Classical Field Theory
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Lecture – 9

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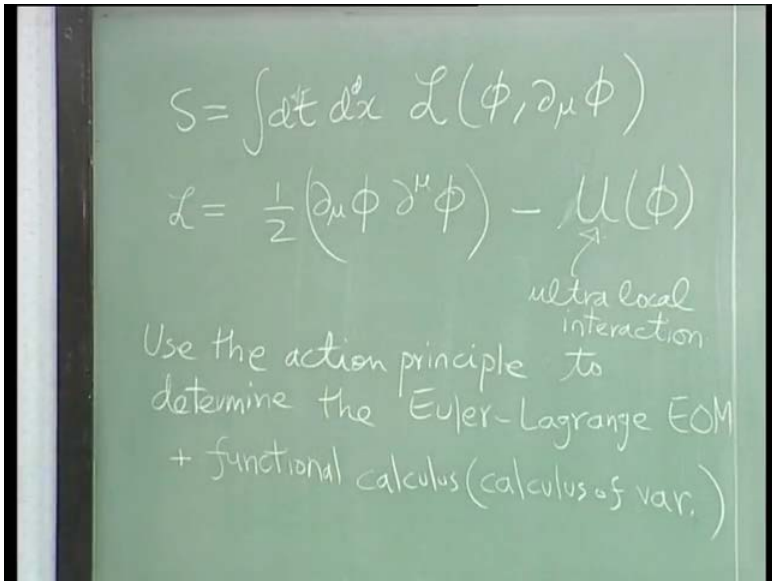

Last lecture, what we saw was that action for a field can be written as an integral over time as well as space over some Lagrangian density, which is a function of this field. And its derivative again, remember the derivatives include both special and time like derivatives. And we also say that the structure of l, should be such that the special and time derivatives combined to give you, this Lawrence scalar and all that remained. If you restricted yourself to two derivative pieces out here is a part, which we, which corresponds to the ultra local interaction. So, in continuum field theory, is one loosely refers to this term as the kinetic energy term, even though it has the local part of the special part, derivative part of this is actually not truly a kinetic energy in the sense of classical mechanics.

So, but nevertheless this is the general structure we call this the kinetic energy and this we will call the potential energy. So, the thing to do next is to use the action principle to determine Euler Lagrange equation of motion. Equation of E O M in short for equation of motion, so this is what we need to do. So, fairly straight forward but what we will do is make a slight detour into what I call functional calculus, but it is probably better to call it calculus of variations because apparently there is something else in mathematics which is also called function calculus, which is not quite the same as this calculus of variations.

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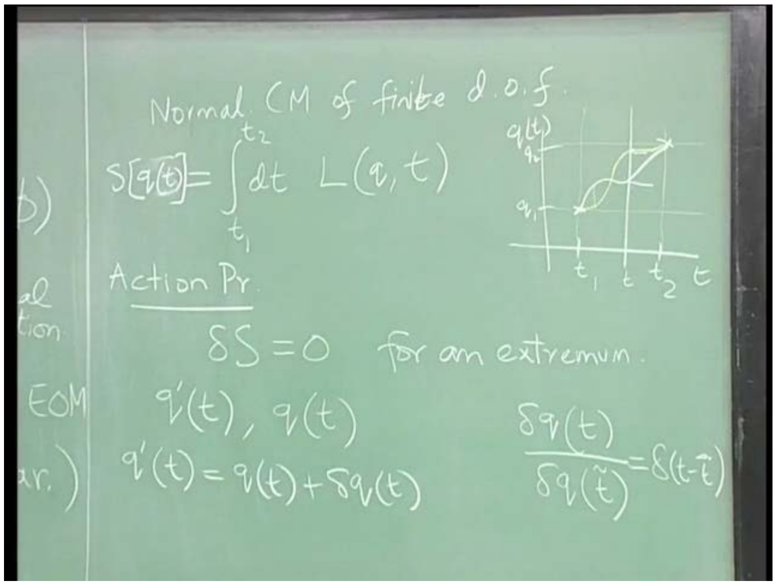

So, let us go back to just normal classical mechanics of single degree of freedom or finite degree finite number of that let us use finite degrees of freedom. So, in such cases the action principle is quite straight forward, so as usual I will use the one degree of freedom to draw pictures, but later we will generalize. So, the problem is as follows. This something which we discussed long ago and the may be in the first lecture.

So, you have some initial time t 1 and later time 2 and you want you are given that the say the system is at t 1 at initial time t q of t 1. Let us call this q 1, q 2 and there is some trajectory, so for every trajectory this is a functional so you write this as S of q of t. So, now the action principle tells you that if you take two different you have to extremism this action. So, it is like taking the first derivative of this thing with respect so delta s equal to 0 for an extremum.

So, what you are doing is you are playing around with these trajectories, so the argument of this thing is a trajectory so for every trajectory you get a number and we keep varying the trajectories. So, suppose we have two different trajectories, so the key is so the these are two different trajectories and let us pick some arbitrary time, what you see here is that the trajectories here differ and this thing…

So, let us say let us call the two trajectories q of q prime of t and q of t and important thing is that we can compare these two things at the same time like I have done here. And call this object delta q of t so we could write it two ways. We could say that q prime of t is equal to q of t plus delta q of t. See the important point to remember is if you take two different times the delta q at two different times are not correlated they are independent of each other.

So that statement we will use that to say that delta q at some time at t and delta q at some other time tilda is would be 0 when t is not equal to t tilde, but it should be equal to some non zero quantity, but it has to be the direct delta function. We do not say it is one if you discretize time it will become 1, but this is so this is just the statement that at two different times, the delta cubes can be completely independent they are completely independent of each other.

For instance, I could even chose a trajectory which has the following property that it more or less follows these things and just has some deep or something in a small region and goes on, and it is more or less the same. So you can always make that deep as small as you like and in that sense and so, in a limiting sense you will require exactly something like this. So now the question is we have to ask, so what did we do if we had a function this is what we would have done the analog as.

