**Classical Field Theory**
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**Lecture - 6**

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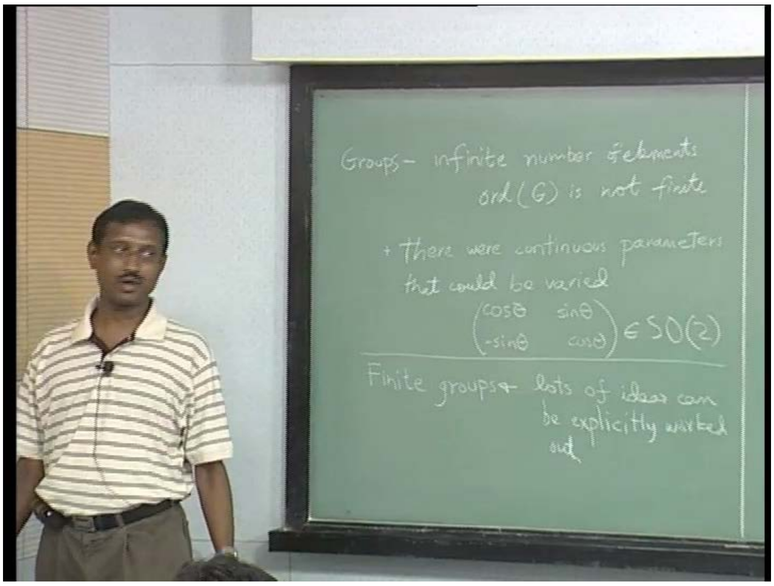


So last lecture, we were considering groups, which had a infinite number of elements, I mean by that mean what I mean is order of the group is not finite, the easy way to see that is if you consider the rotation, the angle take the continuous variable, and so we also use when there is some parameters (( )), even more this thing is there was there exist there were parameters, continuous parameters, that could be that could be varied, so if you have simple rotation into dimension also, we would write. So, for every value of theta line between 0 in to phi, you get a distinct element, and so this is group, which order is infinity.

So, what we are going to do is for we are going to discuss, what I call finite groups, mainly because it is they are simple to handle, and you can I can lot of ideas can be explicitly. In fact in most math text books, you will find that finite groups is discussed a lot more, while in physics book, you would find that what I will call continuous group are discuss a lot more. And this has to do with in the sense of application or whatever you want to do in physics, this is what appears but this is I mean it is not true that finite group do not appear in physics. Of course, it does but somehow this is something which we learn first, and we learn this later.

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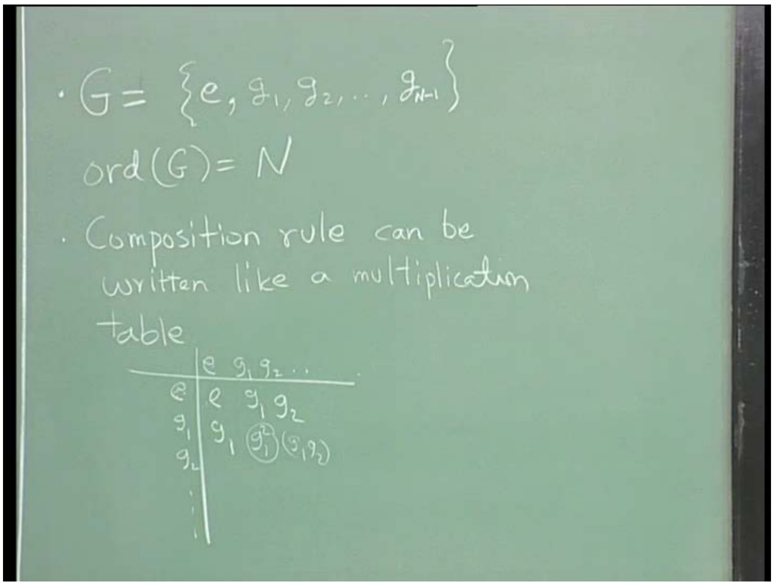


So, so what I am going to do, now is so if so let us go back to the definition of a group, and so let us say every group has to have a identity element, I will call it e, and after that just a bunch of the elements g 1, g 2 so on, so 4 and say g n minus 1, now it is a nice set with elements, so order of G is equal to n the way, I have written it and quite clearly the composition rule can be written like a multiplication table.

By that what do I mean, what you do is you write something like this e g 1, g 2 so on so 4 similarly, e g 1, g 2 so on so 4 an out here, you write e let say we have to follow some convention, so the convention will follows that this act on, e dot e compose with e is e, e compose with g 1 is g 1 e compose with g 2 is g 2 similarly, g 1 composed with e is g 1, this will be g 1 composed with g 1.

Since, there is no ambiguity sometime, you just write g 1 square of course, g 1 square should be one of these are the are the relevance because of the closer but, I do not know because, I am not specifying my group I do not know what it is, so I am just sort of writing it out but, if I know the group I would write the corresponding element out here, so this is like the group composition law of course, you were its not enough to write some random set of entries, here it has to satisfy the occur axiom so, what we will do now is to do a very nice thing is to ask can we if a small value of n can we construct some groups, which has that property ok.

