Classical Field Theory

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Lecture - 8

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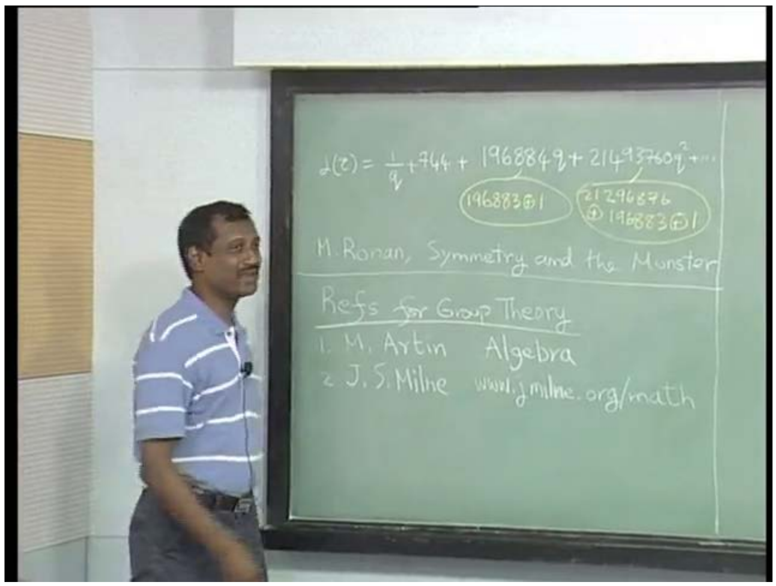

So, last lecture I had mentioned the j function, and but I did not remember the exact coefficients. So just for completeness, I mean you are not expected to memorize this numbers or anything. Let us look at the coefficient of q, which is 196884 and it can be written as this. And what this implies is that this particular number is special to the monster group. If you try to find a matrix which non-trivial matrix representation of the group, the smallest one you could find, which is a reducible is of this dimension. So, it is very difficult to cook up I mean this number is not just cooked up. And 1 mean single - singlet, it means it is trivial. So you just take a 1 by 1 matrix, where every element get mapped to 1 and then, so that is trivial representation. So this is also the next representation, bigger representation. So, it is sort of mysterious, why these numbers come? And the story it is sort of this is a story it is a popular book with not too much mathematics by Mark Ronan. It is symmetry and the monster it is a fun read and I recommended. And let me give some references for group theory.

(Refer Slide Time: 01:30)

The first one is Michael Artins book on algebra. It is actually an undergraduate linear algebra book, it has some stuff on group theory. It also has lots of things on linear algebra, I would recommend, if you do not know many of the things to work through it. This, and J S Milne has actually lot of stuff on his website, which I have given the link out here. And for group theory has a nice sort of 150 pages of notes, which you can read through and here, is some exercises. And I would recommend sort of working through some exercises, so that you understand the ideas which go there. There is no shortcut to that, I mean you if you just keeps there are so many books on group theory you can keep I can the list can keep going on and on. But what I recommend is that actually playing around with things.

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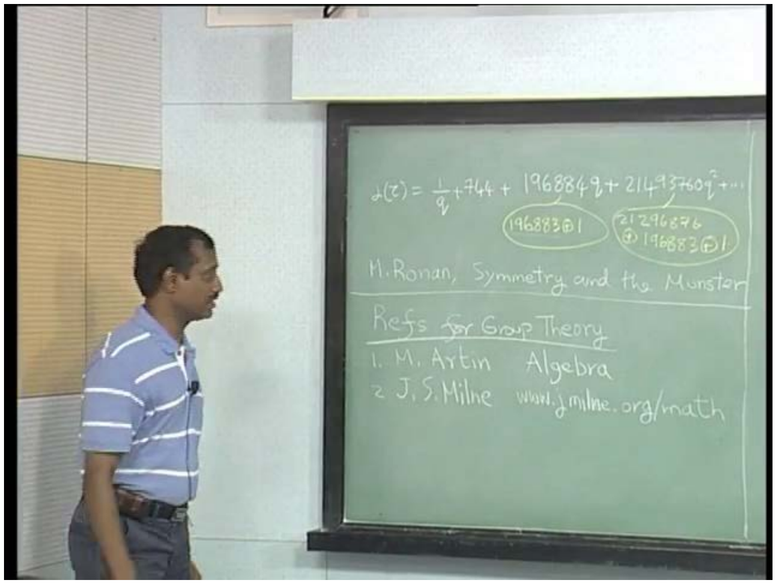

I will also probably give you an assignment on finite groups, so that you can play with some specific examples that I see fit. These all I have to say about group theory, but outside of class somebody asked me this interesting question. Do we really need so much abstraction? And the short answer is yes. The longer answer is that if you there is no getting around the fact that the symmetry in nature is captured by this wonderful idea called group theory. And the first sort of important use of it was made by Wigner in physics.

And since, you will be looking at relativistic field theories, which is usually used in the context of high energy physics. There again the whole development of the field actually, made use of group theory. One important instant is called the 8 fold way. So, people are looking at nuclear resonances and there were plenty, I mean everyday they were finding one and the question was is this a new particle? Is it some you know, maybe I can name it after my wife or whatever. You know it, things like that, did happen. And so but there was Gellman and at some point realize that there was some group theoretic structure. And the group which actually sort of may made him organize things was what is called SU3. And the 8 fold way refers to the fact that SU3 as an 8 dimensional representation, and certain set of objects could be organized in a multiplet of size 8.

Another place where you would see you might have already seen if you done quantum mechanics, you have the angular momentum algebra which you would have seen. Again that is a very important use of the group, rotation group in quantum mechanics. So, there is no getting around the fact that group theory is something which is very important. And the idea here is to do things in abstraction because in some way that is easiest. At the thing is if you are able to digest the fact that there is a representation of a group and it has a certain dimension. The minute I say, something like the 8 dimensional representation of SU3, I mean that is actually a very well defined statement. And as this course goes along, you will, it will sort of you will also learn more about these things. And you will become more comfortable with it. And you will realize yes this is the only way I could say it. You know why should I spend you know, 25 minutes trying to explain what this 8 dimensional representation is when it can be said in one line.

And it is very precise and this is the kind of precession we will need and we will use. So that the Sir John into finite groups was to give you exercises, which you could play with and do something more quite explicitly. There is not much you can do explicitly in general because, you need little bit more of theory to see the structures which we will see as much as we need. And another point were doing the math in this course is because, we would like to I mean if you take group theory it is a full course in a mathematics department. But look we want to just learn certain enough group theory if you get by. And depending on your personal interest you can read up things nobodies is stopping you from that. But what the goal of my lecture here, is to fill in the math background. Because most of you I mean have a mixed background and you know something, you do not know something. But at the end of the day I cannot assume that everybody knows everything. So, I that is one more reason to actually spend some time on the math that is required. So actually today we are ready to actually start working towards description of continuum field theories.