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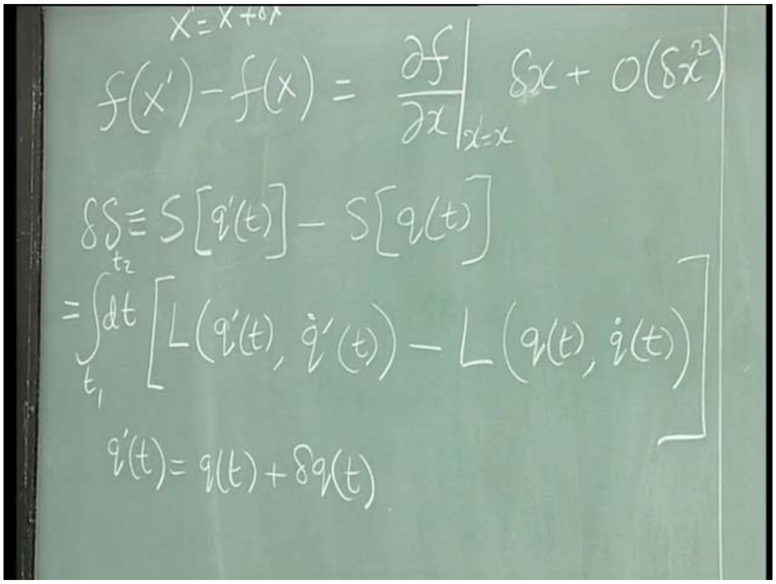

So we would write something like a Taylor series. If you have two phi if two points so let us say f of x prime minus f of x. So, the analog statement to this would be x prime equal to x plus delta x, so we would write this as say valuated when x prime equal to x plus times delta x plus order delta x square.

So, we will repeat the same exercise replacing these functions with function x. Just definition, so now we look at how we write out the action which is out here. So this is what this is nothing this is just taking the definition and putting things in here. And now comes the nice stuff which is to realize that we know that q prime of t is equal to q of t plus delta q of t.

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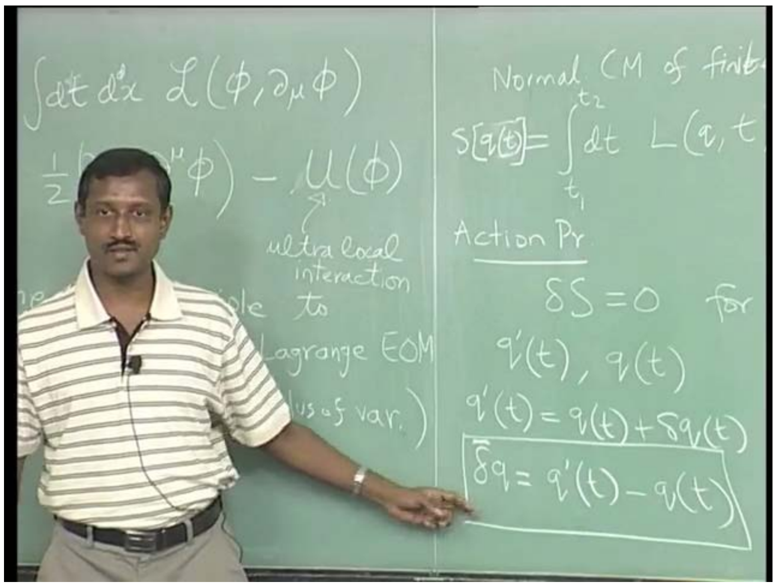

So, we can rewrite this if you wish as delta q of t. So what we are doing out here is look at look what is happening out here, the argument is the same, so all that all you are asking is how this function has changed. So, this is the change in the functional form of this q of t. So, just for later I will this bar is to just emphasis that this is the change in the functional form. We will see that this is important.

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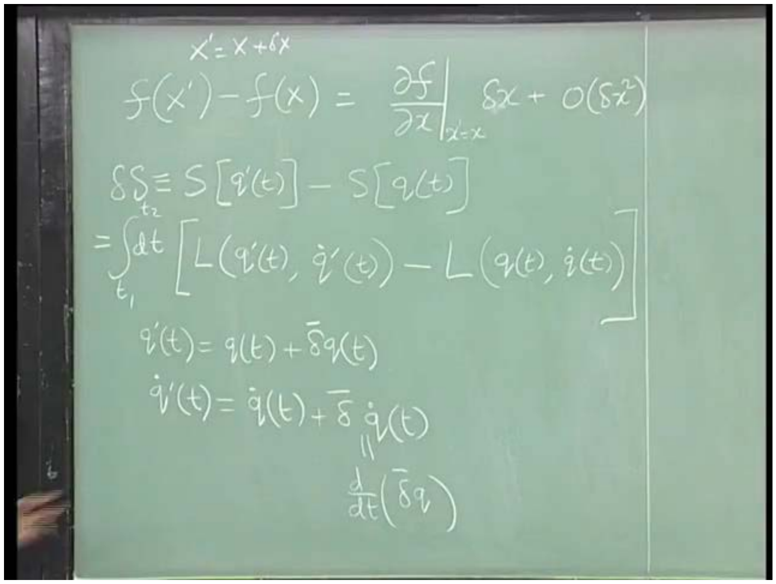

So, one important property so I will put the bar out here just reminding you that its only this thing. And so then you would like know what is q dot prime of t. So if are given a trajectory obviously you can take a derivative. But the important thing to ask is just delta bar commute to taking derivative. I would like to recommend that you go home and think about it delta bar is such there it will necessarily commute by its definition. So, q prime would be q dot of t plus delta bar of q dot of t, which I can write now this particular term can be written as d by d t.

So, I am allowed to exchange the order of these two things operations or equivalently it says that delta power of determines not just q of t it determines all derivatives. It is just not for one derivative even for hundred derivatives we expected to go through assuming function are smooth etcetera. Let us not get into those important details. So, now you can see that the there are two contribution which come here, one from the first argument changing of the and the second argument.

