**Classical Field Theory**
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**Lecture – 7**

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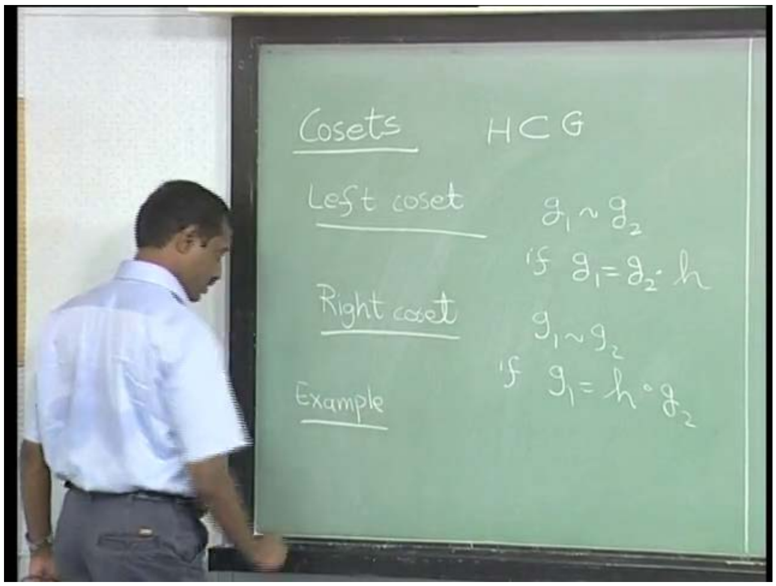


So, last lecture I discussed cosets, but for some reason I always exchanged the definition of left and right cosets, but so but one should stick to the standard conventions. So the left coset, so H is some sub group of g as I defined, so left coset is defined by the equivalent relation where you say two elements g 1 and g 2 are equivalent to each other, if g 1 is g 2 times some element of h. While the right coset is again an equivalent relation, but so as we discussed each both of them give a rise to partitions of the set g. So, that goes through, but in general left coset may not be equal to right coset.

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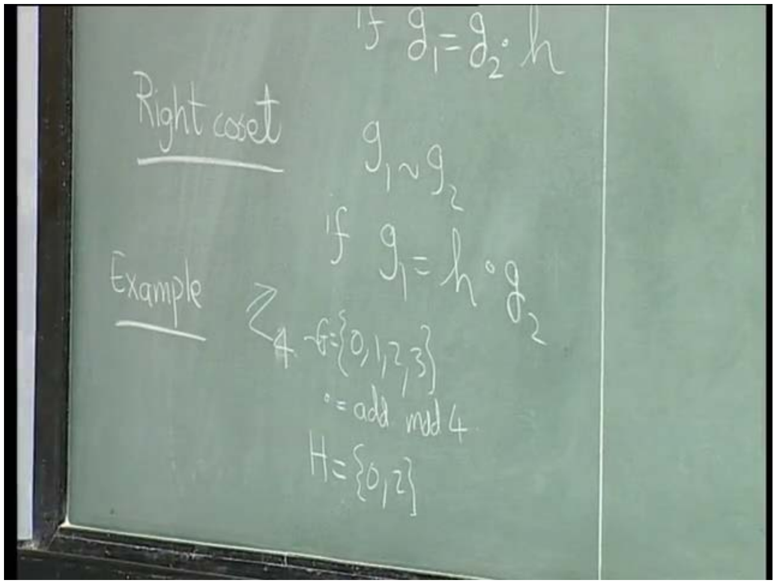


And, so nice example to play with this, take the example of z 4, so let us take that to be given by the set of the following four elements 0, 1, 2, 3. The set, so g is this and the operation is just addition mod 4. Just giving you a realization and let H be the subgroup with element 0 and 4. I leave it as an exercise for you to rather than example. Actually what I mean is I want you to work out.

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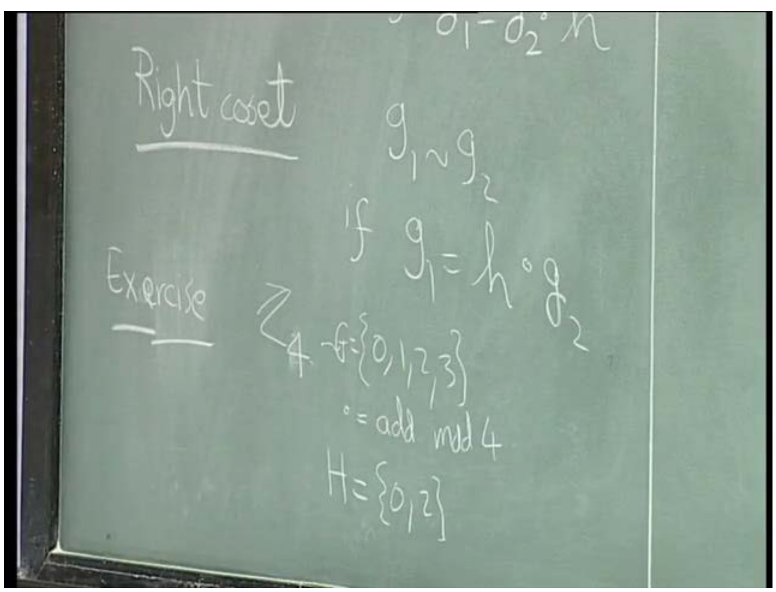

And in this incidents you will see that left coset and right coset will be equal, equivalent to each other, will see why.