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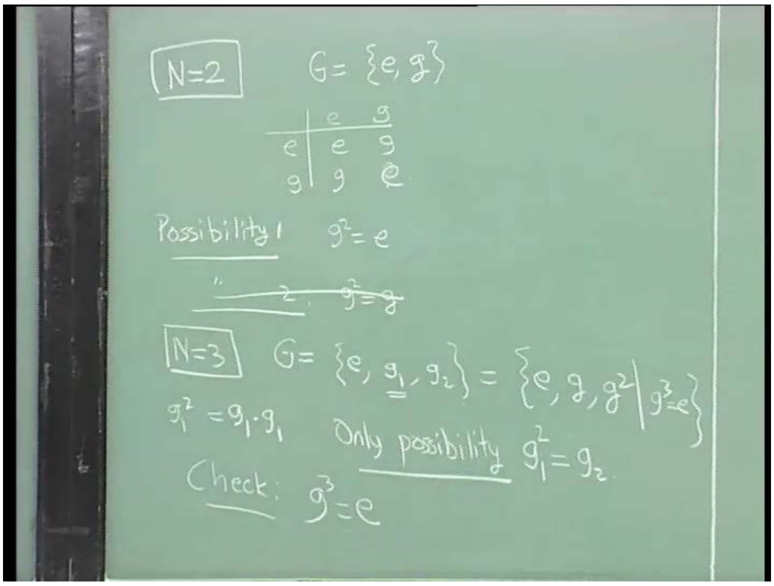


So we start with n equal to 2 that means the group has only two elements, let us call them e, and other element should be g, now we need to write out the multiplication law, which is very easy this entries are easy to write so the question is what we can be write out here.

So, there are only two possibilities, possibility 1 is that g square is equal to e, and possibility 2, that is it is not hard to see that this would require I mean this would implies g equal to e, so then it become a group of order one, group of order one is boring it has only one element, which is e they arrange the element, so this is rule out, so this is so the table is complete.

So, now comes what about n equal to 3, one can go ahead and write out multiplication table, which would have 9 entries, and some of them again would be fixed very easily but, then there will be other entries but, we can ask we can do, we can pick anyone of these element, so let me choose g 1, and ask what is g 1 compose with itself, this what I call g 1 square out here, so what are the possibilities for g 1 square g 1 square, if g 1 square were e what will happen you will go back to the earlier group, it close its over there is no way of generating the other person close your tells you, that it is over, so g 1 square cannot be g for the same reason of this but, g 1 square cannot be e because with we want a group, which has three elements not two, I hope you understand that if I put g 1 square equal to e, it is not like I get then what is g 2 multiplying, these things you will see you are into the contradiction, so what is the possibility, so g 1 square, so the only possibility ok.

So, you can see for n equal to 3, I can go just ahead and write, instead of writing something like this I just have 1 generator g, the other one, I will just denoted by g square instead of calling it something else, and next step would be what would so we need to ask, so we have generated all the elements, now all we need to do is what is g times, g square, what should that be that should be e, so check for yourself, that is your only possibility, it is not too hard becauseis g cube were somewhere new elements then you are extending the groups, and you are getting something this actually do not want, g cube equal to g again is similar contradiction, so on so for so this is only possibility is so subject to the relation, so g square g cube equal to e ok.

So, this is completely abstract, we have not even decided what the set is anything, we just wrote two elements, and we wrote out I mean the axioms or the definition of the group, and we find that we get we get that is only one possibility in this two case. But, of course, as n keeps increasing, you will see that that is more and more chance for you to construct new group so, whatever but, the first thing to do is to look at these two groups and see if you have ever seen anything like this, so let us start with n equal to 2, have you seen this particular group in some form or the other (( )) binary numbers ok.

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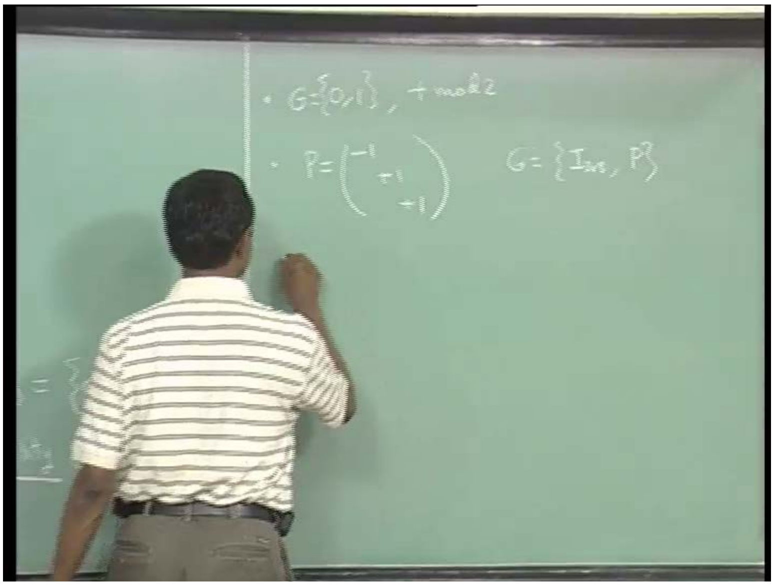


So, that means g this is the group, which is if you take if you take the set to be 0,1 or 0,1 with the operation plus mod 2, I do not know how the rotation is so 1 plus 1 would be 0, so this is one realization but, in this course have you seen, it before have you seen anything like this

Student: (( ))

iso scalar,iso scaling

what is that

Student: (( ))

No, no, I am talking about an example, I claim that we have in a previous lecture, last five lecture. We have seen an example, of it parity, so but parity we realize it as a metrics it was very different but, so you take your set to be the identity 3 by 3 matrixes, and this matrix p again this satisfies the same.