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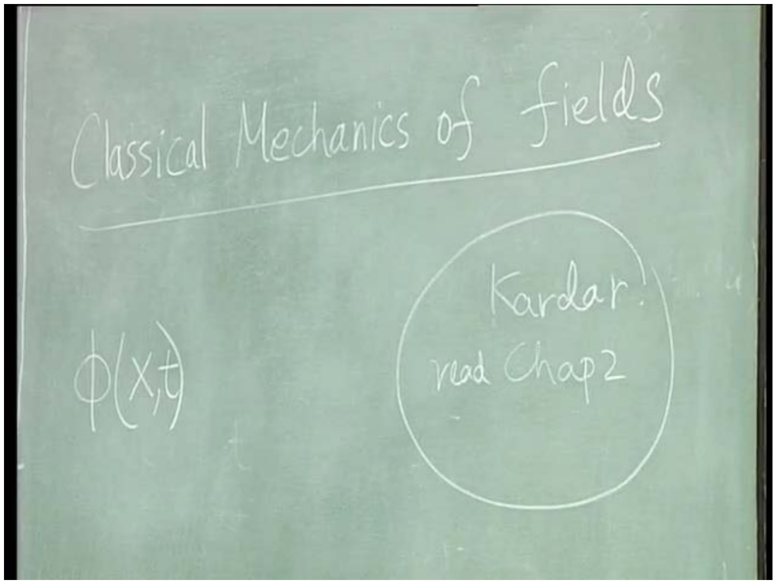

So, classical field theory as we saw in the very first lecture was the classical mechanics of fields. By the ways there is a wonderful book title statistical mechanics of fields. By Kardar I recommend reading chapter 2. It is not a classical the title of the book is statistical mechanics of fields. That we will be doing classical mechanics of fields. So this is recommended reading it is a wonderful book. So, the basic object is going to be a field, so I am just using a symbol phi of x and of course it could be t for simplicity for a few minutes. I will assume that we have only 1 dimension because I can draw I can use the board efficiently there.

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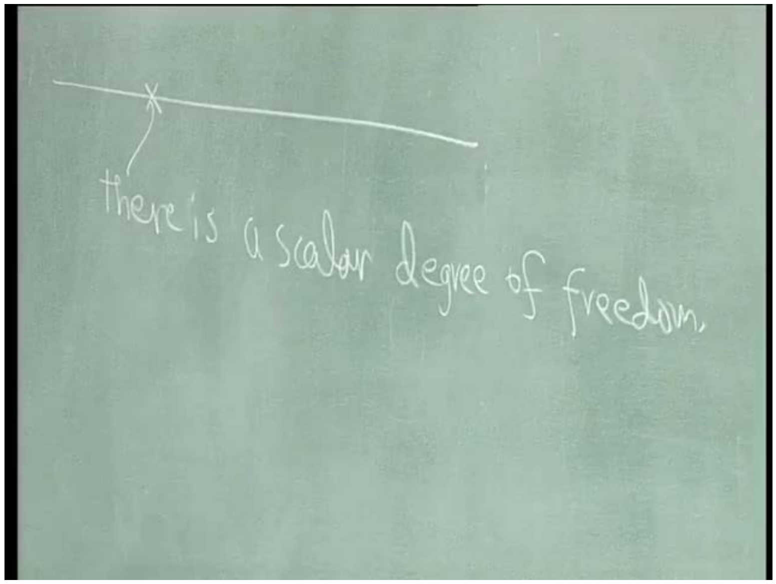

So, in another words x takes values is some real line and at every point. So let us say this is a scalar, so at every point there is a scalar degree of freedom. So clearly there is a infinite degree of freedom. And so how do we do go ahead and handle it. One trick is to actually descritize space and then take the limit where the spacing goes to 0 and that is what we will do, so we will replace.

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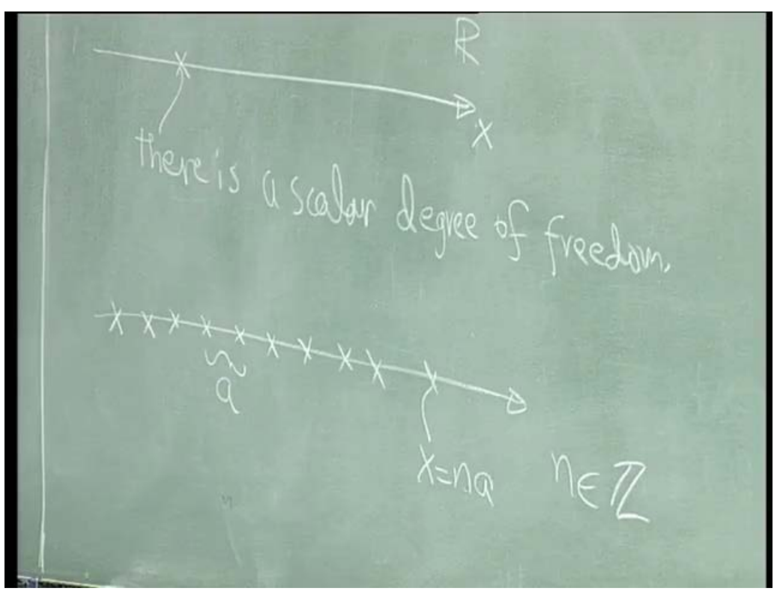

So this is R I will replace this by a discrete set of points so, it is an integer worth of points. And let us say they are equally spaced with some spacing it so, we start so this is something which has coordinate x. But, this would be some typical point would be x equal to some n where n belongs to integers. And important things like, if you believe that there is translation in the x direction clearly it is broken out here. It is broken down to the discrete set of translations in this 1 dimensional lattice. By units of a, but, what we have in mind is that what we will do later is to take a, to 0. So that the spacing goes away and hopefully will recover this sort of a symmetry.