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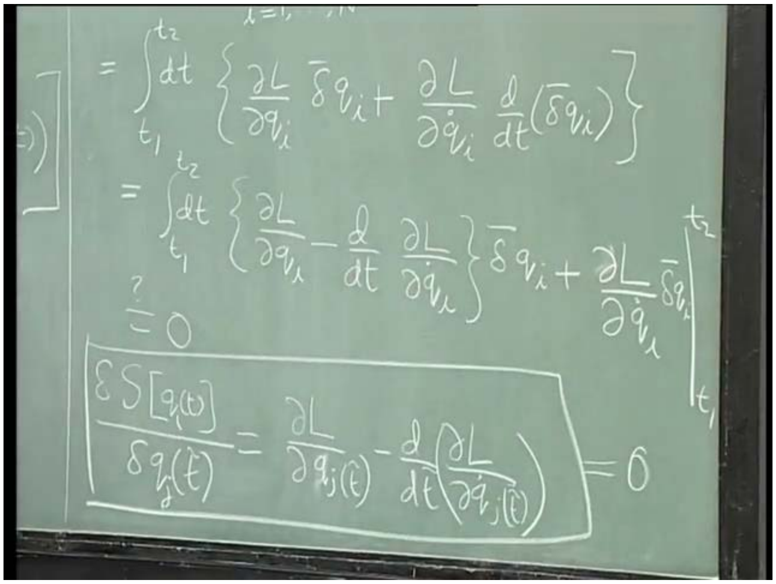

So let us put that in and expand this is just the function of two variables we just need to know we just need to Taylor expands this to first order. So let us do that, before I do that I will not keep writing the zero thought of pieces a will cancel by definitions. So it is only the first order piece which will remain so I will not otherwise I have got a keep saying this is cancel it, I won't repeat this I will just write out the piece. So, there will be two terms one coming from the first argument changing, times delta bar of q.

So, far I am not use this part but I have just written out these thing and now, I will obviously do I will do the following thing I will generalize this for more than one degree of freedom. So that will just tell you that, so let us say I runs from 1 to n, so is just a replacing function of two variables with two n variables. So, for each one of them I have to do the same thing. And the only thing which we need is the generalization of this is I and j and I fix that by putting a chronicle delta. So what this tell so first this original direct delta function tells you that if you keep take a particular I for two different times the delta q are independent of each other, that is what this says.

What this says is there if you have two different fields does not matter even if they are at the same time. There variations are independent of each other. So, that is what captured by these two things. So, now we use the analogues thing out here and we get, so I will not keep writing summation over I it is obvious this is sum overall I. So, I can yes so write it as two steps. Now I can integrate by parts and make d by d t act on this so that I can pull out delta q bar, so that I could write it exactly some exactly like this.

So, plus there is a surface term which is evaluated at the two end points. So and we choose our boundary conditions such surface term vanish. And now comes the this point which I have been emphasizing all through it is a fact that at different times the variations are independent of each other. So out here so this an integral over t; that mean for every particular time inside this if this has to vanish so this should be 0. We already said that we will choose boundary conditions such that the surface term vanishes.

So, it will only follow if these terms are 0 separately, term by term. So what you do so this whole thing can be re written to look like this as follows. So, now you can see that I can rewrite this as delta S. So I am, what I am doing here is pulling a taking a derivative with respect to delta q tilda and I will use this formula which we have which is delta q by the delta q I by delta q z at two different times delta product of the chronicle delta times delta t minus t delta. So, the first thing you can see that since I have t delta of t minus t tilda all occurrences of t becomes t tilde.

And all occurrences of let us call this j and i, i become j from the second chronicle delta so I just rewrite it follows. So, the functional is so this is this object has right to be called the functional derivative because it is a derivative of a function with respect to change in its arguments, which is a part. So this should be 0 of course. That is Euler Lagrange equation of motion. So, now the thing the equation is how do we just go head and generalize it to the case of a field and that is not so hard because all we need to do is to observe that the index j. Actually before I do that may be for once in this course I will write out for you the next term in a, how a next term in the analog of a Taylor series for a function would like for a functional just for completeness.

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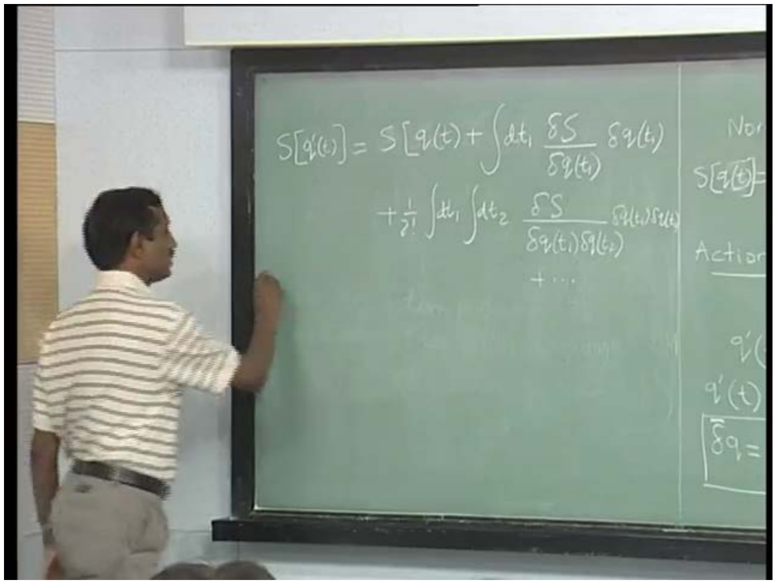

So, for instance so if you have so S of some q prime of t will be equal to s at q of t. So roughly speaking you can just go to a function and of many variables. And what would you have got here is a sum over I; next one would have been sum over I and j so on so forth. And the same thing is replaced here by integrals. You can see that this is a non local guy, it involves non local in time it involves two this things, but a functional this is to be expected. It is just that the first term is nice and local while this is not.

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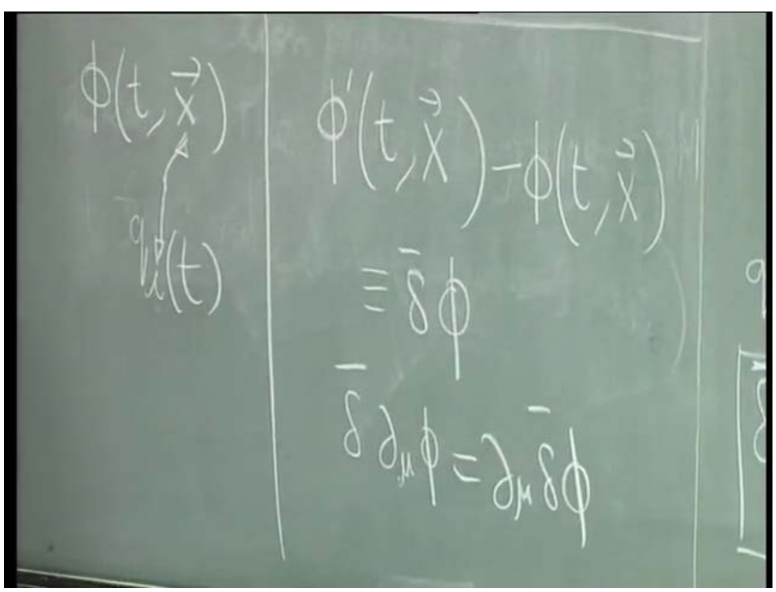

So, now let us just ask how so let me consider the field and for once I will just write out separate out the space and time and space part. So, you can see that this x variable behaves like the I index in some sense except it is continuous. So, this is compare it with q I of t so you can identify this i which is the discrete index and you can think of x as a continuum index and so now we can ask what would be the analog of the statement. So, delta first thing is let us write out, what we would define so what, so changing a trajectory would correspond to changing this phi I phis. And so now so earlier we only have to worry about these thing being local in time, but we also have to choose the so this is a function has two arguments. So, you have to compare these things.