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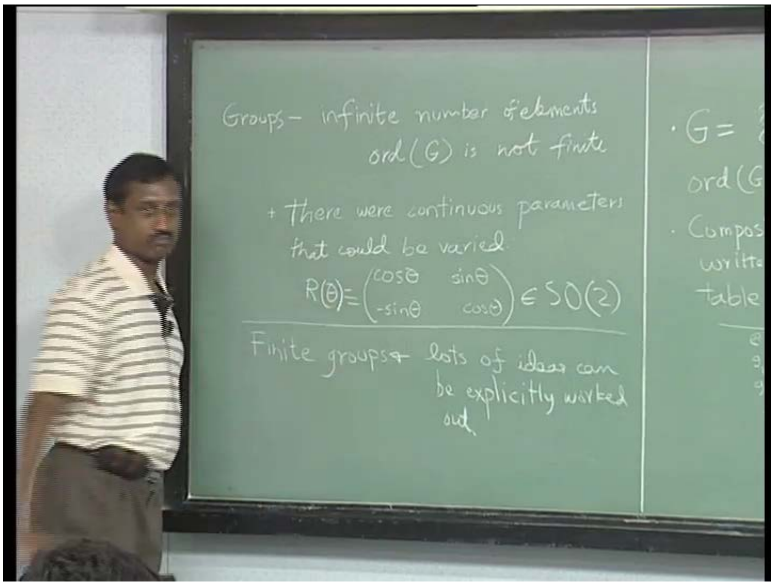


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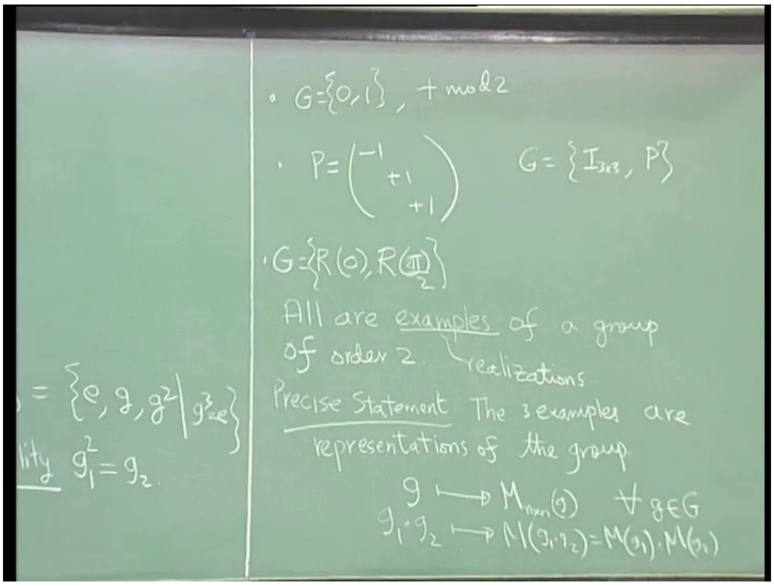


So, let us just one more thing, lets define this things to be R of data, let us take R of 0 take g to be R of 0, and R of phi, so this would be so phi was 0, R of 0 is identity matrix R of phi is minus 1 minus 1, again that is also a realization of this group, so what you see is that, here this is the 2 by 2 matrix, here it is like just one element one dimensional thing but, then there are operation, so in these two cases the operation is just group multiplication but, here it is modular two addition but, there are all from the abstract group that we have written, they are all realization of that.

So, what the statement the mathematical statement is all are examples of group of order 2, I do not need to say which group, because as we say its unique, so this instead of calling examples, we would call them a realization, these are all English words but, the precise statement is that they are all representation of this group, so the precise statement is I still have to define for you what are the representation is but, it just means this sort of things precise statement is that they the 3 examples are representation of the group, so what you should have in mind is when we talk of representation, usually is give me a group element its math to some matrix, so for all ok.

So, every group are mutual give me a n by n matrix, and so such that the product the composition lost, such and you should satisfy closer etceteras g 1 dot g 2, will get map 2, so, n stands for matrix out here, so for g 1 we get an n by n matrix for g 2 we get n by n matrix, the matrix is such that it respect to composition rule, so on this side it is composition rule in the group here, it is the usual matrix multiplication.

Student: (( ))

Yes so the even more precise statement this it act as a linear operator in a vector space, so if you take a linear operator, that it is realizes us n would be the dimension of the vector space.

So we have already seen for instance, when we take the S O 3 rotation group, the vector representation, so you can think of the S O 3, as the some abstract group with some multiplication rule, and we wrote 3 by 3 matrix realization, we also say a 5 by 5 realization, because when we take a symmetric tensor trace less tensor, we saw that that behave so if you work out that thing, that like 5 by 5 representation, and in your quantum mechanics course, if some of you have done quantum mechanics, you would have launch the spin j representation is 2 j plus 1 dimensional.

So, what that means is that for the for the S O 3 group element, will be realize as 2 j plus 1 time 2 j plus 1. So, what I call n here would be 2 j plus 1 n would be equal to 2 j plus 1, so but, let us get back to what we wanted to do we just to go ahead, and so you can see out here again, this think I can I can think of n equal to 3 as a rotation by 2 phi by 3, so phi here I can re write actually as 2 phi over 2, so I can realize this property something, which cubes to 1 is like to rotation by 120 degree, so I have 120, 240, so on so for ok.

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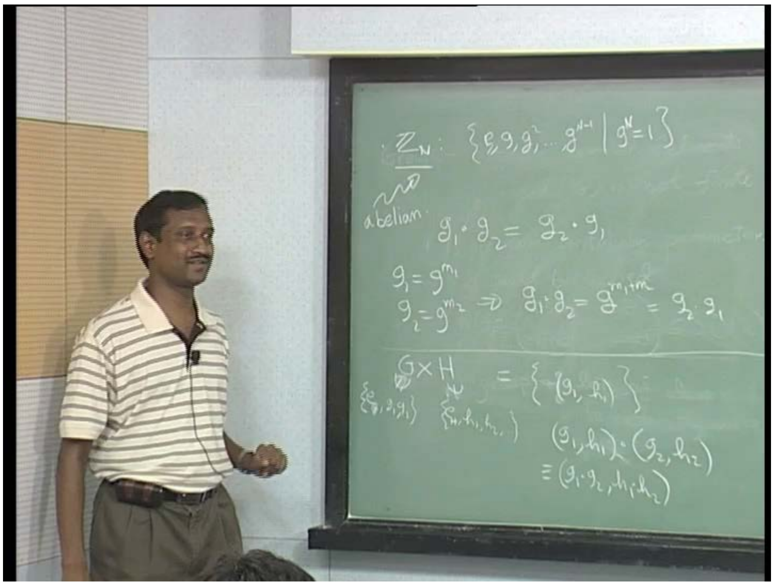