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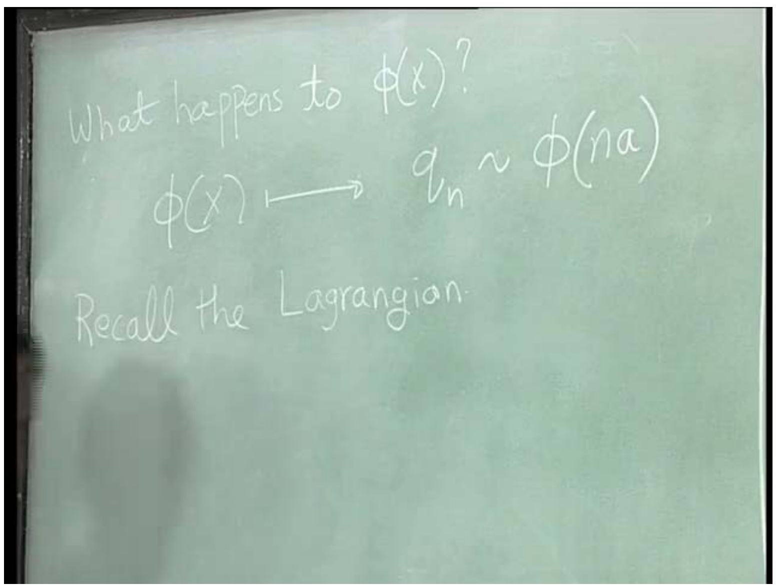

So, first step is to ask what happens to phi of x? So we have done some thinning. So what we are going to do is only know the values of phi at these points. So, let us write out something like this, we replace phi of x with in the discritization process by q of n, which should be morally speaking I am not using the equal to sign, You will see why? We will put the equal to sign later this is e has a same as, the degrees of freedom. So what I have done in this process is to convert the degrees of freedom to make it look like system, which has infinite degree normal classical mechanical system which has infinite degrees of freedom. So the question is what do we do with, the so let us Recall the Lagrangian.

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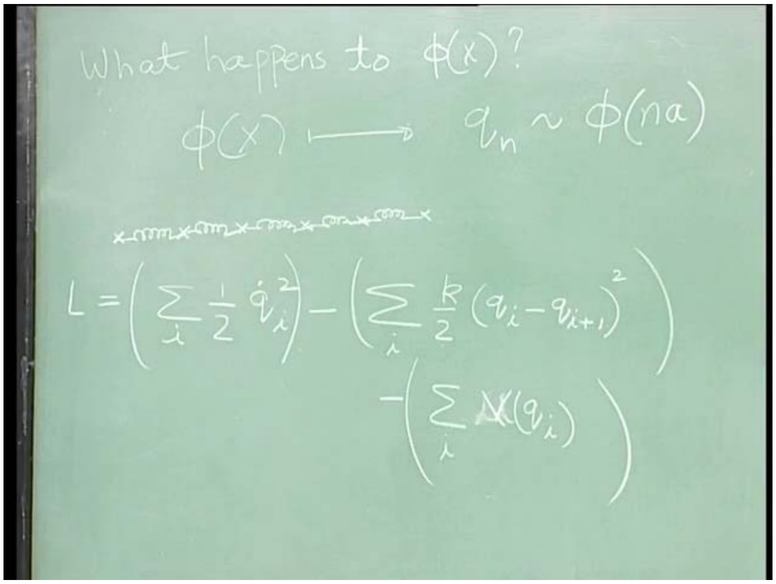

Actually even before I do that, let us take a slightly let us take a thoi model of the kind you would see in Kardar books for instance. So just take bunch of degrees of freedom which lie out here, and think of them. So these are like masses connected to by some springs that means, there interaction is harmonic. So, what I have in mind is we have some so the Lagrangian for such a system would be. So, this would be the kinetic energy part. You could have a constant here but let us say I normalize things such that this constant becomes 1 minus. The potential energy which would come from this spring which would look like summation over i half so let us say k. Let us use a small k k by 2. So, I am just saying that this is some harmonic spring with spring constant k. But, if you think of this s some ion in some lattice it there will be some local potential which will keep it out there.

So, I could write something else which would also. So, let us look at each of these terms again first point you realize is that each of these terms is translation invariance. So we are assuming this infinite change for practical purposes. So, you can see that when the discrete symmetry just i goes to i plus 1. So I have so, it is not like I have different constants for each of these things so this coefficient is uniformly 1. Similarly, out here all the springs are identical. And this potential let everything again side should be identical. So let us look at these two different, so these is the normal kinetic energy term if I had put an m out here, then you would be happy but I let us say I put m equal to 1.

So, let us look at this particular term, this term involves two sides but this involves only 1 side. So, now you can see that but these are all local interactions. There is nothing which says that you know something here and something which is like million lattice points away there is no interaction between them. This is kind of intuitively what we could we would write we could have I mean without losing sleep we could have made this as i plus 2. But I mean, but this is more natural to this kind of problem.

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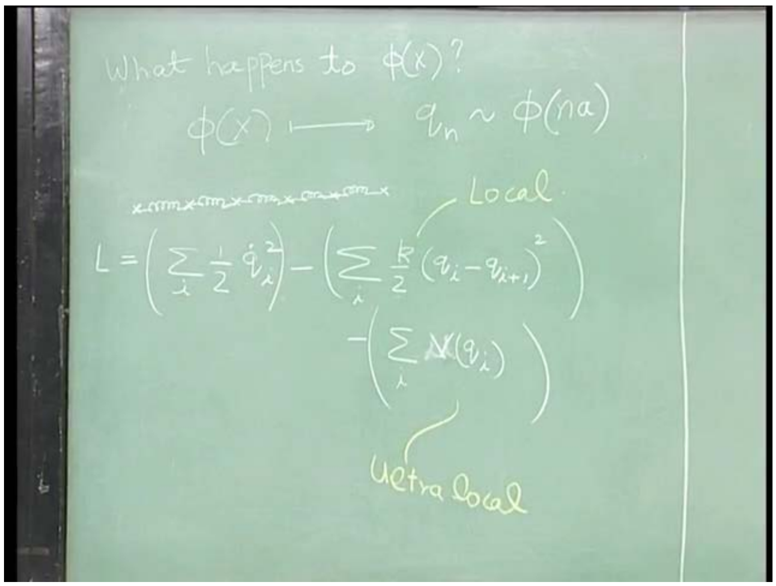

So will distinguish, but there is a difference between these two kind of terms, this term is local this is also local. But this we will call ultra local because it is determine by the degrees of freedom at a particular site. So will call this term ultra local all this is in space all the statements while a term like this we will call local.