Even here we did something similar, you would not write if I put the I index into the story you would see that you only compare things of the same kind. So, now you can just forget about that detail and you can see that what we are looking at is a function of many variables t and x, but we comparing them at the same point in spare time and space. Better to say that we are comparing them at the same point in time space time. So now we because we have a continuum we have notion of derivatives with respect to space as well.

So, now you can convince yourself that delta bar of d mu of i equal to this statement does not need anything beyond this two lines, I mean I it is not it does not refer to anything because we are comparing guys of the same argument. So, this has to happen that delta bar and d mu commute with each other. Not one derivative again we need a betters but for our purposes this is important.

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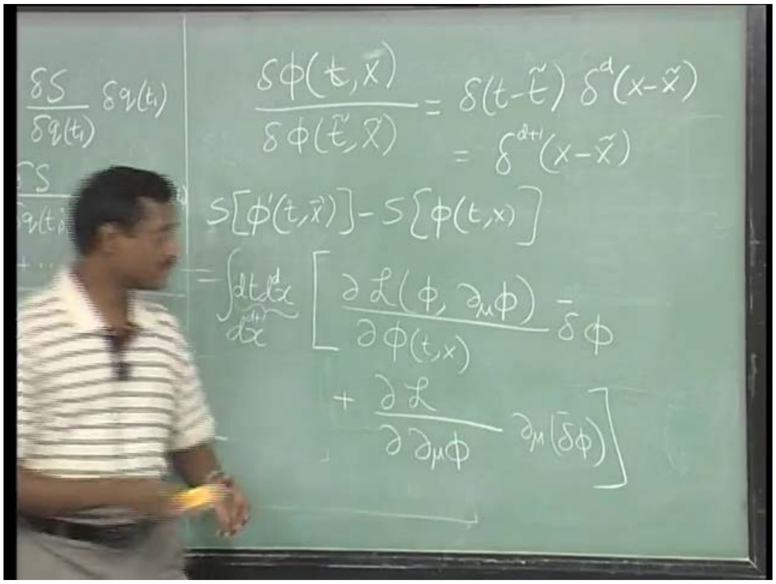

And further we can see that analog of that statement would be phi of t and x. I will not keeping writing of a vector on top of it it is as many as it is enough for me to mention that there is a special dimension. So, earlier what we had was a single delta function in, that came from this part, but here again at two different point we expect the field to be independent. Special points that is. So, you will get a direct delta function replacing this sort of a thing a d dimensional direct delta function for every dimension you get a delta function.

So, but what we will do is we will just combine all these things and usually write this as a single delta function of just the four vectors we think in terms of d plus one dimensional vectors. And if the dimension is specified one does not even write these thing is taken implicitly that it has all the delta functions. So this is natural generalization so I and j become continuum, so the chronicle delta here became this product of delta functions. If you are unhappy with this, what you can do is to go back and look at discretize say time or space just to understand these examples.

And see that the if you remember when we did integrals we saw that summation over I went over to someone by a integral dx. You may wonder should not I have factors of a floating around you will find that things will go perfectly fine, you will see that there will be a compensate why should I write 1 by a here there would come an a out here. And at least dimensionally you can see the delta function has the correct I mean so the dimension etcetera will work out perfectly.

So, you will get a compensating factor of a etcetera and it cancels away. And so that is the nice way of understanding how chronicle delta gets replaced by a direct delta function.

So again we need to do the same thing and carry the whole process through, but it is not so hard. Again the rules the hardest part of the rules are only these things, which we already have worked out. And little bit of notation this curly delta is different from this partial. So, let us look at this out here see this is the functional derivative, while this is a normal derivative, l is a function of its normal partial derivative. So, we distinguish between these two things by in the by this notation.

So, once you see something so whenever you write equations of motions in terms of Lagrangian. So, now we will see in terms of Lagrangian densities, then we will be writing things in terms of ordinary partial derivatives. So, we need to do the following thing we have to again workout how. So now the first change is that we have a derivative I mean a integral over both space and time again. Most of the time we just write this as d d plus 1 x times. So this is a Lagrangian density, so we need to work out so there will be it is a function of phi and d mu of phi. So, we have to so now again it is like a function, which has two arguments I can just take a derivative with respect to that.

But now for every derivative there is a the so there is not just phi dot there is also special derivatives. So, unlike what we had out here, we get extra terms, but actually if you think about it is even that part is there because there is a summation over I. It is just that so it is there implicitly, but this what you get and so now I where d mu is the derivative which includes d by dt and this kind of stuff. So, this is what we get which is analog as to that, but now we can again use these thing and I can exchange these two terms. So, now I can again integrate by parts, but now I have to carry out several integration by parts, one for every derivative.