So, let us just define a group for any n I can do this for any N, I can so we have actually by looking at these two example, we will get for all N, we can write a group like this, and I will so let us call this group Z N, and the this is true for all, there is no restriction on N, so for any given n, I can write such a group, so you can like I said again realize, you can look for representation in terms of just two by two matrices in terms of rotation, you can also think of them as mode N addition, there is no problem with that but, the question is are they I mean this is this the only group very boring, and look at this groups because they are brilliant because, it does not matter, how you so let us say, that if you take any g 1 dot g 2 is equal to g 2 dot g 1, how do you see that, any g 1 can be written as g power some m 1, and any g 2 can be written as g power m 2, so implies that g 1 dot g 2 equal to g power, so this is a group which is abelian ok.

So, now the question is can we construct more group, and the answer is the yes, so the first example, would be to look at Cartesian, so you take any 2 groups, so G 1 let us call G into H, you know what the Cartesian product of this sets are I just need to tell you, what is the composition rule, so let say that the set of this is element, which I will write as e of the group g wrong.

so this would be in terms of set theoretical language, it would be a generic element, which would look like g 1, g 1 and some h 1 for some, write this is what would be a typical element of this, so we just need to be ask, how it will be multiply to elements and it very obvious to converted into a group, I need to just tell you how to multiply, how would you do that, any ideas because, we know how to multiply g 1, and g 2, we also know how to multiply h 1, and h 2 just do it, that way. So, this is the natural one, well this dot here is the is the composition in this group, and the second dot is the composition in that, and so this is just definition you can see that the identity of this product group is e g e h is the identity, so this forms a group, so we can again, you can see starting from this, we can construct a few more ok.

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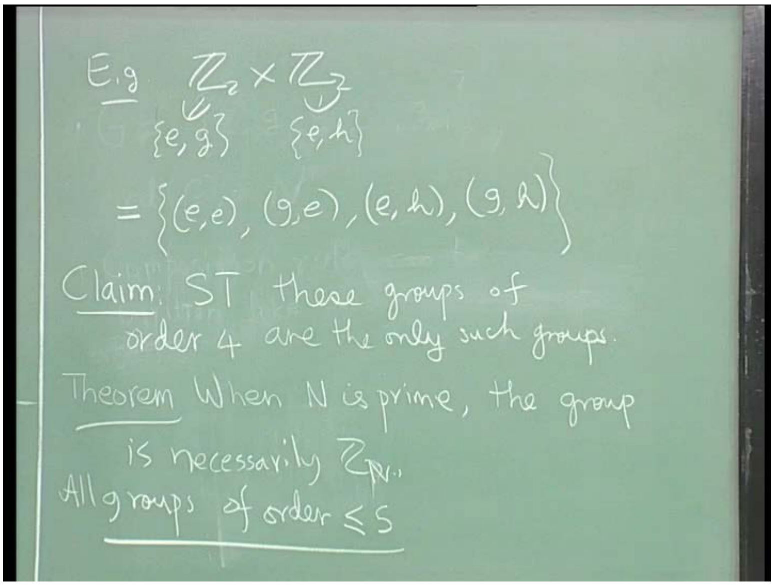


So, an example would be and you can see order of this group will be order of g times, order of h, so this would be out of four group, so we already have 1, here which is Z 4 but, this would be also an order four group, so let say the elements of this are e I will not put the sub script here but, I will use a different symbol for this e g out here, I will call this e h, so this set will have following elements e, e g, g e, h g, h.

So I have an excises for you, which is the following, which is to take consider an element group with element with four element, we have 2 example, the question is there any more the answer is there are no more, so there exist, so claim. So, that these two examples that is of groups of order, actually just call these 2 groups of order 4, only the such groups, so I mean one horrible way to do it is to go ahead, write out the composition table, and convince yourself that there cannot be anymore, that is another nice, I mean theorem or whatever, when N is prime ok, when N is prime, there is only the group is necessarily the Z P Z N there is no other example, and the way to do it, can anybody think of how to prove it, so prime what does it mean if something is prime it means, it cannot be factorize.

So, let us let us just arbitrarily take, the there is so you write out the group with N elements, where N is prime pick an element, which is not identity, and then start taking powers of it you keep on going, and at some point it has to terminate but, the only point it can terminate, without any contradiction is when it hits N ok.

So for prime numbers actually you are done. So, now you can see that with my claim, and this thing I am actually completed all groups of order less than or equal to 5, I have completed everything 2 is prime, 3 is prime 4 is not prime, it is composites, so there were two possibilities, which we I mean if you believe my claim and 5 is prime, so we are done, and so we can see that until order 5, all groups only abelian, there are no non abelian groups, ok.

