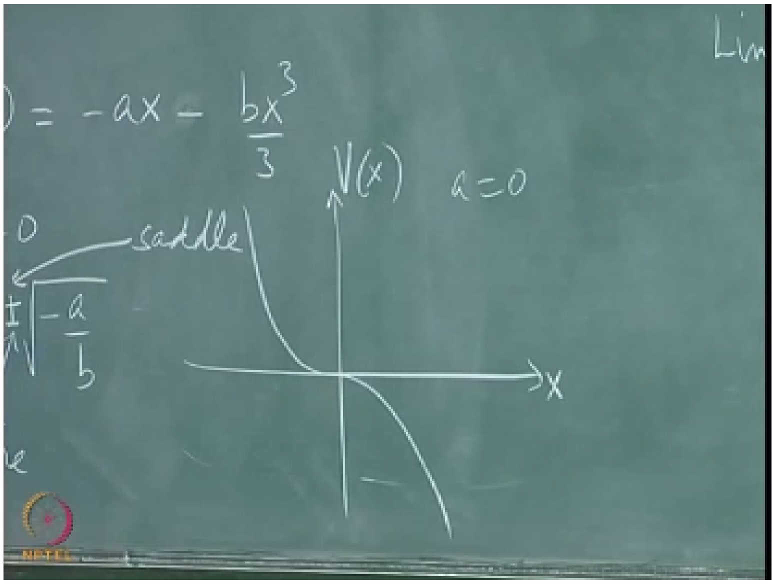

So what I intend to do is to consider a particle of unit mass moving along the X direction and look at these dynamical equations as equations of motion for this particle in some potential V of X so the equations of motion are X dot equal to P if I consider a unit mass this is the momentum here and P. = - DV over DX which is the force on the particle. And I am going to tune this by writing different kinds of functional forms for this V of X and asking if I can examine bifurcations in this framework here.

Now what would you say is the simplest of these forms that we could write down here typically I would write down various kinds of polynomials for this force here and the simplest of these that one could write down is perhaps to say that this is a constant + a linear term in X but it is quite clear that if you have an A + BX here the only critical point is at P = 0 and X = - a / B and that is it is just a single critical point and there is no possibility of any bifurcation no coalescence of singularities.

The next non trivial case would correspond to putting in here some parameter and its various symbols could be used let us use a and let us put a be X square what kind of potential does this involve what is the shape of V of X in this case if I plot here X versus V of X remember this is the force it is - DV over DX, so this would imply that V of X itself equal to - ax + B X cubed over three - four integrate this and change the sign I end up with this what kind of shape is that it is a cubic curve in clay we would like to plot it and we should like to know whether a is positive or negative or what let us fix the sign of B let us take B to be positive for instance you could simply redo the whole thing for B negative there will be no essential change in what I am about to say.

So let us suppose that B is always greater than zero and a could be negative or zero or positive let's look at all possible cases now what happens if a is negative what is the shape of this curve so I plot this for a negative and the shape of this graph is approximately linear at the origin with a positive slope and therefore the potential looks like this here but eventually this term takes over and that's a large negative term.

So it is evident that this is going to go down and fall down on the other hand when X is large negative this term dominates and becomes positive in sign and therefore the potential has a shape of this kind. So one knows immediately that the critical points of the system occur at P = 0 and this quantity equal to zero and what does that look like what would this do, it is clear that there is an equilibrium point here at this point and that is a saddle point because it is a maximum of the potential and this point here is a minimum of the potential and therefore this point is a center since there is no friction in this problem you would have small oscillations about that point about the minimum of the potential.

Whereas the maximum of a potential in the absence of any dissipation is always a saddle point and this is all that a Hamiltonian system could have we have here a Hamiltonian system in which the Hamiltonian is P^2 over 2 for unit mass + V of X and these are Hamilton's equations that I have written down and this is the picture that we have, so we clearly have critical points at P = 0 and the values here correspond to X = - a over B square root of this with a + or - and this is the picture for a less than zero remember.
The + corresponds to the - this point here corresponds to a Center and the + corresponds to a saddle you can write down the 2x2 liniearized matrix and check out at each point that this would correspond this Center would correspond to a pure imaginary pair of Eigen values and this would correspond to one positive and one negative Eigen value if you liniearized about these values of x we know how to do that now. What happens if a is exactly equal to zero once again if a plot at a equal to zero I plot the potential V of X versus X what would this correspond to this term is gone and you just have a - B X cubed and that at the origin is extremely flat it has an inflection point and it falls off in this fashion again for B positive.