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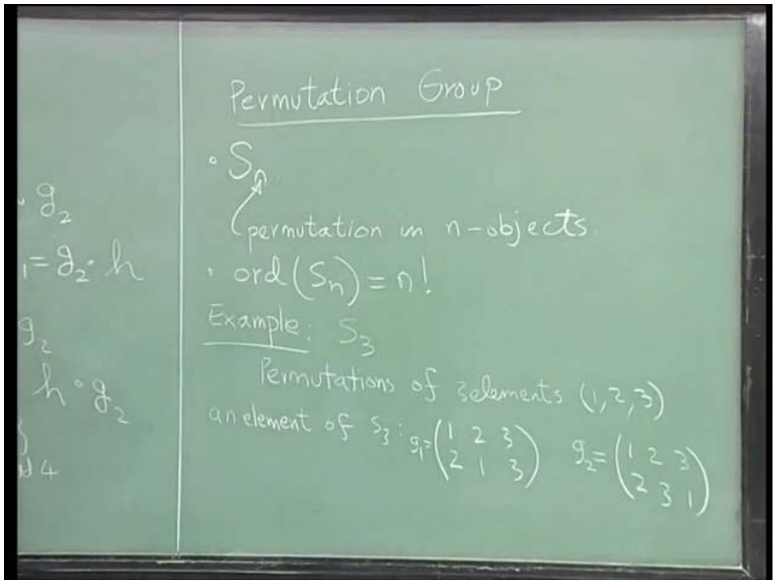


But meanwhile what I want to do is to introduce you to one of the most important finite groups called permutation group. Actually the permutation group is not one group, it is a family of groups and they we will call it S n, and this n indicates at it is the permutation and in n objects. So, I will describe it through, see through its operations rather than give you a giving you a list of elements and order of S n will be n factorial. It is a number of permutations. So let us just take speak as specific example here of S 3. And we will so it should be acting on something. So, we what we will out with three objects, so we will just say that let us say that we think of permutations of three elements which we will call 1, 2 and 3.

So, what we will do is, so an element of S 3 I will write in a is given by something like for instance say 1 goes to say 2, 2 goes to 1 and 3 is invariant. So, this operation you can see is just the exchange of 1 and 2. So, let us just call this element will call it g 1. Let us write another element, let us call it g 2 and this time I am going to choose something where 1 goes to 2, 2 goes to 3, 3 goes to 1. We what is call this cyclic permutation.

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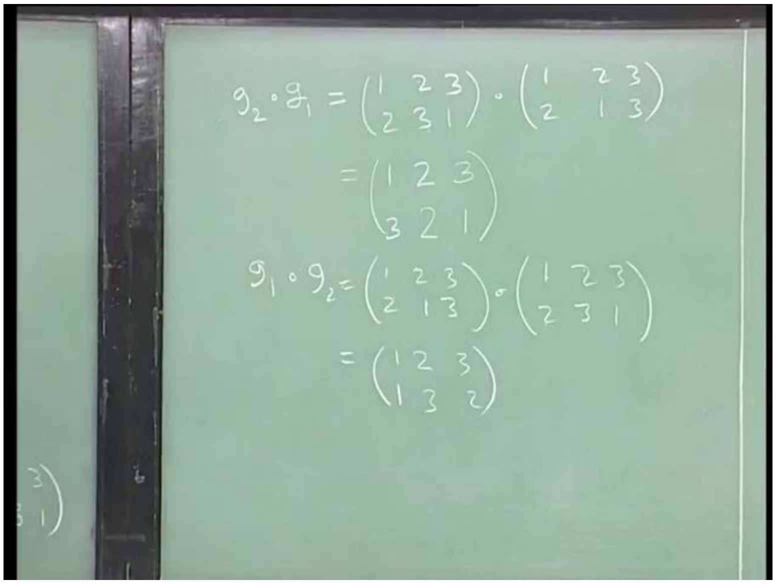


So, now let us see how this group is and let us do the composition of these two elements. And the this way of writing things is very makes life easy in the following sense. I can do the composition by just tracking things. So 1 goes to 2 and then I come here 2 goes to 3, so 1 goes to 3. Next 2 goes to 1 then come follow it up 1 goes to 2. So, I started with 2 and ended up with 2, and then goes to 3 here, but 3 goes to 1.

We can do the other order as well. 1 goes to 2 here and 2 goes to 1. So 2 goes to 3 and 3 goes 3 and then. It is non-abelian. g 2 dot g 1 is not equal to g 1 dot g 2. And what is your identity should do nothing 1 should go to 1; 2 to should go to 2; 3 should go to 3. So you can I will not explicitly write out all the six elements. I have written most of them by thinking out here written 1 2 3 4 and 5. There is only one missing I guess. So I leave it to you to figure out what that is. But now comes the nice stuff ,we can ask what about subgroups of this of s 3.

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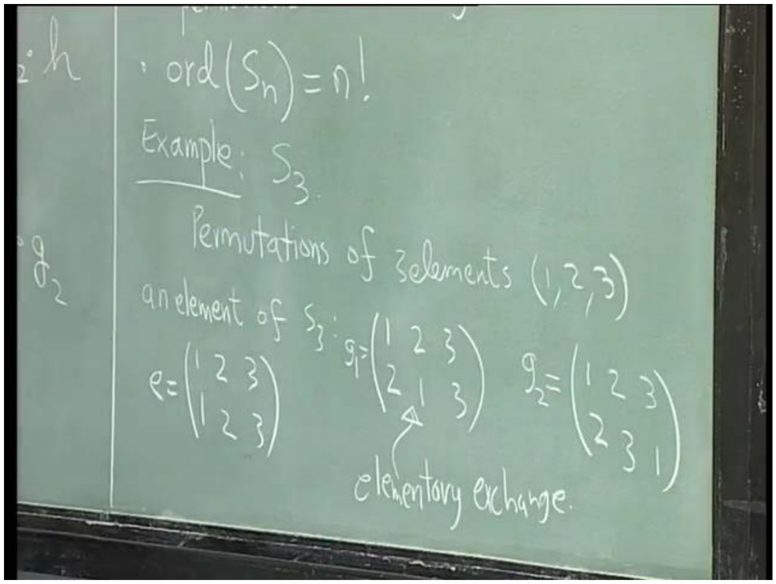


So, obvious subgroups are see I have taken element like this, it is 1, 2 exchange and so it is operated two times it does nothing. So we will call this sort of an operation, we call this an elementary exchange. So, I just before even I get on to this now, you can see that it is very easy to generalize that to arbitrary Sn. You just take a set with an element and then you write out all possible permutations in this fashion.