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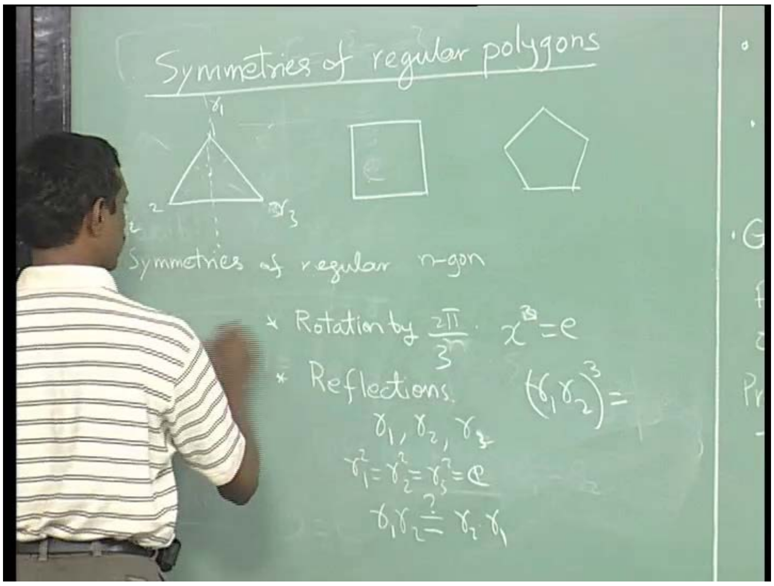


So, let see one more example, which comes in physics quite a bit, which is which is looking at symmetries of polygons, regular polygons. So by that I mean take something like this a triangle, equilateral triangle take a square, then the next one would be pentagon, hexagon, so on so for. So, ask for all the symmetries of this, so you look at so one symmetry is you pick the center of this object, and rotate by we will not locate translation, so we will preserve something, so you take a rotation by 2 phi by 3 here, 2 phi by 4 etceteras, so symmetries are of a regular n g on, that is polygon with n sides it is a rotation by 2 phi over n, which is like a z n.

And there is also I can pick something like this, I can reflect about this goes here, this goes here but, it is a reflection, so it is a Z tw2 but, there is nothing special about this I could have chosen this, I could have chosen this also see well are things, you could see out here you could take for instance here take something like this, so you also have reflection several reflections, so let us pick this particular case and lets call this reflection r 1 1 2 and 3 r 1 r 2, and we will call this r 3, there are all reflections, they all satisfy a nice condition, which is r 1 square equal to identity, I will write it equal to 1 but, I mean write it identity, these are operations, but now the question is what about r 1 r 2 is it equal to r 2 r 1, so go back and go back home, and convince yourself, that is not truein fact you can show that r one r two compose r one composed with r two is equal to rotationyou let us call this rotation we will call it x so this would be something which would satisfy x power equal to let us put identity, that is identity x power n equal to e you can sure that r 1 r 2 cubed equal to so 2 phi by 3 here, right because we are looking at for oh I was doing in n gone so.

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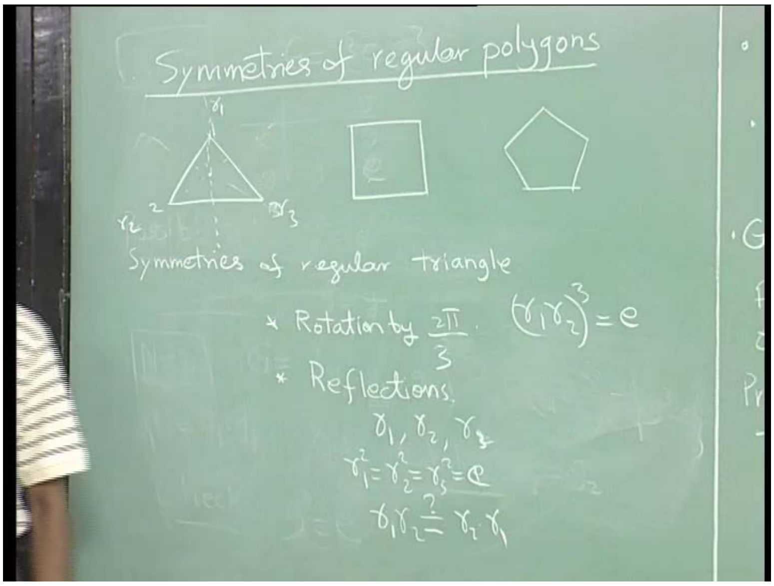


Sometime, I will get very ambitious, so let me let us leave it as triangle, we will write the general answer later sorry. So, what will have is r 1 r 2 cubed equal to identity, so it is one of these rotation but, so in other words this guy can be generated by all these things and you can even convince yourself, that I can generate given r 1 and r 2, I can generate r 3, so homework is go back and play with this triangle example, and you will see that everything can be generating starting by any taking two of these guys, in fact it generalizes to the n gone, where there will be the analog of this r 1 r 2 r 3 up to r n all of them square to this 3, ok. So, this group and it is important thing is r 1 r 2 is not equal to r 2 r 1, it is a non abelian group.

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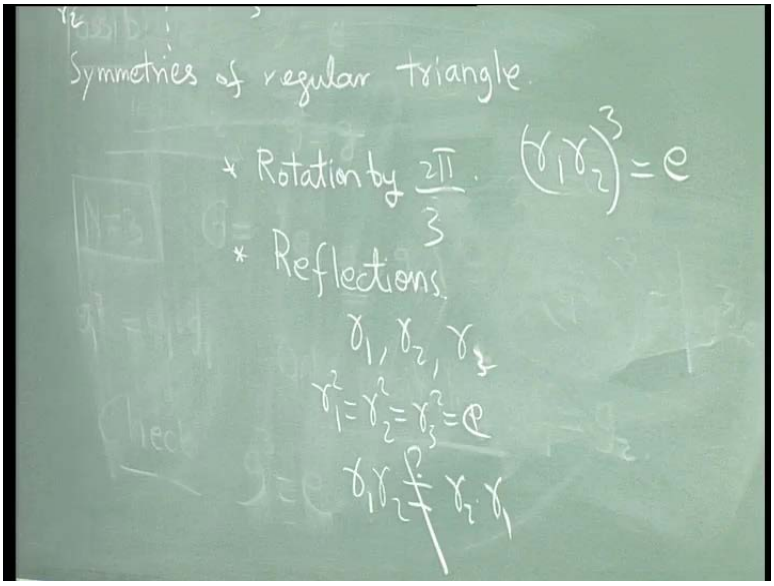


And let me write out the group for you, so the group this group symmetry of a triangle.