So this point here has arisen because this maximum and this minimum of the potential have come together at the origin, and it is become an inflection point where the slope is 0 in the first second derivative the curvature is also zero at this point. And now finally what happens when a is bigger than zero.

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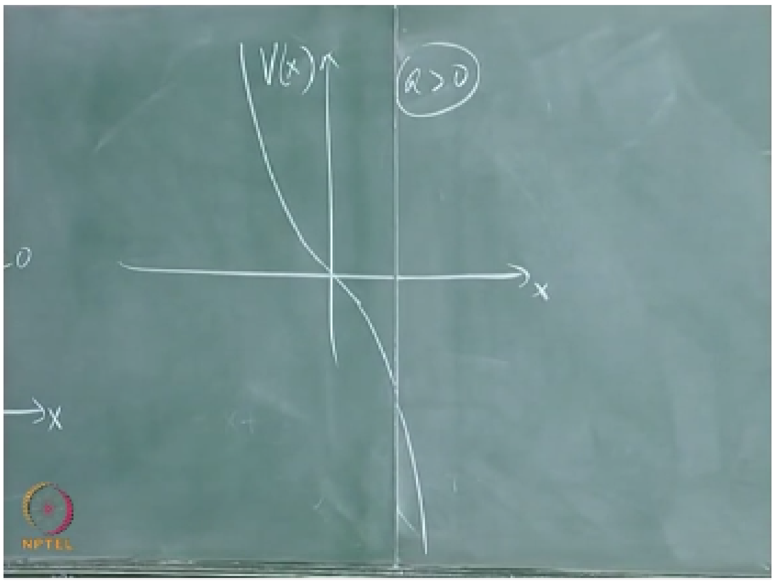

So we look at the picture a > zero that is the third case and if I plot X versus V of X then when a is positive you have a negative slope here and therefore this curve looks like this approximately linear and then of course as X becomes larger it falls off like a cube and it increases here like a cube.

There is no possibility of any equilibrium point at all in this potential there are no maxima or minima at all there are no critical points in this dynamical system because if a and B both have the same sign there is no way this quantity can vanish so that immediately tells us that you don't have any critical points in that system you have a degenerate critical point in this system and you have here two critical points which are separated out.

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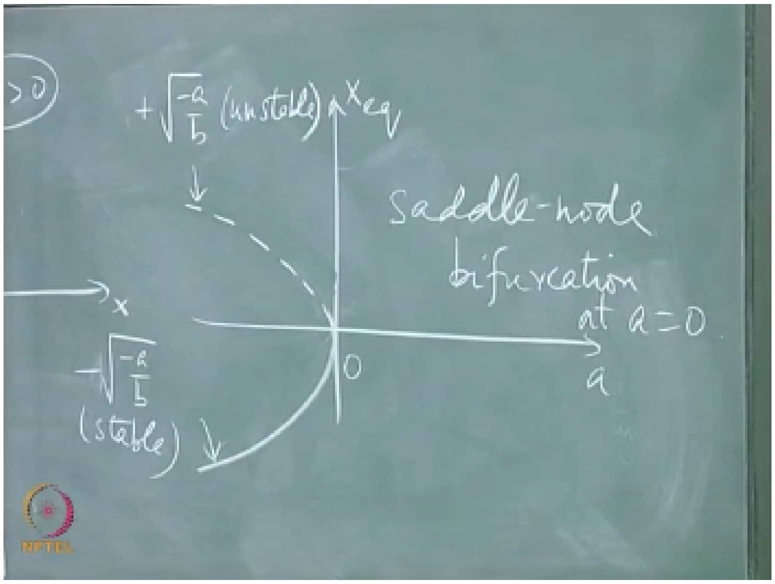

And therefore if I now plot the equilibrium values let us plot X equilibrium here versus a parameter and the critical parameter in which you have this variation is in fact a to plot this I need one more direction I need P as well coming out of the plane of the board. But since P is always zero at the critical point I ignore this P it is always zero so let's just plot it in a two dimensional diagram here X equilibrium versus a and what is the picture one has for a positive nothing for a negative you end up with a saddle point at X = + √- a / B and a center which is stable and - the same value.