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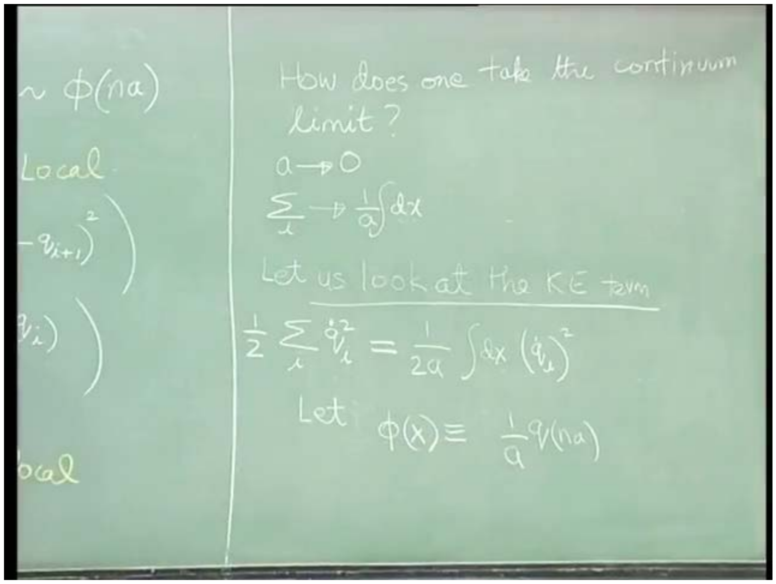

So now comes a question, How would you go about taking the continuum limit of something like this for status? So of course the easy answer is say take a, to 0. But now comes the issue, how do we take a, to 0? So, first thing is to realize so it is not enough to take a, to 0. We need to know, What is it we have to do with summation over i? Does anybody have any idea, what I should do to that? You replace it by an integral. So, but this has dimensions of length and this is dimensionless. How do I fix it? You go back to the old definition of things. So you put a 1 by a, out here. So, now let us go back and first pick up the kinetic energy term. So let look at the KE term. So this says that I should go ahead and replace summation by 1 over so this just becomes 1 over 2 a, integral over x. We are not done yet we need to take the continuity of limit. But you can see that if I take a, to 0 just naively what do I get? I find that it blows up. But we can take care of that by realizing making advantage, taking advantage of this particular thing I said q may not be exactly equal to this. So, we just and go ahead and define so, we say that let phi of x be defined to be. So you can see that what I have to do is to that you have to take the limiting process or whatever but if I define it this way I get, So x equal to…

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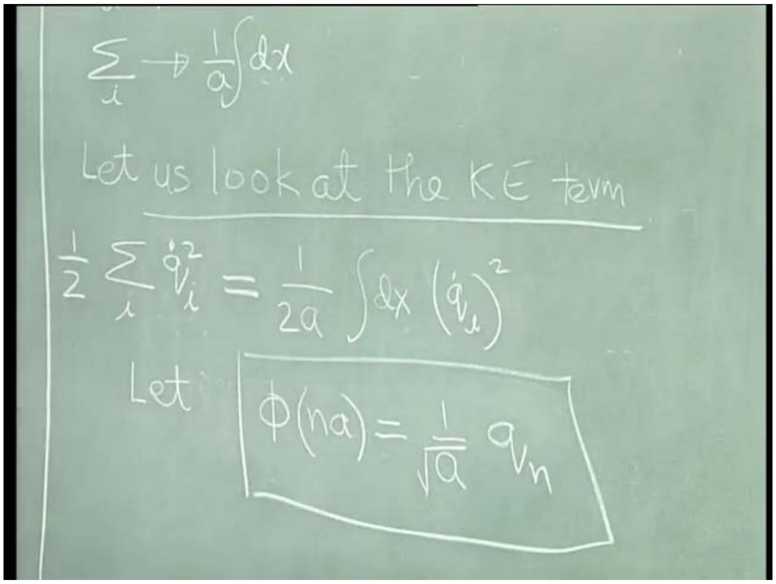

I just need to put a square root right, because there is a square root here. So it is a important note that this is your continuum field and this is the discrete fields. So there is always some power of a, which goes with the field. And in fact suppose we had instead of 1 dimension suppose we had d dimensional space time then, let me write use a different color. And write that so for d dimensional space time you would not sum over just i you some over the d dimensional lattice for simplicity.

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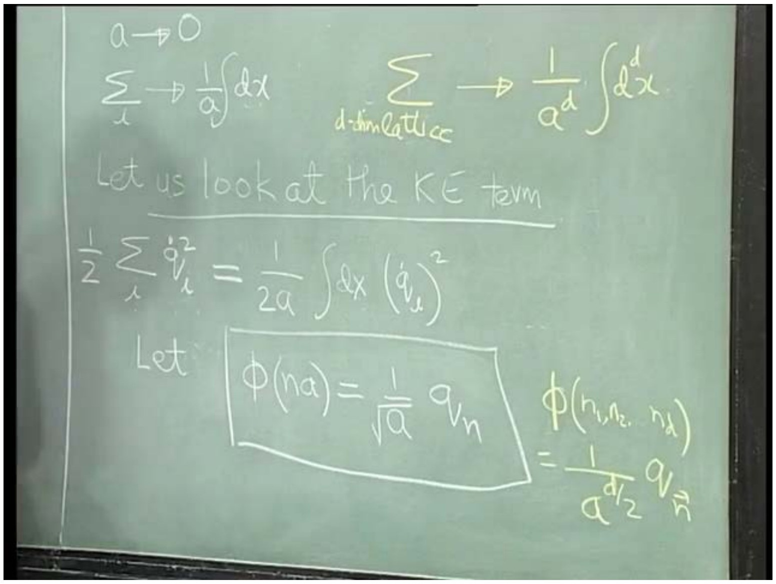

Let us, assume it is a cubic lattice of size a, then sum over the lattice. So d dimensional cubic lattice I will not write cubic it should go to 1 over a power d. So this is what you get for in the summation. Now you can see that the analog statement which, are again right now, that would be that phi of now I will just use x, because they would I require many indices n 1, n 2, n 3 etcetera so, let us do that. So let us say that it is n 1, n 2, n d. So what I have in my mind is do not want to write out. So, I will just, in this notation this would be just n1 would be just there will be 1 n. So, that be of many of this guys out here, which should be equal to 1 by a power d by 2 q of that vector n. So you can see that the dimensionality of your space is actually buried into this. So let us get back to the rest of the terms. And we will again work we let us see what we have to do to take the continuum limit of these terms.