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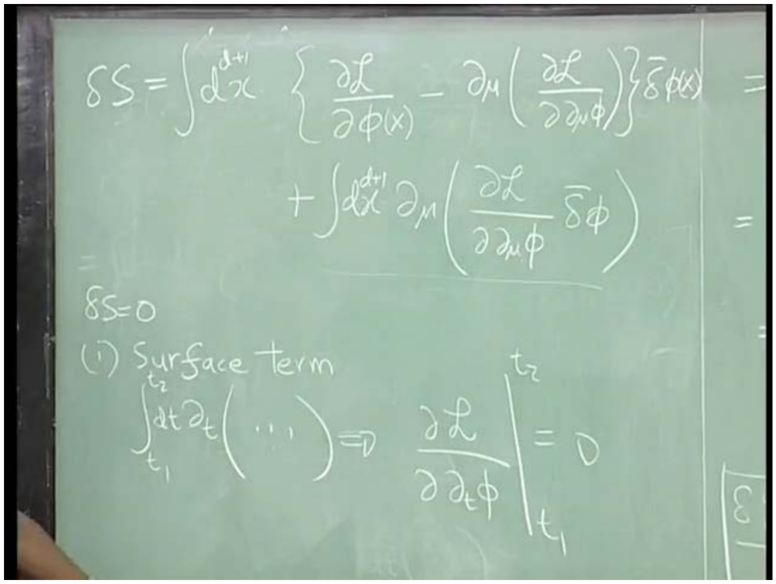

So, we get delta s equal to, so I am again now in the from now on I will not separate out t and x I put them all together. The total derivative piece which I write out here. So, this is what we get. So, we have to worry about more, more terms out here, so the first part first sorry delta is equal to 0 we will do it in two parts as always we will say that object in the curly brackets curly braces is 0. But we will but we will also need the surface to vanish. So, let us look at the surface term first. So, of course mu equal to 0 which would give you the time derivative will look exactly like what we had before.

So, let us look at such a piece so da, so this is one term which looks like a d t of whatever this should so this should be 0. So, this would so you will require that at so this implies that the combination that you get dl by dt of phi should be 0. So this exactly like what we had earlier, but we have some new parts new terms, which were not there earlier and that comes from the special part. So, we have to use the analog of the gauss divergence theorem for that.

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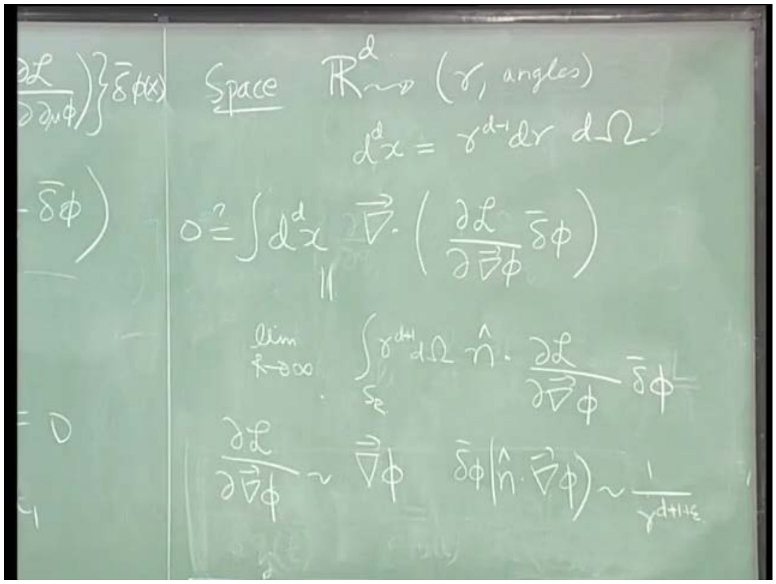

So, let us take let us assume that space is just R d, d dimensional space time space Euclidian and what we do is I will convert these things. I will write this in I will go I will change I will think of this as follows, I think of working in the analog of hyper spherical coordinates because, I am not specifying d. So, d equal to 3, it will be normal spherical polar coordinates. You go you chose coordinates which is r and a bunch of angles. For three dimensions it would be theta and phi the usual coordinates. And the d special derivative would you would write this if you wish in dimensional grounds it will looks something like this r d r minus 1 d r times some angle integration which I write as d omega.

So now the other part is the special part, so which would look like d d of x times some gradient d dimensional gradient dot this is the term I am just writing out that part. So now this should be equal to an integral over sphere of radius r. This is easier than it looks. It just tells you that this is the radial derivative. So, this will pick out so if you have a sphere. So, what I am doing is thinking of getting R d as by taking as sphere the integral of sphere and making the radius of sphere go to infinity. I could have done it by putting it in a cubical box life would be easier but there is a reason for doing it for later discussion.

So, we will want this thing has to be 0. Even this term has to vanish so now you see the there are conditions you get on these derivative of this special derivatives, how they have appeared here. It has to be such that it falls of faster than some, so now if you go and look at our particular example let us lo at what that term would be. So d l by d of grade phi would be up to a sign would be gradient of phi theta, right? If you look here Lagrangian density which has which was there several boards ago, what you see that is that you would get something like this. So, what this tells you is that the radial derivative should fall off. So, n dot so we need n dot grade phi to go to 0 faster than plus some epsilon, where some epsilon is some positive thing, if that happens then I am guaranteed either that or actually to be we need something weaker than that delta bar of phi this thing has to vanish.

So, usually what we will do is we will impose conditions, such that these terms go to 0 faster than this. All I need here is epsilon should be some positive number then it will vanish. So, we cannot so if you are looking for equations of motion, what it tells you it is not enough to specify things just that you know initial and final time thing go to I mean fixed the variations. But you need to also worry about special boundaries. I leave you with an exercise which is to consider just r which is one dimensional space time.

One dimensional space; two dimensional space time, where this integral is just a normal integration by parts and you can convince yourself that this just tells you how things fall off. So there this radii there are no angle nothing r is just x the coordinate. So, you need so you can see that to make your problem well post, you better have certain fall off behavior. We will come back to these behavior, this is so it is important to remember that there is some fall off you cannot just say I will just solve this whatever set whatever is inside this curly brackets equal to 0 and I am done. That is wrong your problem is not well post.

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Yes, it is d minus 1 is a surface integral the d r goes away. So this is some hyper sphere of radius r. And I just take r to infinity then I am getting the full boundary. Is this yes there is no integral over r because I have integrated by parts in that, but usually what happens is that you will want you know the energy density energy in your system to be finite not energy density to be finite. So, in such cases you usually find that these conditions are more normally satisfied. In terms of what we will call physical boundary conditions, so that is why quite often we end up not worrying about these issues we just take the equation. So, obviously the object in curly brackets can be should be z to 0, so I will write that as a functional equation.