And this is changing or increasing in magnitude like the square root of - a therefore it is a parabola which goes up in this fashion and falls down in this fashion, however we know that this route here which corresponds to + square root of - B over a is unstable and this is stable so this branch which is put - √- a / B it is stable this branch here which corresponds to + √- a / B is unstable because I have had this parabola looking the other way had I put a + ax here then the role of - a and a + a would just get interchanged but what is happening here clearly is that if you imagine changing a in parameter space from positive values to negative values no critical points at all in this region.

And all of a sudden at this point a pair of critical points gets created and what is happening is that if you start with this potential and start flattening it out then at the value of a equal to zero you have a cubic you have a second-order zero here you have an inflection point here X cubed and then below that the inflection point unfolds into a maximum and a minimum and you have this shape here and of course you go on changing a making it more and more negative these points will move out further exactly like a square root of - a and this is what happens here this bifurcation we are out of nowhere a stable and an unstable critical point you merge and move off is called a saddle node bifurcation.

In this case at the value a = 0 you see immediately that this is different from the bifurcation we looked at the exchange of stability bifurcation altogether different. So a saddle node bifurcation is one where as you go across a critical value of the bifurcation parameter a pair of critical points is created typically one of which is stable and the other is unstable. So much for this simple form we could make this a little more complicated let us do that in the next step is to take this potential and ask what happens if it is okay.

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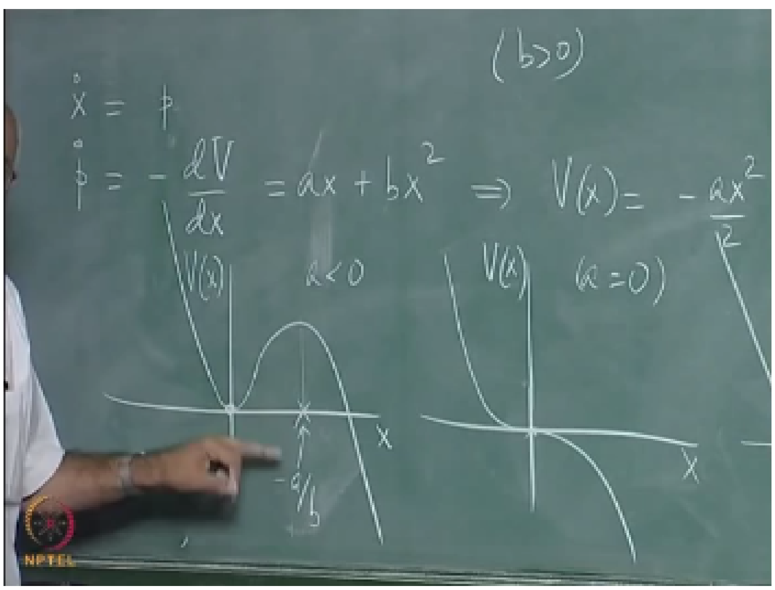

A X + BX quick this would correspond of course to V of X equal to incidentally the signs of these quantities a and B have taken them to be arbitrary here it doesn't matter which way these bifurcation diagrams look the physics is essentially the same thing in all cases now what happens to the potential it is - ax ^{2} /2 - B X ^{3} / three if I integrate this once again we can predict what is going to happen we start again with a less than 0 a = 0 and finally a greater than zero and plot in all cases the potential V of X as a function of X out here for a negative - ax squared over two is an upward looking parabola because it dominates for sufficiently small X.

And therefore this curve is going to look like this, but then once X becomes sufficiently large this negative term is going to dominate and bring this potential down in this fashion on the negative side this always remains negative and for X negative this number is also is positive, so what happens there you seem to have made a mistake here. So let us keep let us keep B positive this term is going to dominate so it is positive yeah there is no problem this term becomes positive yeah.

So that is fine this goes up here because eventually for large negative X this term dominates over this - X^3 is negative and - X^3 is positive, so it goes up in this fashion, so this is fine this is fine when a is 0 exactly 0 then it is just a cubic curve exactly as I drew earlier - X cubed which has an inflection point here and goes down in this fashion and for a positive you have a downward parabola here and of course for large negative positive x this is going to become a large negative quantity go down there but then it has to eventually turn back and go off in this fashion.