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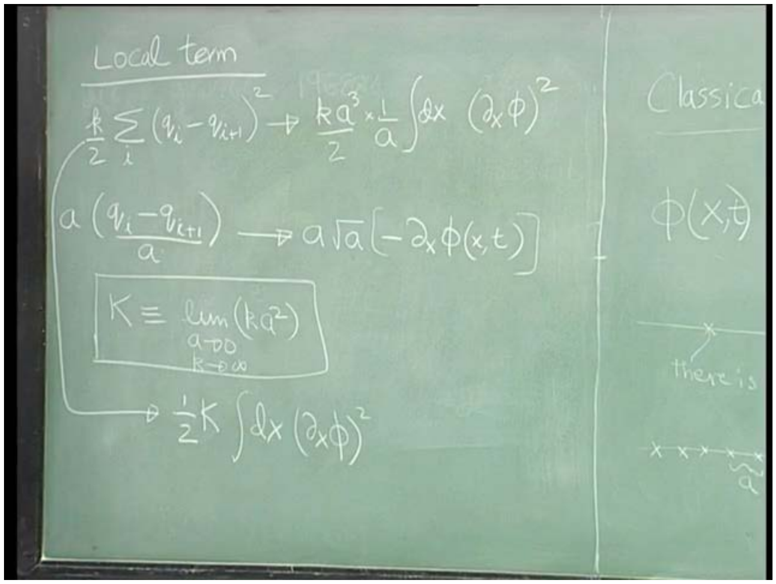

So the next term would be the Local term. So now, again we can see several things first thing is we can we know that the summation has to be replaced by the integral. But now we also have to take in to account what happens to this difference. So, let us first work out what the difference looks like, again I will do it for just 1 special dimension. And I think it is simple exercise for you to work it out for the other dimensions. So, let us look at what happen to a term like this. If I divide this by a, and multiplied by a in the continuum what will happen you can see here is that we have seen the appearance of the discrete derivative.

So in the limit that q goes I mean a goes to 0 this becomes the derivative. And putting in that other factor that we have out here. That is square so nq is equal to square root of a times phi. So this should become this there is 1 a coming from here. And there is also a square root of a coming from the scaling which you need to go from q to phi. And this way and the way I have written it is qi minus qi plus 1. So, that will be the minus of the derivative. So, let us just now we can put we can combine we are also we will get also some more powers coming from this things, so this term now. So you get since, there is the square of this, you get this is what a power 3 by 2. So you get a cubed coming from this part and the summation will give you a 1 over a.

So now we can see that so we so again taking a equal to 0. You can see that this term is now going to 0. Now you could say fine that that means such terms are irrelevant. But, that is not quite true you do not want to lose your local interactions. So again you scale things in other words take k to 0 k to infinity, such that a becomes a goes to 0 such that this combination whatever ka square is some constant. So the limit is let k is defined to be limit a tending to 0, k tending to little k tending to infinity, such that this combination, which is k this should have a finite limit of course by definition, which you also know that springs becomes stiffer as you make them smaller so this is not it is not counter intuitive.

So this so this tells you that this is another thing. So now we can see that local term becomes the local term is now nice it is half k. There is some minus sign in the way it appears there. But, I am just looking at this term is now half k integral we still need to handle the ultra local term. So what I will assume is that the ultra local term has some power law. So let us say let it will V of q b some, let me use some lower case it is some power m. If you want several of them you can sum over them. But, I am just choosing for simplicity just 1 term I called this coefficient bm.

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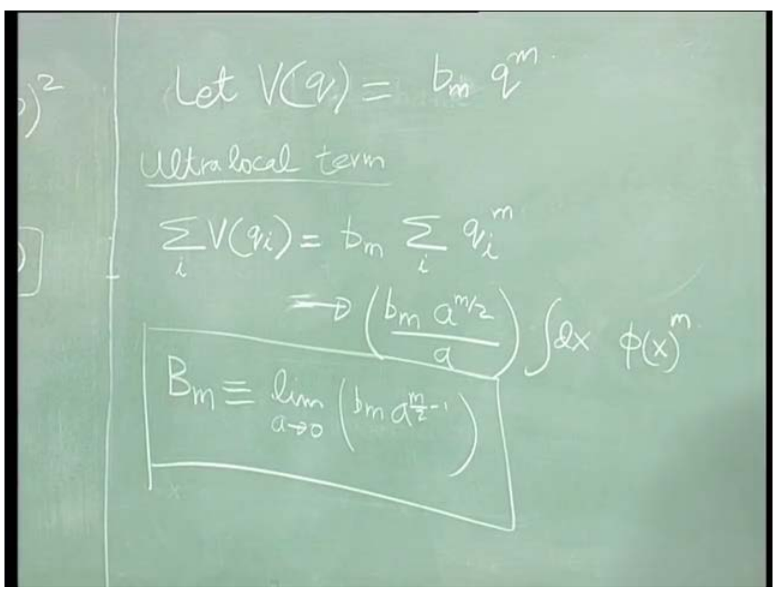

So now so that the ultra local term now has the following part which is half no half out there, So it is just summation over i V of qi this is equal to. So now, we can quickly do things we know summation will give 1 by a, and q each power of q will give you a square root of a, on top. So this will just tell you that this is equal to or goes to bm. This gives you a power m by 2 divided by a again. It is obvious how to take the limit, is to just take the limit such that. So you take now b goes to so you basically take the chose the limit of b such that this thing becomes a constant. So, just define so I will not worry about how the limit crosses goes we just define bm to b. So, now we are ready to write out the Lagrangian. What I will do now is to write the action first just like the Lagrangian it is let us write Lagrangian.

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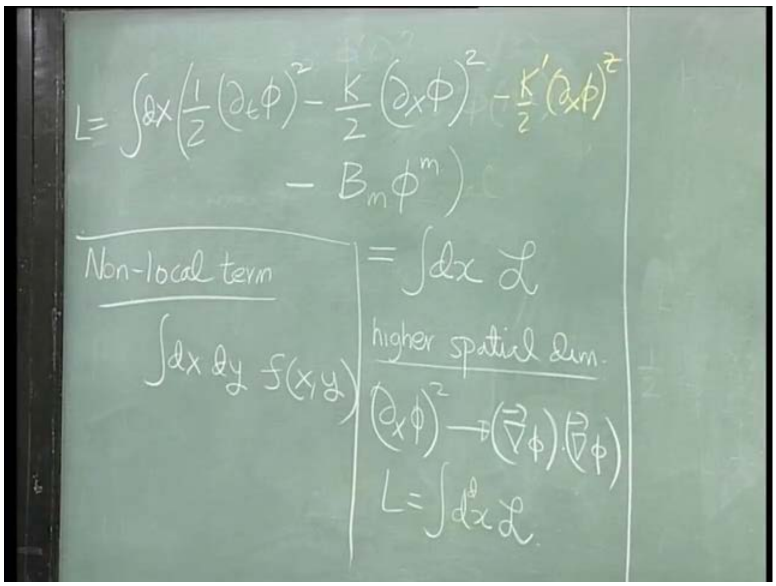

So, the Lagrangian in the continuum limit is the sum of three terms. The first one is this, so it is just half. And the important point here is that so it is very clear, how to do this for an arbitrary function you can scale things properly and get everything to work out. If you have some polynomial kind of things; so this what you get. And we have already put in locality in space. However, we done if we have seen that we have only taken nearest neighbor interactions, and we have got an integral over only dx. An example of an non-local term would be something, so it is useful to see how an non-local term looks like, I am not saying this is the only non-local term I am giving you an example of a non-local in space. That is if I have something like this, times some function of x and y. So, it depends on two points. You can make things even worse you can make it depend on n points, but this depends only on 1 point.