Some if you get very bored with writing it in these things, we just write the denominator part because the things comes for the right. There are other ways of writing out elements of the permutations group, but we will not do that. But there is a nice, first thing is that I will give a claim. I will make a very simple claim. The claim is that S 3 is isomorphic to the group the dihedral group D 6. So, I leave it as an exercise for you to prove it. There are many, many ways of proving it. One way is the horrible way is to go ahead write out the full group operation, group multiplication and write out a map from these thing to the other, but this but there are other ways of doing it as well.

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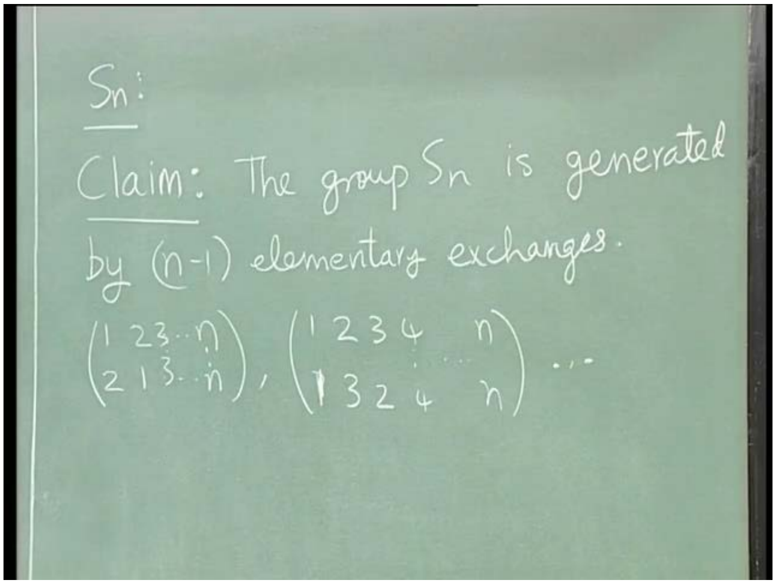


But in general let us look at S n, if you take S n I will make a claim here again or I can say it is a theorem. The group S n is generated by n minus 1 elementary exchanges. So, let me show you what I mean by that, what I mean is by an elementary exchanges so 1 2 up to n, so 2 1 this is so 3 goes to 3 all the others identical. So, this is the first elementary exchange, where exchange 1 and 2 alone. The second exchange where exchange 2 and 3 and it is just an inventory exchange, but involving only neighboring element and that is it. There are n minus 1 of them.

Coming back to this example there are exactly two so I can so this is one element the other element would be the 2 3 exchange, which we have some idea at this element. I claim that these two elements are sufficient to generate the whole group. So, here the next one would be 1 2,1 goes to 1, 2 goes to 3, 3 goes to 2 and 4 goes to 4 of course. You can rewrite out n minus 1 element. So, this is intuitively you can see this, but in general you can work this out.

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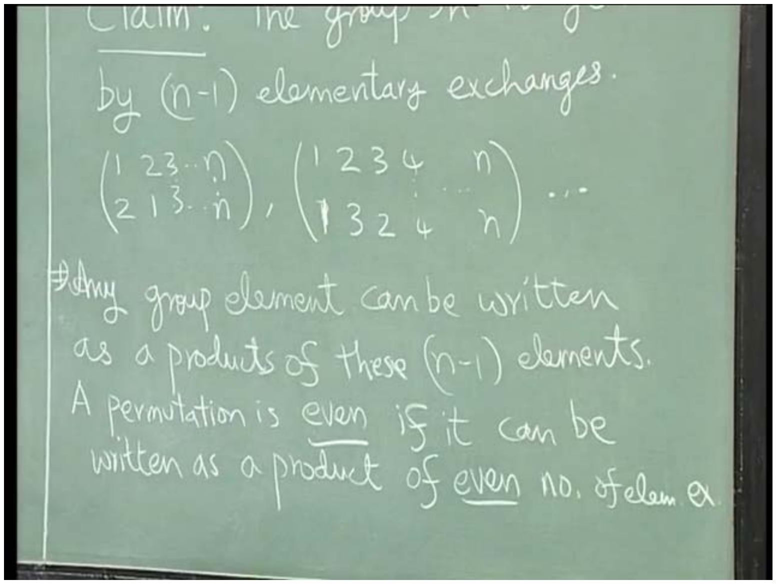


Now, so assuming this claim is true, what did tells you is that any group element by group I mean group element can be written as a product of these by product of these n minus 1 elements. Now, the thing is that we will call a permutation, an element, a permutation element even if it is made up by composing even number of these guys.

So, a permutation is even, if it can be written as a product of even number of elementary exchanges. This way of writing its does not mean there is a unique way of writing. You could you know write a whole different ways of writing it, but what will not change is the fact that you can never convert something, which is an even permutation to an odd one. The other ones are odd obviously.

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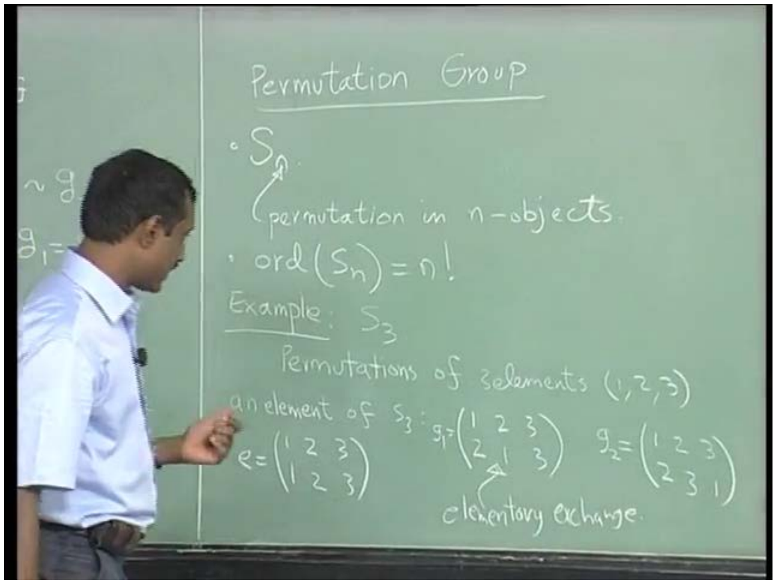


So let us see so an example of an odd permutation and of course an elementary operation they involve only one, but you take this g 2. Now, this can be obtained, this is an example of an even permutation and the way to see that is that the way you can do this is to first you take 1, 2 exchange you get this, and that followed it up with a 2, 3 the second and the third exchange then you will come to this. So, it is an even permutation. Now comes a nice part.