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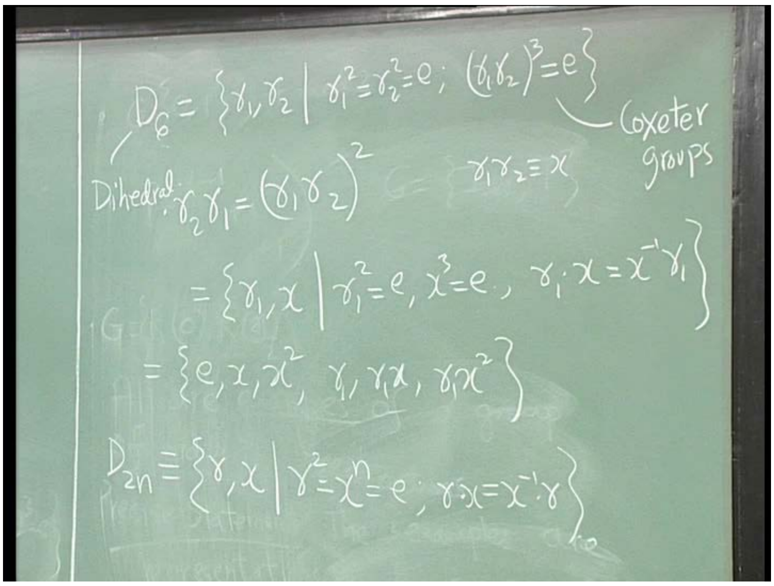

So, we will just call it D 6 we will see what the six means generated by 2 elements, r 1 and r 2 with square, which have the following properties, and it is not it is not so hard to see that r 2 r 1 is equal to.. So, I generate even this other things I leave it as an exercise, for you to determine, what r 3 is… So, this group actually has exactly 3 elements, you can also rewrite in a slightly different form I just pick r 1, and I call r 1 r 2

Let us call it x with the following I still need to say, with this way of writing it, you will it is not hard to see, that I can write out all the elements, it will be identity x x square I cannot write x cube but, I can write r r x r x square, all the others anything else, I write will gets will gets simplified, so I drop this r I have not written one out here, maybe I should, so we have nice group, which has 6 elements, that is why it is call 6 D stands for the dihedral group ok.

So, more generally I will define D 2 n, it is exactly will like this except, I have put if I put n equal to 3, I get D 6, you can convince yourself that the number of elements is exactly 2 n, these are non abelian groups, so you can see the non commutatively straight away built in here, so actually there is a nice.

If you see here that are in this first way of writing these group, it was written in terms of elements which are all square to identity they are like reflections, there are exact two but, their composition behaves away has some property like this, so this also has a name, this is call, so such groups are called cogcetor groups, so by I am not writing I mean it is quite different from what we did earlier.

We try to write out the all the elements, and work out the multiplication table here, we are writing in terms of some generators with some relations, so you can see that this is the natural generalization, even this is written in this way in that presentation, it is just saying that it is given by one generator with one relation but, here it is somewhat different in the sense that all the generators are always Z 2, they are also called cogcetor group so, this is actually in fact the smallest non abelian group ok.

We will see another whole family of non abelian group, again one can one can write another order 6 group, which would be abelian, which is 2 cross Z 3, so here is a case where you can use the non abelian nature to prove these group or cannot be equal to each other, there is nothing to compute Z 2 cross Z 3 is by definition, it is abelian so ah.

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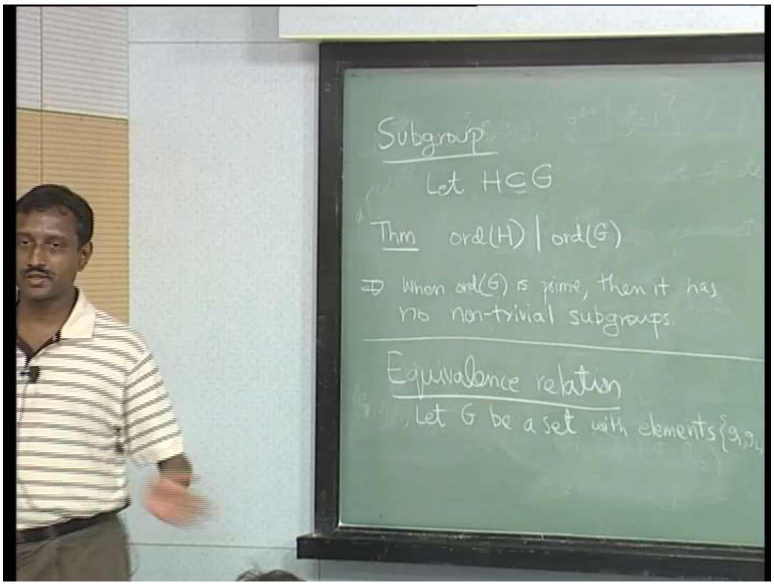


So here is another nice idea, which are already discuss the idea of sub group, sub group is that let H be a sub group of G, so H and G by the way, whenever you mention a group and I do not I will not keep writing originally my definition of a group at G with some operation, it is sort of annoyance to keep writing the operations, so you just usually one just refers to the group G, and until it is necessary you do not specify, the operation on context you know what you, what one means so here, what I mean is that G with some dot is a group and H with the same dot is also a group but, it is also a sub set.

So, here is a nice and important theorem, which says that order of H divides order of G necessarily so, you can see, now if G if order of G is prime, it cannot have any subgroup, it can have subgroup one trivial subgroup, which is element with 1, 1 element e or itself, so there are two I mean, so I have not I am not I mean in principle, it could be equal to that any group with it is own subgroup, obviously order of order of G divides order of G, so you can see that so this implies that, when order of G is prime, then it has no non trivial subgroups ok.