So, we have already put in locality and it shows you that locality in space implies that there will be only two kinds of terms. That you could add one is a local term that can be expressed in terms of derivatives of x, I mean phi and its derivatives, I mean derivatives of phi. And the local pieces are the derivatives of or no derivatives but just ultra local terms, which are function of phi.

So this sort of gives you the idea what kind of Lagrangians you have. So obviously since, what you have in inside this inside the braces it is natural to call this object a Lagrangian density. Because, dimensionally it is really per special thing right. So, one writes these thing as integral dx we use script l. In fact there are more general local terms that, you could write I could write higher powers of phi. And there are models where people do look at such terms. So I am just for I will write a something like that, let us call it k prime over 2 over some power let us call it Z, where Z is some where Z is equal to 2 it is this Z can be greater than 2 anything higher powers.

So now we can ask this was we did only for 1 dimension the generalizes to d dimensions is quite easy. But once you go to more than 1 dimensions even if, you are in two dimensions you have additional symmetries to think about, which is rotation you would like to think things to be isotropic in space. So things have to be rotationally invariant and you may wonder, How do I implement that? We can implemented just like, we did by implementing translational invariance we required things to be invariant under translation the action to be invariant under translations. So a term like this, you would just replace it by the scalar product. So for instance in higher dimensions higher special in higher special dimensions dx of phi. So for instance into in we could have taken in one dimension dx phi whole cubed there is no issue or so it seems you can take an odd power. But if you got to higher dimension you will have to take square roots to do that.

So dx square would go to some kind of gradient, let me write is it as a dot product. And you can see that notes no particular direction is special in such a term isotropic is taken care of. So if you want to put higher powers you can but, it has to be some power of this sort of guys. And needless to say, L will become so, the continuum limit of that simple model of spring tie together would be some would be a string with some tension right. And so in such cases this k, what would k be in such in that example? k would be the tension it related to the tension and also related to the speed, speed of sound from that thing.

So k is some kind of even by simple dimension analysis you can see that k that k should be the square of k would be the square of some velocity or speed to be more precise, so, that just dimensionality. Now we can come back I mean there well all these I raised one of these things the question is. Can we figure out is there some way of making sense of these numbers? This powers of a so, let us write out what are the powers which we got. We got two from here; we got m by 2 minus 1 from here. And the last bit was with phi had a square root right. So these roughly the so this is this has. So, now question is can we come back look at these Lagrangian and derive the same thing. And I will show you how to do this and the way to understand that is at the lattice level I could have done the following thing.

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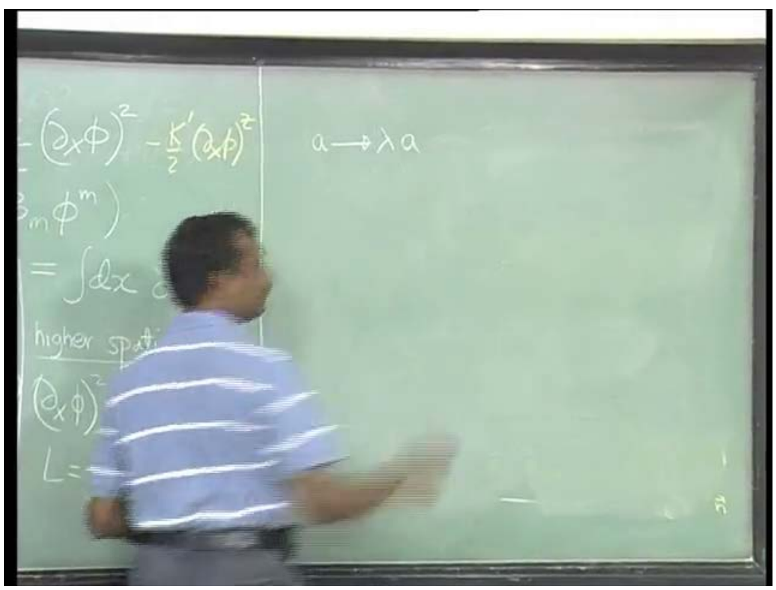

I can take a to lambda a, I scale it and so we can do the same thing in the continuum by just realizing that I just go ahead and scale x.

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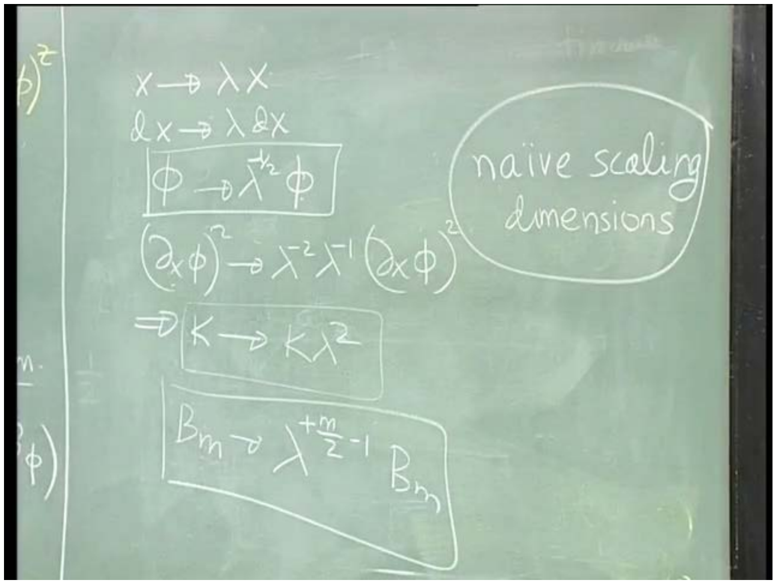

So I scale so, I replace this, by x goes to lambda x. And you want to write you want to associate by demand that every term is scale invariant in some sense. So, we start from this guy and let us, look at this. So x so this implies that dx also scale like lambda dx. So you look at this part t does not scale so, only phi has to scale. So implies that phi should scale to cancel that phi should scale with. Now, that this term has fixed the scaling will call a, this thing we will call the k half or minus half actually to be precise is a scaling dimension of the field phi. What about here?

So now you can see so there is d by dx here, so this will go like 1 by lambda out here. So there are two of them, that will give you a 1 by lambda square and a phi gives square root, so you gets a 1 by lambda. If I goes this so then k, so k so now you can see that k so dx of phi whole square will go like this will give you a lambda power minus 2. And phi will give you a lambda this will give lambda. So the only way the things can be invariant is if k is scales like, so this implies that k scales like; it has to cancel this k lambda cubed.