Now it is easy to see what is going to happen at a equal to 0 you have a higher-order critical point here for a negative you have a maximum of the potential which corresponds to a saddle point you have a minimum which corresponds to a center at the origin on the other hand for a positive the origin corresponds to a saddle point and the minimum which occurs here corresponds to a center.

So what has happened what kind of bifurcation is this, and where is this point this point mind you is not at X = 0 but it is equal to at - a over B this point also is that - a over B it is as if the diagram has moved but what is happened is that the critical point at P = 0 and X = 0 which was initially a center has now become a saddle and the critical point at P = 0 and X = - a over B which was a saddle point has collided with the center at the origin and has now become a center they have therefore exchange roles.

And what is this bifurcation this is an exchange of stability bifurcation, so it is evident immediately in the bifurcation diagram if I plot the parameter is a if I plot this versus X equilibrium as long as a is negative this center at the origin is stable. So we have this picture this is stable and once A becomes positive that point becomes unstable and therefore you have a dotted line here on the A axis.

But the critical point at - a / B this thing here which is just a straight line with slope - 1 / B and we have taken B to be positive something of this kind is unstable there and is stable at this point that's the location of this critical point and for a negative it was clearly unstable and for a positive it is a center and is therefore stable and we have an exchange of stability bifurcation at k = 0 this bifurcation has another name it is also called exchange of stability or transcritical bifurcation the saddle node bifurcation incidentally is also called a tangent bifurcation this is a matter of terminology but there are these alternative names we have looked at two distinct bifurcations.

And one could go on and ask are there any other bifurcations of co dimension one because the critical parameter here is a the one that you are tuning well the next step would be the following one could play this game continue.

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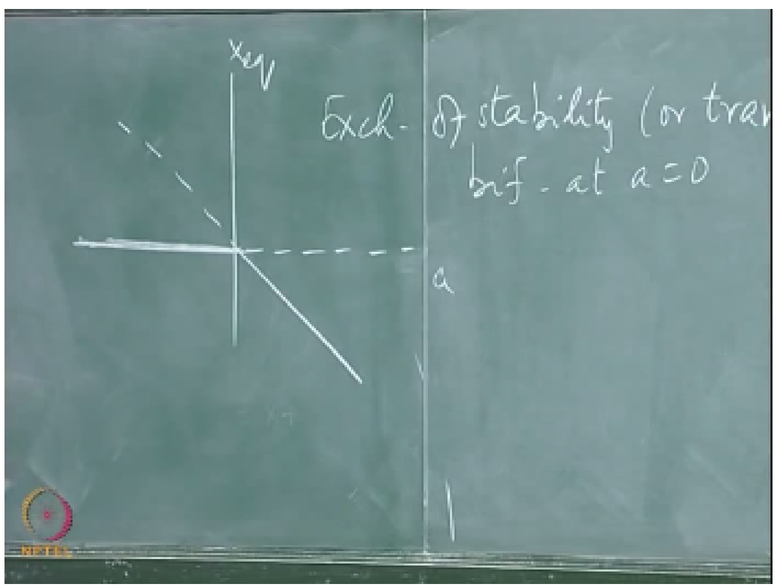

The next step would be to say what if the shape of the force or the potential corresponding potential what if this was a x^2 and this was B X^3 what then this would of course imply that the potential V of X let us make this a little stable.

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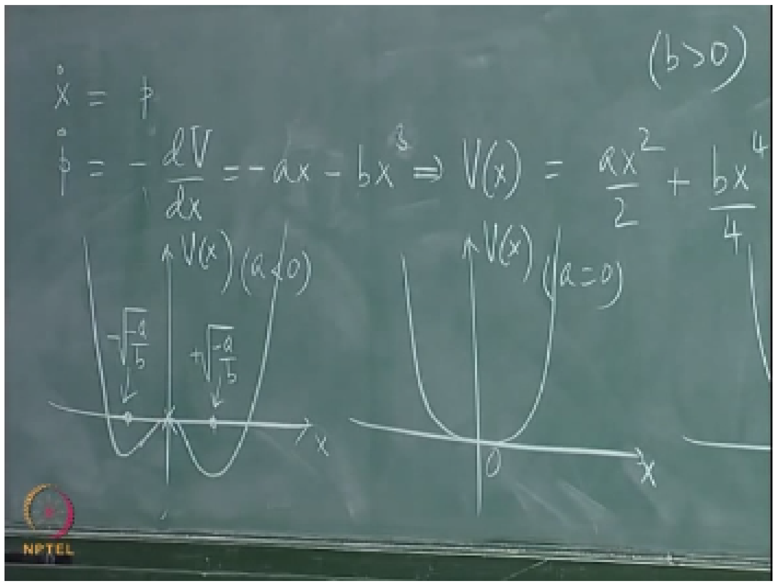