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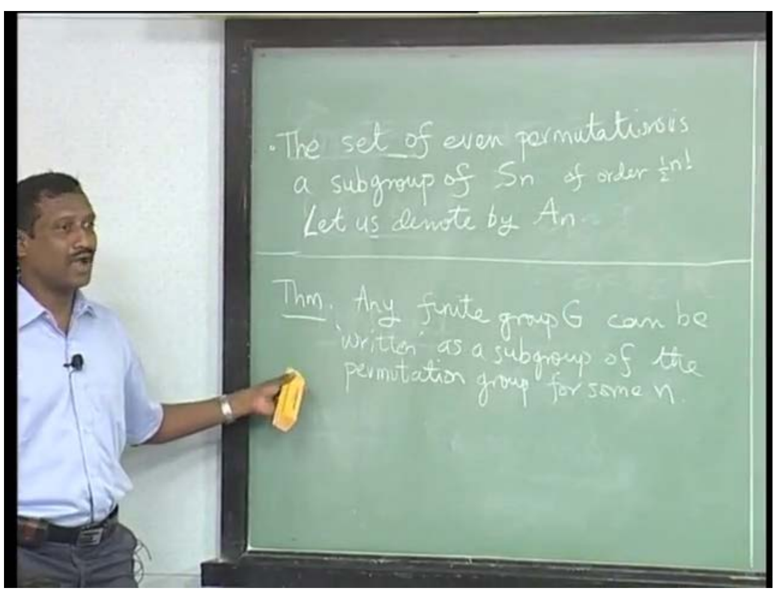

The set of even permutation is a sub group of order subgroup of S n. What will be its order? It will be half of the, it will be half of n factorial. This group has a name we just call it A n. So, are there any questions so far, because we have been just defining things. Now we will see in many ways that every group any finite group can be embedded into the permutation group. So, this is a very nice thing and it gives you a very what is called the permutation representation of the group.

So, here is the nice theorem, any group any finite group g can be written. I do not know the bad English but written as a subgroup of the permutation group for some n. We will fix the value of n in a moment, but so, what this theorem implies is that, in some sense all groups all finite groups are actually, if you can study subgroups of S n.

For arbitrary n they are subgroups of I mean that you would see all the groups. But this is only this has some nice properties, but I do not think it will let you search for all the groups in some ways. So, how would one do this, so like I said you need to start with something which you should the permutations groups acts on set of an objects. So, I will give you the set. So consider the set.

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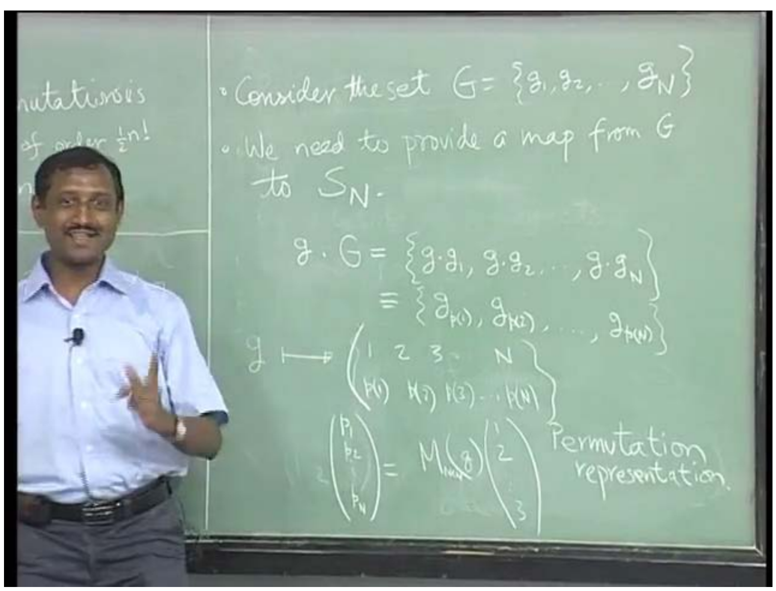


G and let us say that g s capital N elements. So now what I need to do is to for now, we need give a map from G to s and now I will make it precise to S n where n is the order of the group.

For every element that follows. So, the way you do this is, let pick an element g and let it act on consider the following object. What does it do? It takes…

So, now what is happening out here, is g is acting on g 1 it will give you, because it is a it is an element of the same group, it will give you another group element. Similarly you can you will generate all the other group elements. You can convince yourself that it has to be only a permutation of this group.

So, now I will write out the permutation, so it would be let me just use some notation out here. Let us say this will call g of p 1. We will call this g of p 2. So, for this particular element we got this thing. So what we get now is a nice association 1, 2. Now, you need to check that it is a it is consistent with the composition of the group, but you can go back to the original definition and you take two elements and you will see that the answer will work out. So I leave that as an exercise.

So there is another way of writing these you can write this you can think of 1, 2, 3 to n. So you can also write this as an enbion matrix, which is a following which so yes, so what is it? There some, now this will be some matrix, which will I have only 1's and 0's in its entries. It is a very simple matrix because 1 goes to p 1, so you go to the first row where at the p 1th element you put 1 the rest of the first row would be 0.