There is a lambda from the dx, which will reduce this number to 2. And k so let us forget this. So we are look at bm so b m will have to go like phi m is like it goes like lambda power minus m by 2. And but it has to also I have flip sign I think. So, you can see that is not as if these things we scaled away. So, let us see what we get out here, is this I cannot see that. So you can see that you get the same scaling dimensions which you had out here.

So, it is not like by going to the continuum, we forget all the memory of those limits. We can actually recover these things. And these are called the naive scaling dimensions. They are called naive because, quantum mechanically these dimensions can change. And they are not something which you will do in this course. But so, there is some memory even after you take the continuum limit, in this Lagrangian. Is this clear?

So now, but we are interested in constructing Lagrangians for Relativistick theories which have Relativistick symmetry that is a Lorentz group. So, we have already inputted special rotations we have taken care of. Even if you look, at the Boncary group we have even got taken care of translations. In fact translations in space and time are taken care of. If you go look at the level of action, so action you can see is an integral over these things. So, now the question is. How are we going to so? We need to ask what is the value of k such that k or all these constant such that we end up with things which are Lorentz invarient. So, we need to only ask, How they transform under Lorentz boost? So first thing is let us start what I will do is now is to start from the easy part. We will let us, start from the ultra local term and ask the following question.

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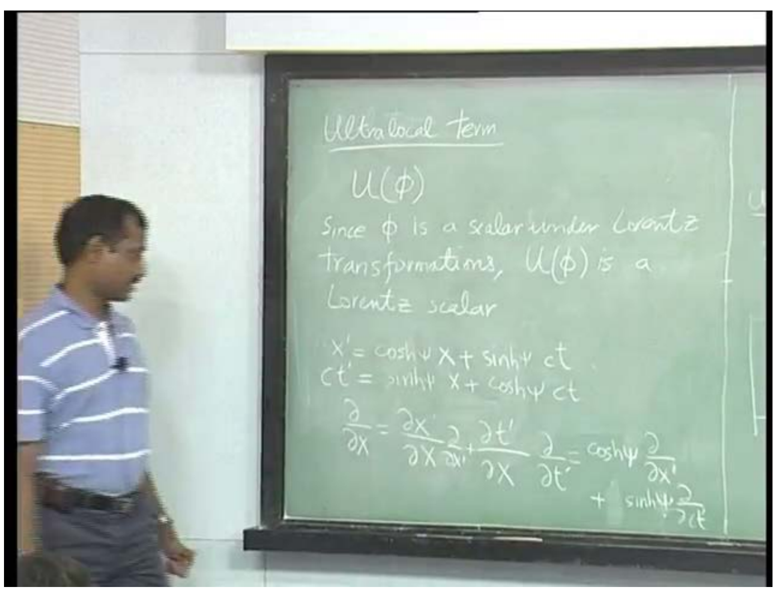

So, let us take the ultra local term and what I will do more generally is that, you can replace this by a function, which we call it u of phi. I call it u because, u is transfer ultra local to remind you, not v but, u. So the ultra local term is u of phi and we have said phi is a scalar field by that we mean it is a scalar. Since is a scalar under Lorentz rotations transformations, any arbitrary function of it is also a Lorentz scalar.

Now, you may think that I can construct I can consider the local term, I can just write out a local term, which I look at only the local term and try to make things work. But, we know that under Lorentz boost space and time mix. So clearly, there I mean terms like this can become terms like this. So we can ask how things change. So let us go back and look at how so we have let us say that x prime is equal to so these Lorentz boost.

So, the question is, what happens to d by dx prime? Now, we can use the chain rule and may be I will go the other way. I will go from d by dx. So we can just look at this, so dx prime by dx is just cos hyperbolic plus. This should give you some sin hyperbolic with some factor of c d t prime by this is 1 by c that may shift the c this side.

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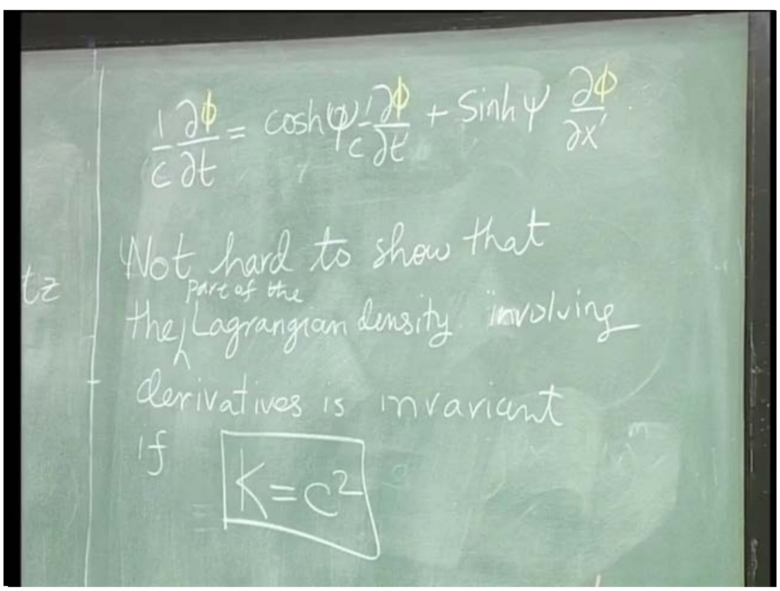

Let us put psi. So d by dx is x prime and t prime. And it is not hard to work out what d by d prime is. So, now comes the nice stuff, so this we can see. So the question is so we have to ask, How this transforms? So by acting on these things you can see that I can put the phi inside this.

And not hard to show that the Lagrangian density involving derivatives the part of the is invariant under the Lorentz boost.

So that is the case so basically what you will see that the cross terms vanish and the nice identity cos hyperbolic square plus sin minus sin hyperbolic square equal to 1. And the nice thing is that there, is this minus sign. This minus sign is very important, the reason is that from our if, we go back to our discrete system this was part of the potential. Then you also learn one more thing, if you put a term where like this, where Z is not equal to 2 it is not invariant. You cannot choose some random thing, which you could have if there was no say nothing which, mixed space and time.

So theory is will like this, would usually happen in non relativistic settings. These are Lipschitz these are called Lipschitz models. And there is no there space and time do not have to mix in this fashion. So you can see that arbitrary terms like that are completely ruled out. So, now we are sort of ready to write out the most general relativistic.