So it looks physical let me make this - make this also a - sign you will see in a minute why because I want to draw convenient pictures in this case V of X then becomes equal to a a x^2 / 2 + V X 4 so the three cases that we have looked at were constant and quadratic function x^2 then we had the next situation was an X and an x squared in the next case I am looking at is an X.

And an X^2 here the potential corresponds to - the integral of this function the primitive of this function which is $ax^{2/}$ 2 + BX 4 / 4 right differentiate it and take a – sign I get precisely this what kind of picture do I have now and what would you expect once again the simplest way to do this is to plot the potential as a function of X we plot V of X in all three cases so V of X versus X and let us do the same thing V of X versus X 4 respectively a less than 0 a equal to 0 and a greater than 0 I am actually going the other way but it does not matter it is easiest to plot this potential.

Because it is always positive we have taken B to be positive always, in which case for sufficiently large X this is going to dominate and that is just an X for curve moving up steeply and near the origin this is a parabola turned upwards concave upwards for a positive and then after that it increases more sharply so it is parabolic at this point and increases very sharply this fashion so one immediately knows what the critical points are there at the origin there is only one critical point and that is at P = 0 as well as X = 0 and it must be a center.

Because this is a minimum of a potential at a equal to 0 this term is absent and you have a very flat potential at that point not only is the derivative 0 the second derivative is also 0 as it is the third derivative it is a minimum but it is not a simple minimum because you can see it goes like this point we X = 0 at this point. And what happens here when a is negative for small X this term dominates and since a it is negative is negative it is an inverted parabola so it is a curve like this but eventually the x4 term will dominate and take over you have a symmetric graph of this kind.

Unlike these cases where you just had a minimum at the origin now you have a maximum at the origin but you have two minima at these points and what are these points these points are given by the vanishing of this quantity other than X = 0 and those roots are at x 2 = - a a / b therefore X = + √- a / B and this is at - √- a over B those are the locations of these points and it is evident that these are centers and this point is a saddle point in between we can therefore draw a bifurcation diagram now without further ado.

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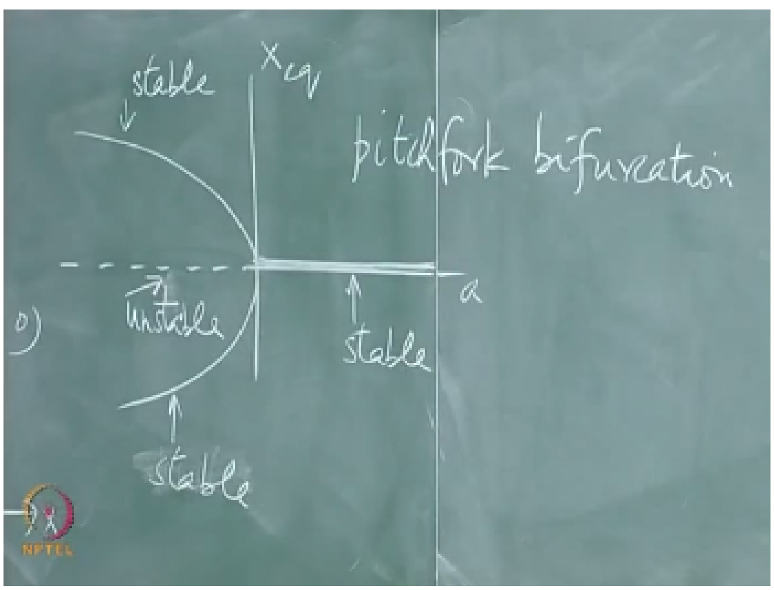

And this as a function of a I plot X equilibrium and what does it look like for a positive there is just a single critical point at P = 0 X = 0 and here since P is always zero we have not drawn it so X equilibrium is zero and it is this unstable. So this fixed point this critical point is stable when a becomes negative that critical point becomes unstable. So we have to replace this with a dotted line on the other side and this is unstable, however two new critical points emerge and move away from the origin and they are at + or - √of - a / B.