Second row would have only the p 2 the element equal to 1. Third row will be this way that way we will see that you get a nice matrix. So, this is called so now this fits in with what we had earlier. If you remember we said that if you give me any matrix which satisfies a group multiplication row that is like a representation. So, this has a name, this is called the permutation representation. So if you…

Student: (( ))

Yes, it is a group yes, but I am thinking of the yes it is also a set it is also a group. So this is the this is a set on which g is acting. So, it has a dual role absolutely right. It has right now this here this is just a set of an elements with some ordering. It does not matter what ordering you choose, you choose some order and leave it at that. So, here I am just thinking of it as a set with elements written in some order. But of course when I come to this part I am doing the composition within the group. is this clear? Are there any other questions? It is a very good question.

So, let us take S 3 which is already a permutation group, but so but for it the order is 6 so this process will give you an embedding or a presentation of S 3 in terms of 6 by 6 matrices. Which is much larger then in fact much larger than 3. So, obviously it if you are a given a group this method of writing it would not give you the largest, will not give you the it will not give you the sort of the minimal embedding into S whatever is the group, but it will give you something.

But this particular presentation has lots of nice properties, which one would do in any course on finite groups, but this is not such a course we just to show you that these object exists. And there is also going back to the full set of permutation groups. What you have is the S S 3 is a sub has a subgroup which is S 2. It is a subgroup which preserves one element, subgroup of permutation which preserves one element. And of course if you start this whether in three different ways of there are three different S 2's that you could get this way.

Similarly, out here if you start from there are four different S 3's you could get out here. Similarly, S 4 you can get five different ways, S 6 so on, so forth that is an infinite sequence. S 6 as a slightly different property, which I will not which leads to some very nice constructions, but from this view point it is just that out here that I just 5 S 5 you can get into this. But it turns out there are five others, which you cannot see in this manner it called an outer auto-morphism. It has an outer auto-morphism. So, this is like full sequence which you have and you can see that any group can sit inside this.

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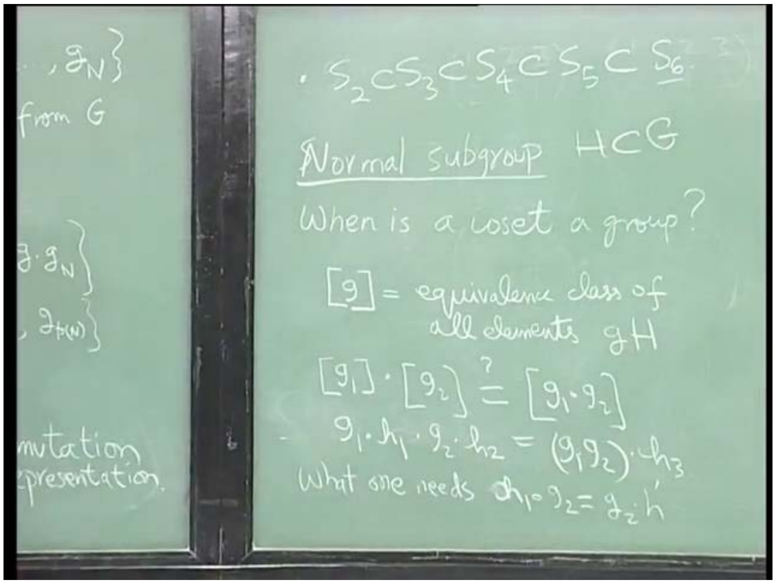


So, now the point here is to ask question about subgroups and so this can be obtained as follows, when is a coset group? And the answer is it is a group, when the subgroup which you are quotioning by is a normal subgroup. So, we have to like chicken and egg we need figure out, I mean choose one of them and work it out.

So, we will ask this question and that will tell us the answer to the definition of this. So, let us let me remind you of this definition, so let us stick to left cosets. So, g this symbol is equivalence class of all elements, which you can write as g H. It is a has a set of H elements. When g is equal to e so you could choose any representative from this set from this and put it out here. So, it represent any of those things. So a genetic element of this would be something g times capital H. So, the question is suppose you are given two different elements, g 1 and the question is does this sort of can I make something like this work.

And I do not want to introduce any fancy and new composition rule. I want to ask if there is a composition rule, is it in used from the bay group into which it sets. So, a generic element here would be writable as say g 1 dot h 1, a generic element here could be written as g 2 dot h 2 and let us just write this one would be g 1 g 2 with some h 3. So now you can see here is that, what you want really is if g 2 need not commute with h 2.

What you need is something very weak, what you need is g 2 what you need? Say g let us say that h 1 dot g 2 is equal to some h prime. Then but h prime is of course an element in it so again will stick to this notation. So if this is true this will follow. And this should hold for all g I mean this is not enough for it to hold for one particular equivalence element equivalence class it should hold for all equivalence classes.

Student: Within an equivalence starts.

It should not matter which element you choose.

Student: Exactly but you have to showed two for all equivalence classes.

Yes, so you can see out here this is the what we are this is the kind of things which we would have used, which we used in the definition of the left coset, while this is for the right coset. So, roughly it tells you intuitively left coset should be equal to right coset. It should be true for all g 2. Is this clear? So, the conditions can be written as follows.

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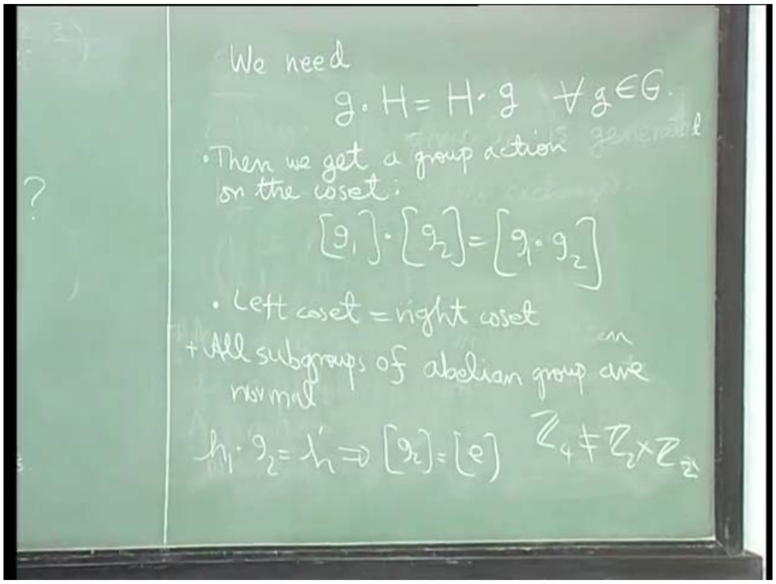


So, let me box this out so we need. Then we get a group, action on the coset, which we can write as follows. This is one thing which follows and of course left coset and right coset are the same. First thing to notice that if g is abelian, then there is nothing the everything goes through there is nothing to check. So now we can see that corollary or whatever, all subgroups of abelian groups are normal. So, the definition here is that if h is such a subgroup, then it satisfies that property we call it a normal subgroup.