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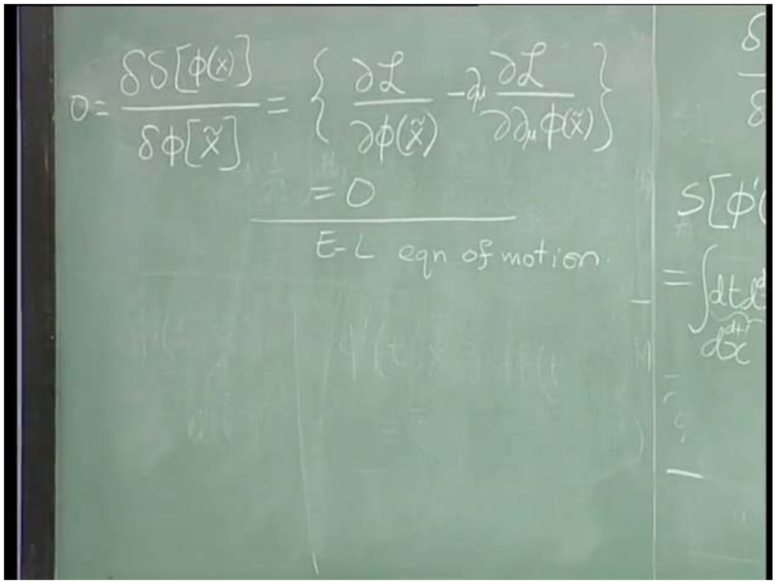

So, delta S of phi this should be equal to 0. So what that will do is again as before you get a proper delta function, which will take care of the integration and which will gives you this term. So, this is the Euler Lagrangian equation of motion. So, we see a slight difference from what we saw in the classical case in the normal that there was there is also special derivatives coming out here, but you can see this is very nice covariance is actually built in. If your Lagrangian density is Lawrence scalar then things are it is this object is also by Lawrence scalar. And you have one equation for every point in space, which is like saying that there has many equations.

So, this is just a equation which puts these things. So this is the Euler Lagrangian equation of the motion. So it is more or less whatever it is just a repeat of what we did earlier for the case of finite degrees of freedom which some small belts in this. So if you lo here all the Lagrangian density I am considering I have not chosen it should be a function to have any explicit time or space dependents. And this is related to the fact that if you want translation symmetry in space and time.

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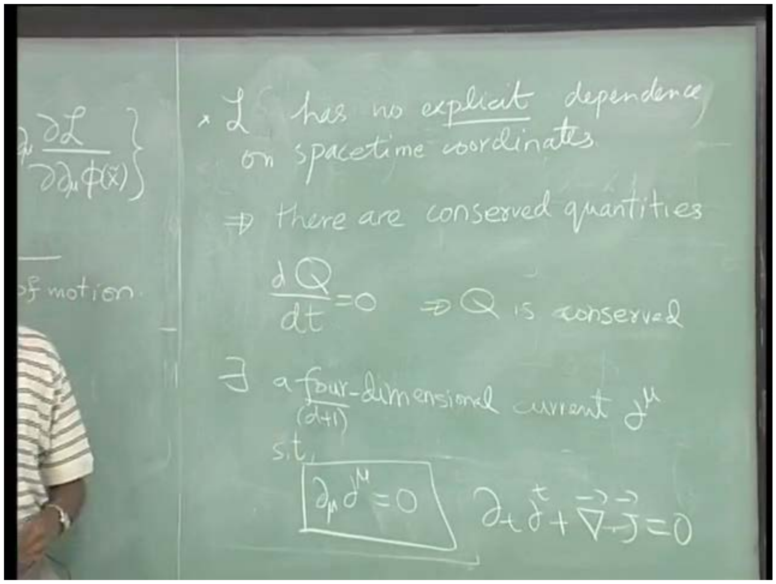

So, has rather say the Lagrangian density has no explicit dependents on space time. And we will show later that this implies that there is a conserve there are. So, let us see how what we mean by this statement. So, what you do is this statement is written as follows. So, normally what this tells you is that there is some object Q, which is a constant of motion, in other words you just take a derivative of it this is what we mean. It remains constant through the trajectory of the system once you measure it.

But when we say something is conserve is it is saying something much stronger than this, and this is just happens in the context of field theory. What we are saying is that there exist, so this statement is that there are exist for four current if you are in four dimension space time, or to be more precise d plus one dimensional current k upper mu such that If I separate out the space and the time part in this you would get something like this. So, this is like the equation of continuity. So, what this implies is something much stronger than just this global statement.

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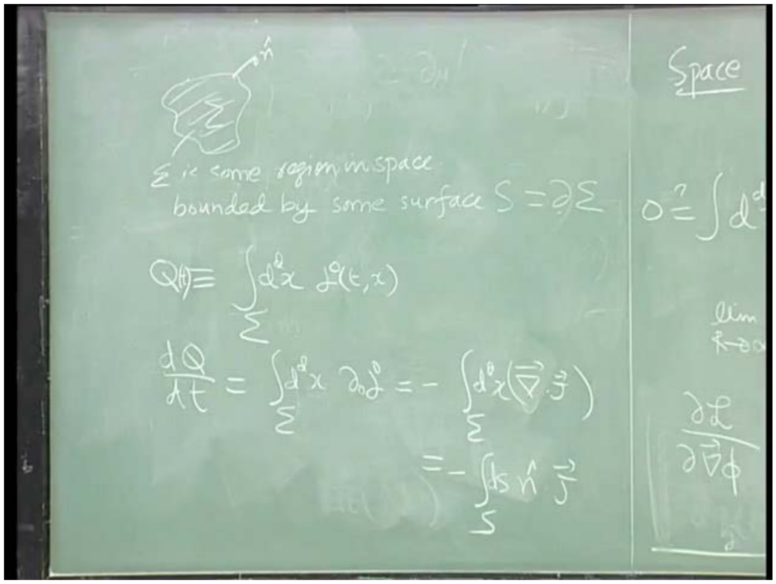

What it says is that so suppose you have some region sigma in space this is sigma, some boundary some region in space, with some boundary bounded by some surface s. Mathematical notation for this is to write d of sigma to write this partial of sigma but for us so for instance an example would be to take say cubical surface and look at the interior region of that. So, now the boundary of that would be the phases of the cube. So, you have something like this. So, what this tells you is suppose you are given me a four a current like this, I can define q to be integral. I take q to be the charge which is inside that volume.