So as a function of a they go like √- a which would correspond to the parabolic shape there and a parabolic shape here and this is stable this is unstable this is also stable, so this is also stable with an unstable critical point in the middle and that is obvious because you can't have two minimum of the potential without a maximum in between so automatically you realize that stable and unstable critical points or tend to alternate.

Now what does this figure remind you of it is a different kind of bifurcation altogether from either the saddle node or the exchange of stability bifurcations, here we have a stable critical point coming along and bifurcating continuing as an unstable one and a pair of stable ones is born what does this figure remind you of it is a combination of both in some sense but this figure because it resembles a pitchfork is called a pitchfork bifurcation. A saddle node bifurcation or a tangent bifurcation had no critical point at all and then the creation of a stable unstable pair an exchange of stability bifurcation was when a stable and unstable one collided and exchanged stabilities.

And a pitchfork bifurcation is when a stable bifurcation bifurcates into a pair of stable ones and an unstable one in the middle, well one could go on and ask suppose I go on increasing the powers here what would happen while it is true that you could in principle get what looks like new kinds of modifications the fact is these are the only three generic elementary bifurcations of co dimension one in such systems in continuous-time systems.

So the three elementary bifurcations of core de mentioned there A the saddle node of tangent B the trans critical or exchange of stability bifurcation and C the pitchfork, we have Illustrated these bifurcations in the framework of a simple mechanical system drawing pictures using these potentials but as we saw right in the beginning when I gave the example of the transcritical bifurcation they could occur in dissipative systems as well.

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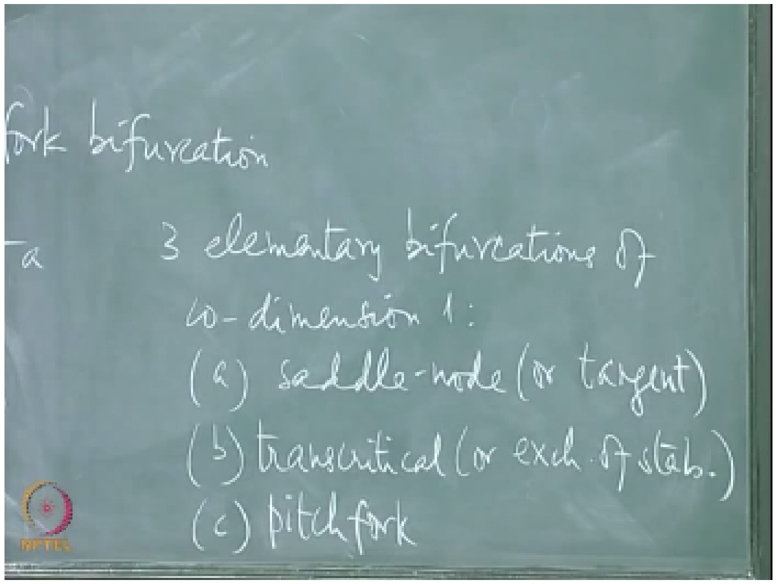

So there is nothing which says these are exclusively restricted to Hamiltonian systems or potential problems or anything like that it is the phenomenon that is important and this is what basically happens these are elementary because you could have more complicated Co a licenses of bifurcations. So in exactly the same way as saying for instance the elementary functions which I would have of a variable X expandable in power series would be 1x x squared X cubed and so on.

And from these I can construct polynomials I can construct more complicated combinations in exactly the same way in some sense these are the basic things that happen no other pitch work is not a combination of the others no it is not it is a very different shape altogether you can see from the potential example that I gave there is a very different thing altogether here, if you go back to the potential example you had a cubic potential and that cubic curve could have a minimum and a maximum it need not a general cubic curve need not have a cubic and a specific cubic curve need not have a minimum and a maximum.