So, now the question is how are we going to use this fact, and prove this and this leads to a beautiful contraction, and the so this needs to me to introduce, simple mathematical idea, what we will do is we will if you if H is the sub group of G, suppose I was able to partition G into given H, partition G into equal size parts, then it follows so for every element of H, I will give you 1 part with a label going with that, so that will that will lead to a partition of this side, and if I can show that each partition has the same number of elements, then I am done, we will do that so, we need the idea of something called an equivalent relation, it is a beautiful it is a beautiful idea, so let G be a set; this is one of the cases being formal writing out the thing is better than anything with elements of course, g 1g 2 etceteras, ok.

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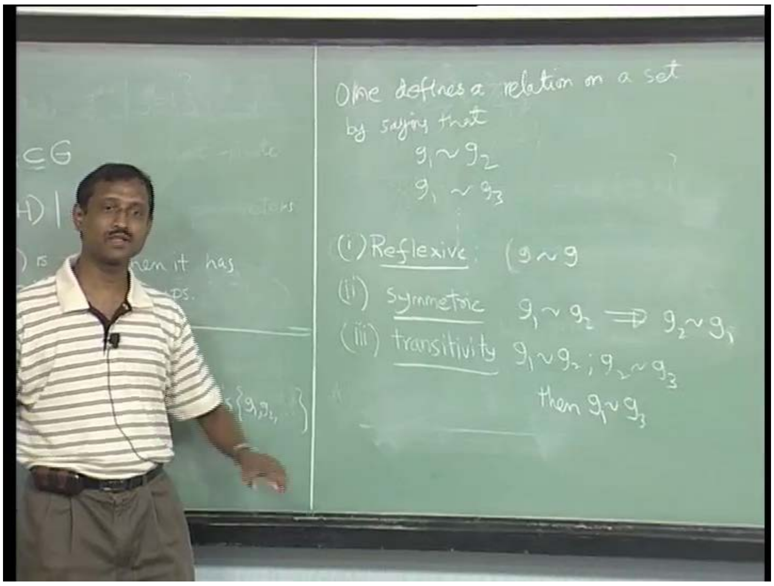


Then we can define one define a relations, set by saying that g 1 is related to g 2, and so on so for. So, that might be two elements, which may not be related, so it is sort of it say that g 1 is related to g 2, may be g 2 is also related to g 3, so on so for, if every element is related to everything, it is not very interesting, so there will be so we have to the relation specified by giving all the all possible relationship in this groups ok.

So an example would be, this does not require, this to be a group or anything, so for instance if you if you if you take a population of a country, and they said that we define two people are related to each other by some blood relation, or whatever if you do that thing what will happen is that you will find, that is like putting a relation in this set, so there can be two people are not related to each other is that clear so, we will so that if a relation satisfy certain condition, we will call it un equalence relation.

So, the first one is so these are relation is said to be reflexive, if any element if g if any element is related to itself, second one is symmetric, if g 1 is related to g 2 it implies, see for instance if you said the relationship is say brother, somebody can be the brother of somebody but, the converse need not be true, why is that that could be sister, so it is not a symmetric relationship.

The last one is transitive with t or whatever, if g 1 is related to g 2 and g 2 is related to g 3, then if all this is true then I claim that its it is the same as partitioning, this set into bunches each of basically people, who are related to each other, the group elements, were related to each other, it is like the family is this cleared, you seem very unhappy

Student: (( ))

Yes, g 1 so you pick, so what you the partition is as follows, let us do it pick one elements for the set, let us call the first element g 1, then low as scan through the whole thing, and pick all the elements, which are related to g 1.

Then you are done now you look at what remains, there will be some elements, which are not related pick the next element, again you look for all, the elements, which are related to it again you form the second set part, third part, so on so for, it gives you a partition, and you need to think about it, that you need all these properties, otherwise you will run into problems, you would not get a partition, so what an equalence relation, does for you is to take a set and partition it ok.

So, now what I will show is precisely that I will show that, given H, I will define an equalence relation, so that you will see what that is so let us so I need to defined any equalence relation, so we go back to original problem H is a set and g is a also another set.

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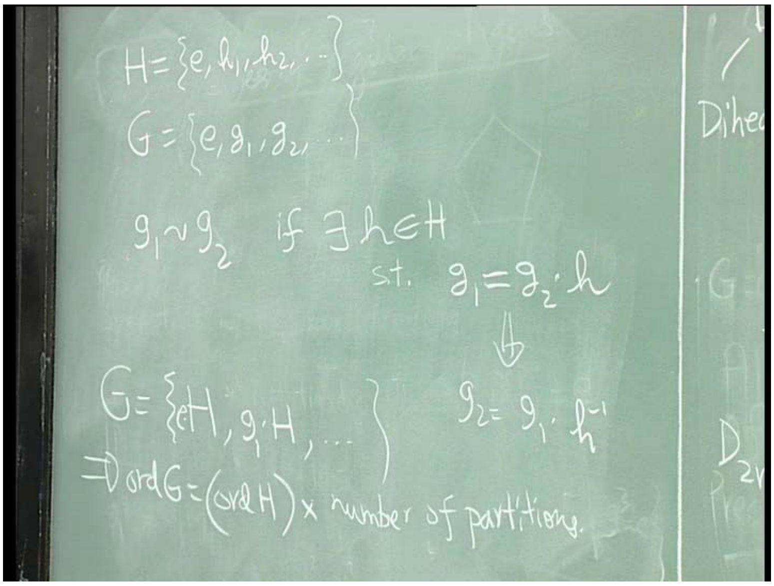


So let us let us define, so let us denote all elements of H of course, identity must be on both, so I have to say that g 1, if there exist some h belonging to H, such that so reflexive is trivial, there is nothing to choose h you have chose e, so that is done symmetric again it is not so hard, I can just this implies g 2 equal to g 1 dot h inverse but, that is guaranteed because, why is that guaranteed, yeah inverse exist h is the group transitivity again is not so hard, it is trivial, so it satisfies the equalence relation.