Now the thing is that what would be the charge inside the volume and if you ask how does it change with time that would just follow by saying. So, note that even though j zero is a function of t and x this is only a function of t because you are integrating out all the x dependents. So, is just the ordinary derivative of this, but now if you are going to write it inside this thing it will become the partial derivative. Well now you can see what is happening here, d 0 j 0 or d t j t is minus of these things. So this is equal to using the equation of continuity kind of.

Now, this is exactly similar to what we had here. So now you can again do an integration except here I chose my thing to be a sort of a ball or with boundary s r and then took the limit here it is just some finite object, but his will be this I can rewrite this as an surface integral I will just write a d s and there will be some direction. Normal direction being every pointed some outward pointing, normal with a minus sign. So, what this tells you is the rate of change of charge is equal to how much is this just this is measuring how much is escaping.

So, this tells you that does not matter what is this, how small you take your thing, if you are able to account for all the charge which is going out of the thing and what remains than. So, this is in some sense is a much stronger statement, but of course you could take the integral over take sigma to be all of space and time. And then and if there are no currents at special infinity usually you assume there is nothing at infinity this term just goes away. And then you get the dQ by dt equal to 0, which is what I wrote first. So what I am trying to tell you is that this is the stronger statement than this. I could in principle have total charge conserved but not locally conserved.

So, this is a much stronger statement because you can account for what is moving and moving from region to region. So, anytime we say that there is something conserved in this course, what we mean is something like this. That something is locally conserved. For instance, some beside that the Lagrangian as no explicit dependents on space and time coordinates, so let us focus on time.

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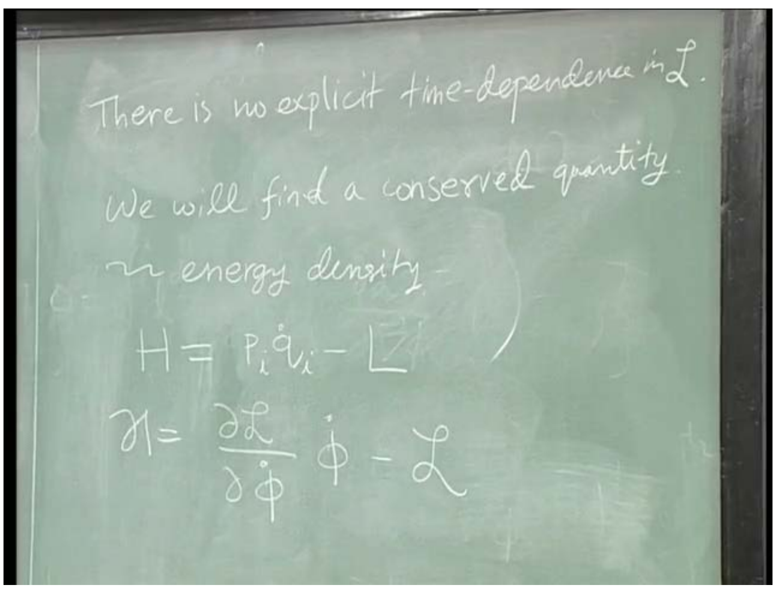

So, in classical mechanics when we had systems, where there was no explicit time dependants we got a conserved, we got a constant of motion which was the energy. So, just lets so here also here we will find that we will get conserved quantity, which will be the analog of energy density.

So coming back to this j 0 will be normally identified with this is we call this charge it would be charge density, so if you have energy you would call it energy density.

So, again we will mimick what one saw and this is what one hand in classical mechanics. So, we will just we mimick like this and I will write we will prove that all this holes the analog of j 0 here will be. So, this will be what we will call energy density. So, I will just use the normal definitions out here. So, this what we will get and let us work out what this is.

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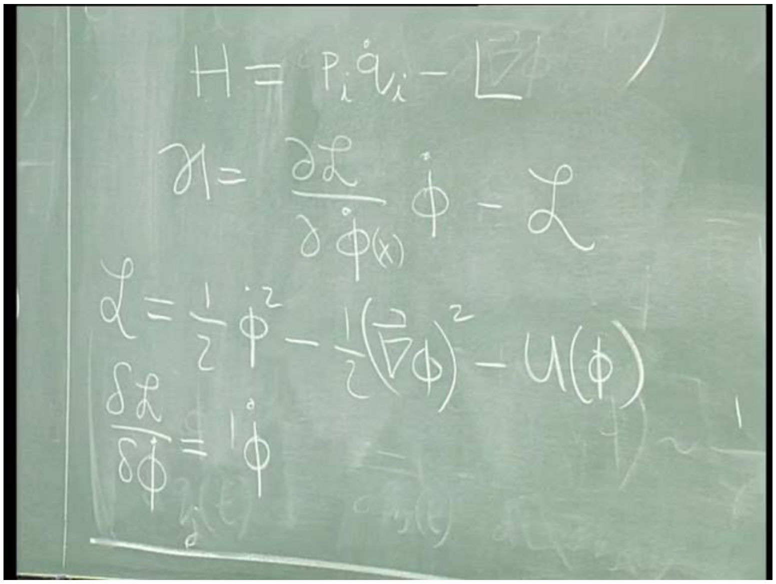

So, if you coming back to our example which chose l to be half phi dot square minus half gradient phi square minus u of phi. Of course this has an argument of x etcetera. We are just be hiding all those things because for what we are going to do here we just will look at where the phi dot appears and here is phi dot so delta l. Sometimes we will use the symbol pi for this pi naught will be half the half goes away will be just phi dot because there are two things, this is what you get. So, we can put this back here and we can look at what the Hamiltonian density is we will prove that this is conserved etcetera, but that is not what I want to do, I want to connect up with something in my discussion earlier.