But it could depending on what the parameters are and that's exactly what led in the K into the saddle node bifurcation similarly if you took a cubic curve then the position of the minimum and maximum could get exchanged this is what led to the exchange of stability bifurcation a fourth order curve of this kind could have three stemma but it could also have one just one right here and as it transits from the situation where it has one extreme um to where it has three which is the largest number it can have you end up with a pitchfork bifurcation.

So clearly what has happened is depending on what kind of polynomial you have what kind of unfolding of these singularities you have you have different kinds of bifurcations and of course one could ask what would happen if I didn't have an X cubed here and an X here but I had an X cube here and next to the five here for example etcetera those would not be the most elementary ones they would be more complicated versions of what we already have, so these are the only three elementary bifurcations for continuous-time systems as I said earlier the number of bifurcations possible the classes of bifurcations possible in general dynamical systems is not known in general especially if the core dimension increases beyond two or three or four.

Then it is hard to classify these bifurcations although a lot of work has been done in this along these lines but for the elementary cases we are looking at these are the basic ones in the verifications we will come across a few more we are going to study in this course a few more bifurcations which are also elementary bifurcations. But which are not of these types and they are current slightly different dynamical systems as we will see there is one more which is very important goes along with these and that is called a hope bifurcation.

And to lead to that I need to introduce yet another concept namely that of a limit cycle so let me do that now, yes okay oh yes the question is what's the correct way of unfolding a given degenerate form and there is an elaborate mathematical machinery to do this all these bifurcations are cast in what is called a normal form namely the minimal form or the minimal expression which leads to that phenomenon the bifurcation these are already the ones have written down here are already the normal forms apart from some constant multiples or scaled marks which could be scaled out some constant factors these are already the minimal forms you could make them a little more complex.

There is a branch of mathematics called catastrophe theory which deals with this problem of unfolding these singularities and writing things in their normal form the minimal form and this is elaborate doesn't elaborate theory to this effect and I've just given a flavor of it a little glimpse of it in this in writing these forms do not so there is a systematic way of doing this of unfolding these singularities now let me go on now to the idea of a limit cycle let me introduce this as follows.

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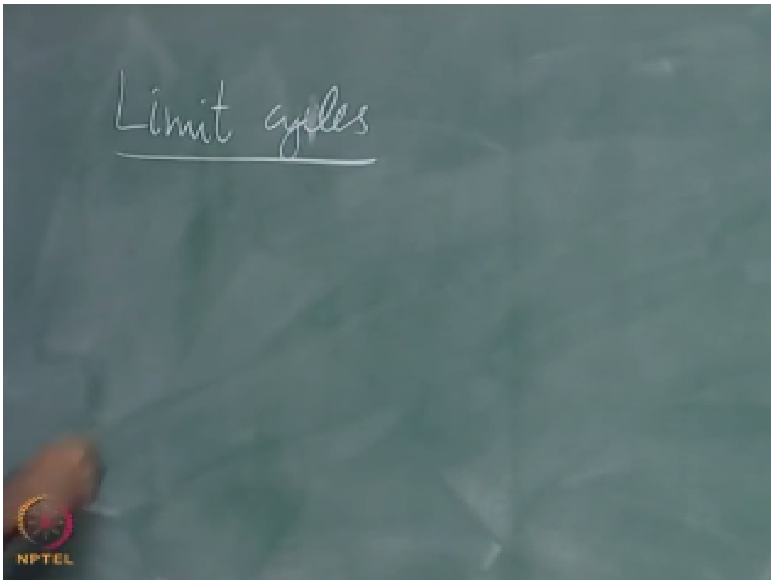

In our study of two-dimensional systems we looked at various kinds of critical points these are points where the right-hand sides of the two dynamical equations vanished one could ask are there sets of point’s continuous sets of points where you have some kind of equilibrium. So you don't have point attractors or point repellers the point singularities in the vector field on the right hand side but can you have lines or can you have whole sets of continuous sets of points where you could have such behavior equilibrium kind of kind of equilibrium behavior.

But the answer is yes because let me do this again with the help of examples and then limit cycles we're going to say a lot more about limit cycles as we go along because this is a typical feature of nonlinear systems and unlike the case of critical points these are much harder to detect and study so it makes them interesting and let us look once again at an example.

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Funded by
Department of Higher Education
Ministry of Human Resource Development
Government of India

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