Now, let us ask what is the partition we would, so the claim is g can be written as so you pick an element of e, and ask for all the elements related to it by the action of h form this side, so what will happen because the first element we have chosen is e it will give as all the elements of H, so the first partition is H itself, the group H.

Next one you pick another elements let us say its g 1, so let me write this is e dot H should remind you, where it came from the next element would be g 1 dot H, so on so for. Until you have done, now you can see each of these elements each of these partition as how many elements, same as H, same as order of H. So, now there are certain number, the order of g is the sum of the these things each of these elements so this implies, order of g is equal to order of H times number of partition, it is a number of partition, where 1, then g and H are same that is we said it is a un trivial sub group, so that is gone so it has to be at least to so this is this group, this theorem ok.

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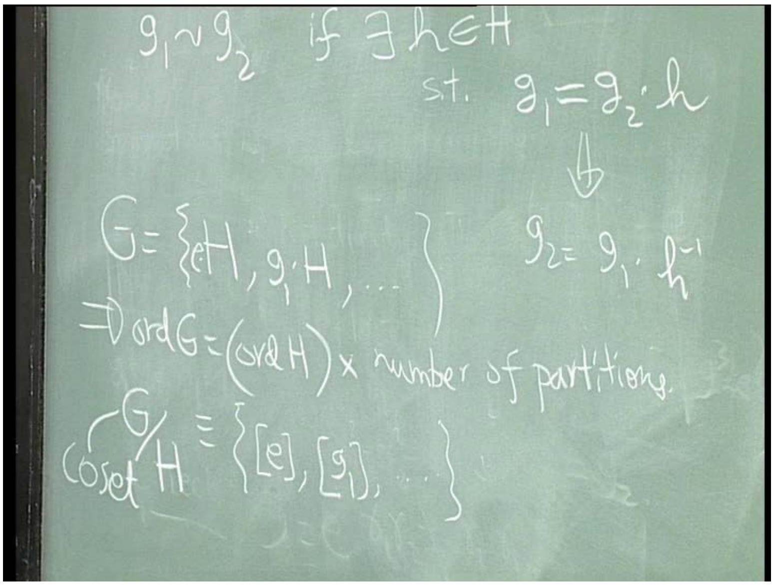


So this the partition here can be understand, I can rewrite I will define something, I will call the partition G mode H, this is the definition, the notation here, is for e is symbol for this partition I group all of them and think of them as one element similarly, g 1 is this element but, actually any it does not matter, which element I choose I chose just one element in that partition, and plug it in there, so on so for. And I will write these things as follows, this is this has a name, it is call a cosset, it is just a set, it is a set of partition and each partition, should be given a label I label it by any one of its elements in particular for this I have said it e but, I could have put in fact I could have put any element of H into it ok.

Now, again coming to the whole set up in this equalence relation, I have put H on this side, on the right side I could have put it on the left side, it will give me a different partition, so they have names we will call this a right coset in general, it is just a set it is a set of partitions of given a subgroup, the other one is called a left coset, so I will stop here, and ask the following question, which I would ask you to think about, can we can a coset be a group is there a natural action, we know that they has coming from some bigger group, where there is a group action.

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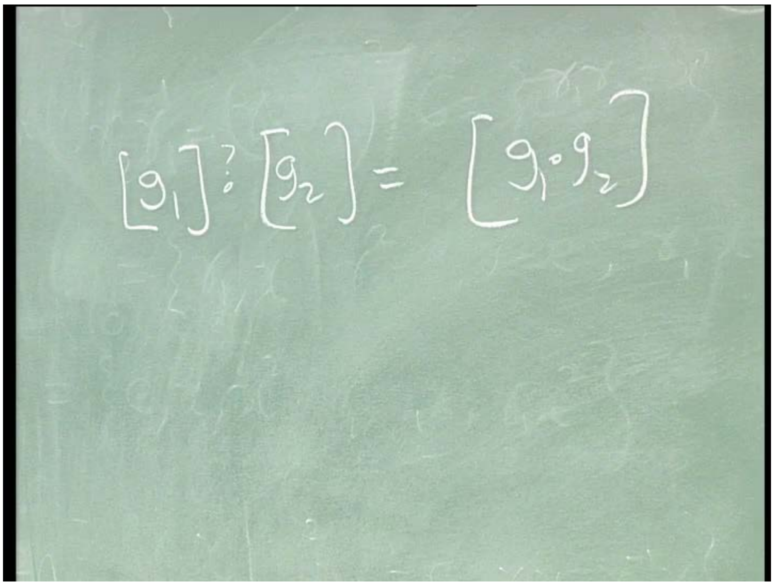


So by that what do I mean, if you give me two elements, can I define was it take me form one partition to anther is it possible, what are the condition under, which it can happen like I said there is this may not happen but, there are some condition under which a coset is also a group, so I will leave you with this puzzle, you should go back and play with it again because, there is none trivial test of what the axioms are of your understanding of these axioms, so you have to ask what will happen, and there will be a nice obstruction or whatever which you see, and, so I recommend that you play with this and see for yourself, how that works out.

I will stop here, but next time, we will look at permutation groups, this in some sense generate, all the all groups can be all finite groups, will sit will be subgroups of a of some permutation group, so in senescence that is like the permutation group is the mother of all finite groups, and it is not a single group, there are it is a family of groups but, we will see I wide that is a very plays a very important role, and so the lots of things of finite groups, can be understood by embedding it into some permutation group, so I will stop here.