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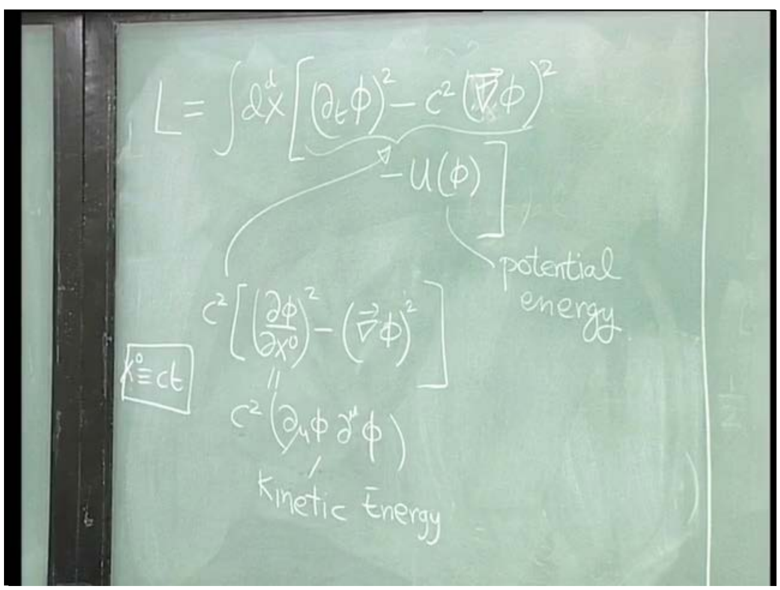

So L and I have just read it, I have just solved for everything. So this is what you need to do. The only change we did we go through away that kind of piece and we have got this. And now, you can we can also see the generalization to arbitrary d dimension is to is just write this and replace this with gradient. But this term actually can be written in a very nice manner by, so recall that I can pull out as c square fully. This will become d by dx 0. We called x 0 is defined to be ct.

And this has a nice way of you can rewrite as C. So in relativistic field theory you can see that, these since these two terms mix. And they can be written as this so this you can see is explicitly that is contraction this is by manifestly a Lorentz Scalar. Do not worry about, this factor of C, you can avoid if you wish. You can rescale, you can absorb 1 C into the d t definition. And the other C can be absorbed into the taken in with the phi, so that the C goes away. It will and you can absorb all the other constant, so that not so important. What is important is that this is manifestly a, Lorentz Scalar.

So in this course for the rest of these thing I will when we will what I mean by kinetic energy will refer it to this term itself as the kinetic energy. So it is a loose imprecise thing because really this is what you would have thought was the kinetic energy part. And it contains the local part as well. So we will call this kinetic energy the standard convection and the ultra local piece, we will call that potential energy so this is how you would write things. And one more things you can see is that if you even though the Lagrangian this measured the dx is clearly not invariant under just Lorentz transformations. The full action is because the action becomes…

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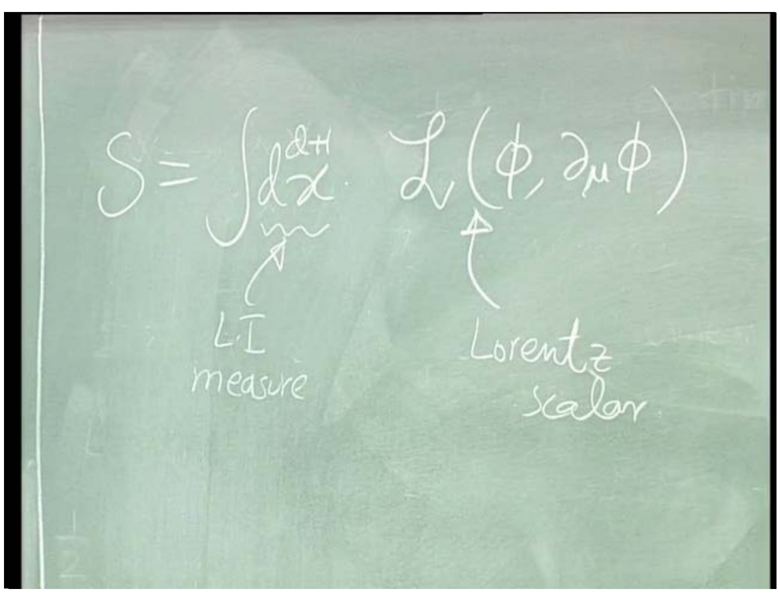

So, I can write so S is integral so I am including time and this thing times. Some Lagrangian density which depend on phi and d mu of phi and so this of course this measure is also trivial nicely Lorentz invariant measure. And these term by term we saw was the Lorentz scalar. So what we are accomplish today is to write out what would be a local Lagrangian which is for a scalar field. So you can ask what about vector field? And the answer is we could write these things. But what you can see is that this kind of things will still go though, that you would write your action some integral over both space and time. And it will be in terms of some Lagrangian, which would be in terms of the field and it is derivative. This structure will still hold.

Student: What are Lorentz invarient scalar field will this be any possible structure? You could take higher powers.

Student: Kind of potential terms but anything else.

Not just in the potential you can write the whole thing power higher power you could write.

But, this combination you cannot suspect.

Student: Some function of the field the potential energy.

Absolutely.

And just a, I mean if you one more thing I want to remind you is that in quantum mechanics. Why did you restrict yourself to 2 derivatives? You could have had 3 derivatives etcetera. Suppose you had q double dot square in your thing. What you usually do there is to introduce an extra degree of freedom, whose equation of motion is just saying so, if there is q triple dot. You call you introduce some other let us say q prime which is equal to q dot. And then you rewrite stuff and then you will get something.

So you add you sort of increase the degrees of freedom to handle lower derivative pieces high derivative pieces and, write it again as a normal Lagrangian So these kinds of tricks can be done. And so, but actually the hardest part in any system is not writing the Lagrangian density it is the first step itself. What are the correct degrees of freedom, for in to describe your system, and that is the hard part actually. And once you have the correct degrees of freedom then, anything of I mean so you could the same system may have a different energy scale different Lagrangian description system.

Classic example is qcd at higher energies that it is theory of quark and gluon. But, low energies is actually a, is more like a theory of mesons and baryons. So you could write some other theories. So, degree the fields which you would describe low energy stuff, would you can write effective Lagrangians which actually do these things. And there is lot of interesting work which works that way. So you can see that the same what you thought was the same system a different energy scales behaves like the degrees of freedom natural degrees of freedom are different.

Another example is the super conductor, I mean in its normal state, it is a system which contains only electrons. But in the super conducting state there are other degrees of freedom, which carry the thing and that, is what lead to the super conductivity. The cooper pair you think of that as a different degree of freedom and it is funny. That, the cooper pair is a Bosonic degree of freedom even though the individual parts are formeyonce.

So, you would describe that by scalar field theory. Even though I mean you might say all the conductivity is due to the electrons move. But that it is like two electrons moving in synchronization, and it does not make sense to think of it as a system of electron. But rather the scalar field which describes this thing the cingulated part in some sense to be precise. That could be the triplet which could gives half that is also a scalar. So that is actually the hardest part in real systems.