Student: (( ))

That means, g 2 would be in the universe will be in the equivalence class of the identity.

Student: (( )) element is belongs to.

Yes,

Student: So that is still (( )).

Suppose, what you are saying is suppose h 1 dot g 2 is h prime, then this implies that g is also an element of h g 2, so they just implies that equivalence class of e.

Student: (( ))

Yes, the one particular equivalence class, but that is only one set of elements its always true for that trivially true for that such elements. But it is not true for elements which are not in that universe in that equivalence class. So now comes the question can we find out all finite groups and you want to sort of you can see that if there is a group, which has normal subgroup, then you quotient it and it some sense it comes from another group.

So, example of this is the one we looked earlier, which was to take z 4 which is an abelian group it has a z 2 which is a subgroup obviously z 4 cross z 2 is is also a group. And because it is a two element group there is only one two element group is also a z 2, but the most interesting thing is that z 4 is not z 2 cross z 2. These again it is an exercise for you to check that these two group are different. And so actually so what it tells you is that if you give me two z 2s there is this direct product, but there is also a sort of much more complicated way of doing it, which should give you z 4. So, these are called semi direct products which we are not going to discuss.

But there is a, so but at the end of the day you can see that by getting looking at z 4 we can it is enough for us look at z 2's. We are more or less we will get we can construct it by taking two copies and doing something to it. So, the idea is to look for groups, which cannot be broken down in some ways into smaller guys. They should not have normal subgroups those groups are called simple groups. As a group is set to be simple, if it has no normal subgroups. Yes.

Student: (( )) any other composition of on cosets.

I mean yes then you in principle you can take any group of elements you can define anything, but the precise statement is not natural. You are introducing some other thing, that way you can give me any set you can take and you can put if you I mean you take at this is only a set I can give any operation random operation and that will give you a group but that is not in some sense. Yes, mathematicians call it natural it is not natural it should come in a this is natural, I did not have to create any new structure.

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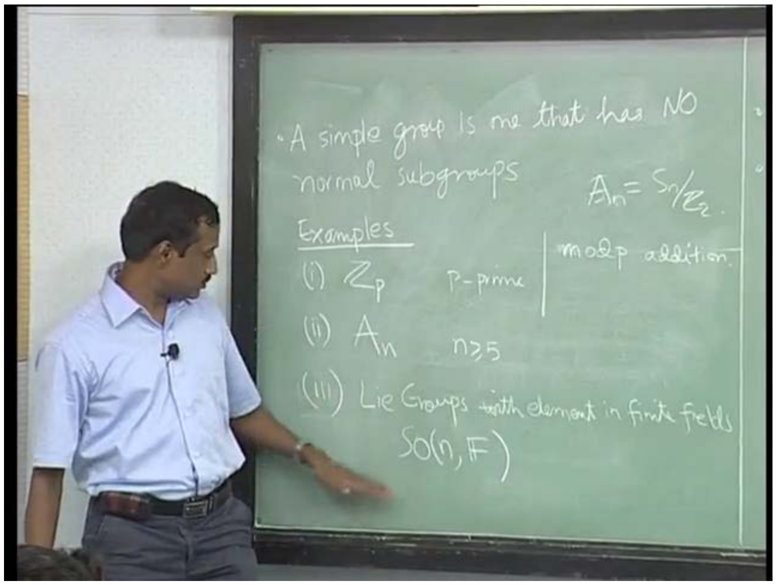


So, a simple group is one that has no normal subgroups. And, so the idea in looking at this is in some sense these are the simplest objects. You cannot deconstruct it into something into smaller parts that is what it means. So the question is so let us look for examples of these guys. So examples, the first one is the following. Z p where p is prime it does not have any non trivial subgroups so it is true.

If p, p is not composite obviously it is like z 4. Second one is A n, so this is a alternating group which is I did not give the name I guess a for alternating, it is called the alternating group is set sub set of even permutation, for n equal to 4 is in so I think its n greater than or equal to 5. They are all simple. So if you take S n, S n is not simple you remove this z 2 which is this thing and quotient and you get a these thing… So you can think of S n sorry A n to be and there is a now easy way to see that the dimension of A n should be half and factorial.

Now, come there is much more a complicated set, which are associated with what we will called lie groups I will explain this terms lie group with elements in the finite fields. I would not explain this in go in detail, so will just take just consider for instance S o n matrices and these were n by n matrices which are orthogonal and the element will suppose to be real. But I could have chosen different field, I could have put complex number also and keep the same definition that will still hold will get a larger set of solutions, but what you do now is to choose instead or r and c choose some finite field.

So, what is a finite field? An example of finite field is if you do mod p, where p is some prime element. So that has only so the number of elements would be 0 to p minus 1 because anything… So, mod so it is called z p I do not know I should call it mod p edition p is a prime. So, a field is something, which has two operations plus and product. So, these are all the same things, and important this is there should be an inverse for even multiplication. And that is why the prime condition comes you can go back and play with it and see if p is not prime it gets violated. So that is an example of a finite field.