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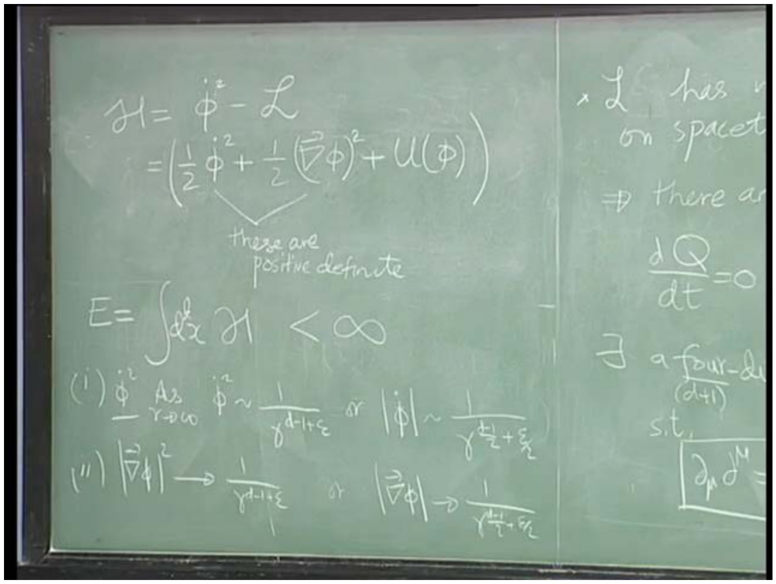

So, this would be just phi dot square minus l. So, now I just plot things in there I get something very, very nice. So, they this two things came here with a minus sign and because it appears out here in this fashion that will science get flipped and so we end up now getting all positive. So, you can see that these two are positive definite but u of phi may not be. So, clearly so let us ask a condition for which, let us say the energy which is the integral over all of space of the Hamiltonian density should be less than infinity.

So, when so what are the conditions which should have such that it is less than infinity. The integral over this. So, it is not enough for this so clearly there is it will impose conditions on each of these terms. For instance suppose you said phi dot were a constant, that looks, but if you come here you put that constant in here you can see it becomes constant times a volume of space, which is infinite. So, you need certain fall off behavior for things to be finite. So, let us look at there are three terms out here. So, the first term will look at is the phi dot square term.

So, again we will see that we can go back and rewrite things in terms of radial direction. So, phi dot square better go as 1 by r power d minus 1 plus some epsilon. Only then the integral at least it is a necessary condition not sufficient if phi dot blows up in somewhere, in some finite region that will be a problem at least, but I am just worrying about the condition of special infinity. Phi dot square should go like this or we should say that magnitude of phi dot should go like 1 by r plus d minus 1 by 2 plus epsilon by 2, but I can just write it as leave it like this epsilon is some positive number so it should at least fall off like this. It can fall off faster than this.

Similarly, gradient of phi square should fall off again, similarly or otherwise again this term will like this. Now going back to this you look if you look at this is stronger than what we required earlier for that thing. Is it stronger or weaker? It is weaker. It is weaker. So this is weaker than that one we will get to that later, but we are not done yet. What about u of phi? So now comes the thing, first thing is if you want things to be positive definite we will require u of phi to be bounded from below.

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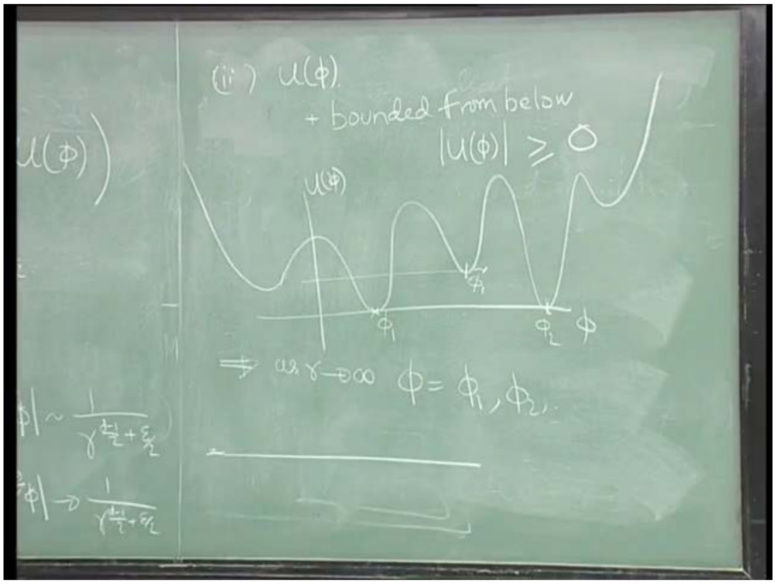

So, the third part is condition on u of phi bounded from below, by that I mean mod u of phi should be greater than or equal to some constant everywhere. For any value of phi just a function. So you should have a bottom but let us ask what kind of bottom it can have can it have negative number then it will be negative so you could end up with negative this thing so you think that I could I could settle for a positive number c can be positive yes.

Student: (( ))

No, it is bounded from the bottom I want energy density from below to get I mean so I i will say greater than or equal to 0. I will if I put a negative number then it can become I want my energy to be positive definite. I can adjust if it is not so what my point here is suppose it were some constant. I can always add or subtract some constants to make this 0. If it is bounded from below, it may not be bounded from above there is nothing. So this is just a condition which I will impose. See the thing is if it were remain a positive constrain it would not valid positiveness, but you will end up getting the minimum value would be the that constant times, the volume of space.

So, you always adjust things that way such that it is bounded from below and its adjust u of phi to be this way. So, it could have you could have a possible taste would be u of phi versus phi. Stuff like this any lets draw some functions, can have many can go off to infinity could be something like this. And it is important that there are points which have 0. So, now we have come back to the fact that we would like the energy to be not just positive definite, we wanted to be finite. So, what about this piece, so this requires implies that u of phi, phi should be so let us call this special that is phi 1; let us call this phi 2 implies that as r tends to infinity phi should be one of those minima, which has 0 which take 0 values where it takes 0 it can.

If it did not then this will start contributing some finite number even something like this you may think finites it is a good thing. I can use this point for let us say let us call this phi tilda 1, but if you put phi tilda 1this is some finite constant and that term will blow up.

Student: (( ))

Yes, no first step is that yes so I am just saying that whatever happens, it better heat this evaluate this thing.

Student: (( ))

Yes, it should heat it should reach one of its 0 values where it becomes 0. It is a necessary condition.

So, yes I mean this is so this is just to ask the following question when is this positive definite and finite? If you do this you see that you start getting conditions on the fields. I am not saying that these are all I mean like for instance this and this are independent of each other of course they are, but you can see. Now, suppose if you had one dimensional space time, what it tells you is that you can have solution where the at one left infinity it could be phi 1; at the right infinity it could be phi 2. But now we can ask what is the lowest energy, which is possible that would be 0, will call that the vacuum.

And when would that happen, when you look at situation where the fields are space independent and time independent. That means they are constants, but what is the constant values it can take it, it can take only phi 1 or phi 2. That would be the lowest energy configuration it is the 0 energy can configuration and so we call it the vacuum.