So, the key point is that there are only finite number of element. So if you look at let us say you take mod 3 edition, it is like looking at n by n matrices with whose elements are only 0, 1 and 2 because 3 is equal into 0 in mod. So, you get only 3 n square element even before imposing the orthogonal conditions, so it is only a so you end up getting finite group. So, what was actually in our case we started off with S o n r, which would be a infinite dimensional group from this view point because it has an infinite numbers of elements. Once you put in finite elements so from that set you get a whole thing, so these are the so this is sort of examples which we understand. And the question is are there more and the answer is yes, there are 26 more.

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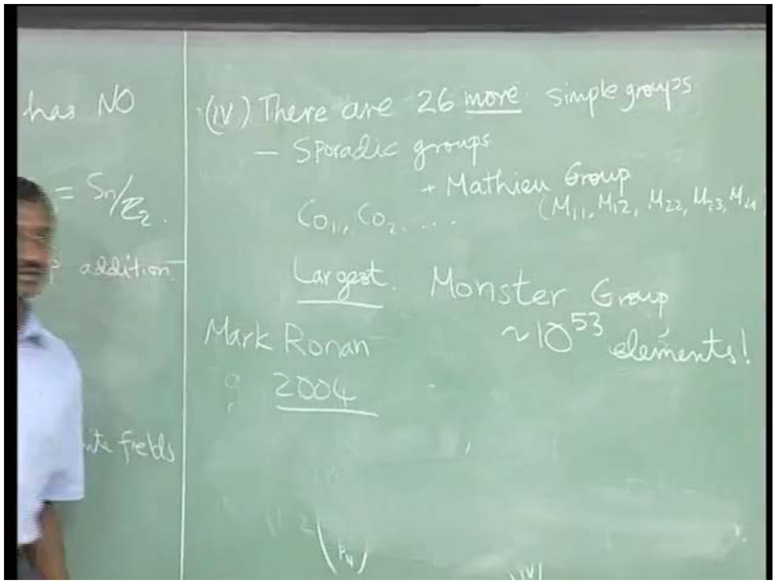


The 4th one I will write here, more simple groups and they are called sporadic groups. In the 1800’s around that time Mathew had constructed a five examples. So examples of them are something called the Mathew groups, this were known for a while. So there were five of them. The symbols they mean something, but here the one thing you can think of it as follows these 12, 22 etcetera indicate that there some subgroups of the corresponding s. So, m 24 is a subgroup of s 24 for instance, but this was and people did not know too many examples. And that is all may be this was it and there was a classification on classifying all simple groups.

And so this was this involves lots of people thousand and thousand of pages of calculations etcetera. And they found in the process they found several more examples and the largest of this 26 is called the monster, it was found by Fisher I guess and it is called the monster group. It has about 10 power 53 elements. I do not know it something like a number of atoms or whatever in Jupiter or something like that exactly, but the amazing thing is that the classification actually found these things.

So and there is actually beautiful book by what is the name by Mark Ronan you can check it on Amazon or something like that. Where he actually discusses the history of how people found these groups etcetera. Actually it was a period of ten years I think 1960’s to 1970’s where actually all these things were found. The various groups or found in bits and pieces by different people. And the convention usually is to name it after the person who found the group.

There is something called convey c o 1, c o 2 so on so forth then there is Yanko group Yanko found them etcetera. So but there are only 26 of them, and so there I think Jef Mayson or somebody have give a proof which had a gap and it got a filled only in the year 2004. Something called the Quazi thin case it got completed. So, in around 2004 a couple of people I think prove that they filled the last gap and so now it is generally accepted that this is it these are the only groups the once which I have written.

This I have been little bit loose, but Wikipedia as an is entry where they write out in more detail what this various groups are. These two are easy to state but the amazing thing is that is how there is no more. And the key point here is 10 to power of 53 is large, but at the end of the day it is still a finite number, so it is still a finite group. So and now people are in the process of a trying to write, what is called the second generation proof of the classification. So that because it was done in bits and pieces and apparently, so far now six volume have been written and there are still going on.

But for us we believed this and its sort of an exercise amazing exercise done by mathematicians group of mathematicians. So, this is all I have to say about finite groups. But I should point out that, finite groups will come, I mean even in physics there will be situations where you would come across finite groups. For instance, if you are looking at a finite or at least discrete groups so if you take.

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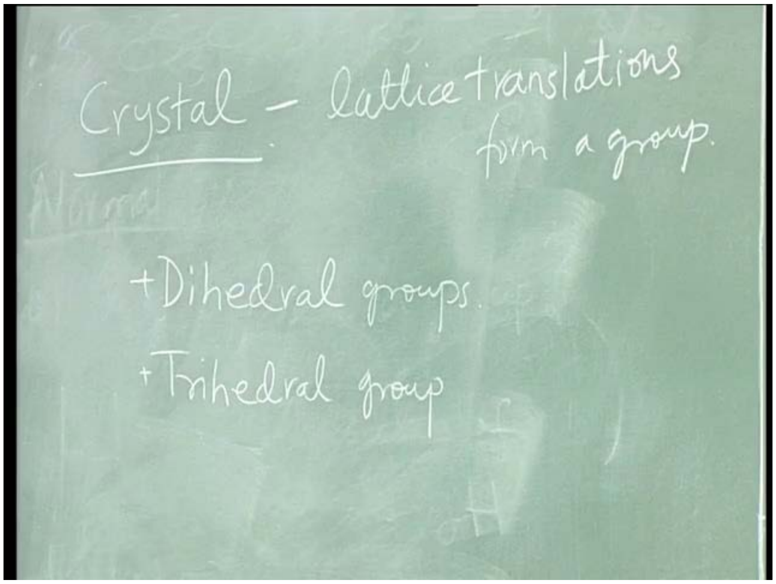

A crystal for instance, so there on a in a crystal your translation is broken to just lattice test. So, in a crystal set of lattice translations form a group. And of course if you look at point symmetry is fix a point and then you start looking at it, you get things like the dihedral group. So, you get things like dihedral, trihedral groups you get from this.

Student: (( )) there are exactly 29, 26 you said 26 more plus so is 3. Yes is it?

No, these are three families these are families, so this is like in fact now you can see that if I take a prime number which is larger I mean larger than 10 power 53. Of course there exist a prime number greater than 10 power 53 that group has more elements than this but it has much less structure in such a groups. So I mean A n of course you can see it is not very hard to take in it was like factorial, so it sterling formulas will tell you the that it will go very fast. I do not even know 100 itself might exceed. Yes

Student: (( )) you can construct any other group as a composition of these or so.

The point here is that now any group, any all the finite groups that you look at, could be obtained from these group in some ways. So, these are the irreducible parts, I mean this cannot be broken down into these things. So again so for instance there are of these 26 group some of them are actually the subgroups of the monster. And there are I think of you which are not subgroups of the monster they called the they called parayas. I mean in India since we are in India we do not need to explain the meaning of the word paraya. We know what that means. So, they are the outliers.

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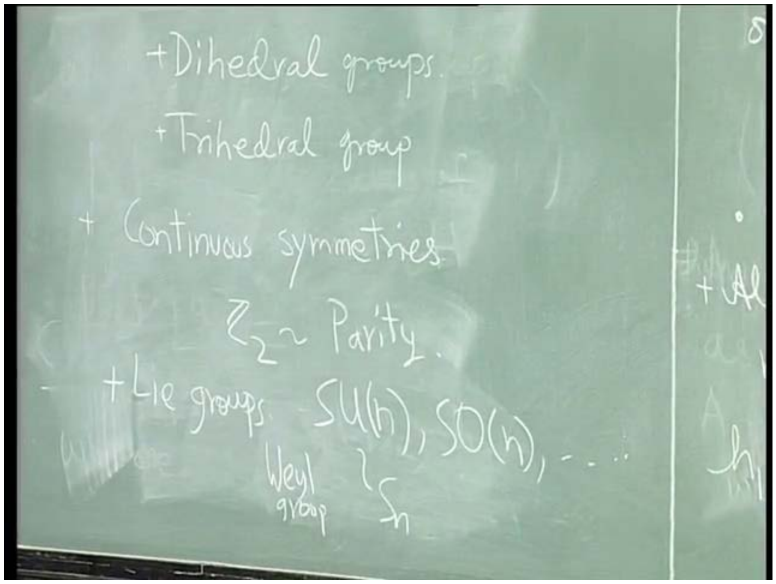


So we do come across a symmetries like this, but even when we look at continuous symmetries, we saw. For instance when you looked at that orthogonal group there was Z 2, which came out which was like parity. And what we will see later is that if you when we are looking at what a called lie groups, such as you know S u n, S O n these are just examples. You can look at what is called they are weyl group, they have something called as weyl group. For instance at the reason I chose as s u n is because it is weyl group is S n.

So, these discrete group come in disguise, they do come in many applications in physics. And by large one things or at least, so I thought that one would never use none of these sporadic groups would appear anywhere. But it turns out in a weird way the monster group made an appearance in something called moonshine.

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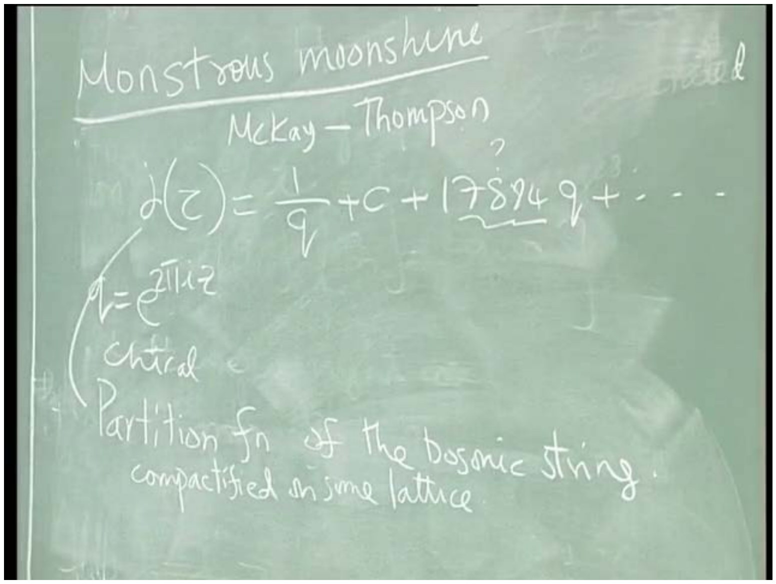


So, it is called monstrous moonshine, so mathematicians are very specialized people. They each person does their own thing and they are experts in that, and they do not quite often look at things which are in other areas, but they do and when do interesting things happen. So there was a mathematician he was looking at their some j this function called the j function in there is modular form.

So, this is a bunch of people who do look at modular forms, but why would a guy looking who does group theory look at this. In general there is no reason to look at it. So, this has expansion so q it is a Fourier expansion, so q is some e power i pi there is some expansion which is 1 over q plus some constant plus, then came the number I do not even remember the number now some 1, 7, 8, 8, 4 or something like I am not sure q plus so on so forth. And these number he looked at it and this look this smallest matrix representation of this monster group was exactly one less than this number.

Then he looked at the next term as well and that also broke up into two representation broke up into presentations of the monster. So, out of the blue I mean this is some wake function in some other part of mathematics and this is in group theory. And the question is how are these two related? And the amazing story is that yes they get related in a very beautiful way and in fact this whole series every term in that has something to do with the monster group. And so this was the observation due to Mckay and who sent his observation to one of this very important names in the classification of finite group Thompson. And Thompson actually verified that it actually work a little bit more and so this led to a how beautiful construction.

But now comes the need stuff that this actually this function also appears in string theory. So, it is the partition function, so this is the partition function or the what is called the chiral partition function of the bosonic string. Compactified on some lattice, some 20, 40 mansion lattice called the least lattice. So, it is actually came so one would think that there is nothing why would a physic person do it, but a physic person do string theory, actually you would see this I am in a very interesting way. In fact this observation led to the proof of what I called the monstrous moonshine conjecture. Now, it is longer the conjecture and it is a beautiful story, which as an interplay of physics and mathematics actually.