Classical Field Theory
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Lecture - 5

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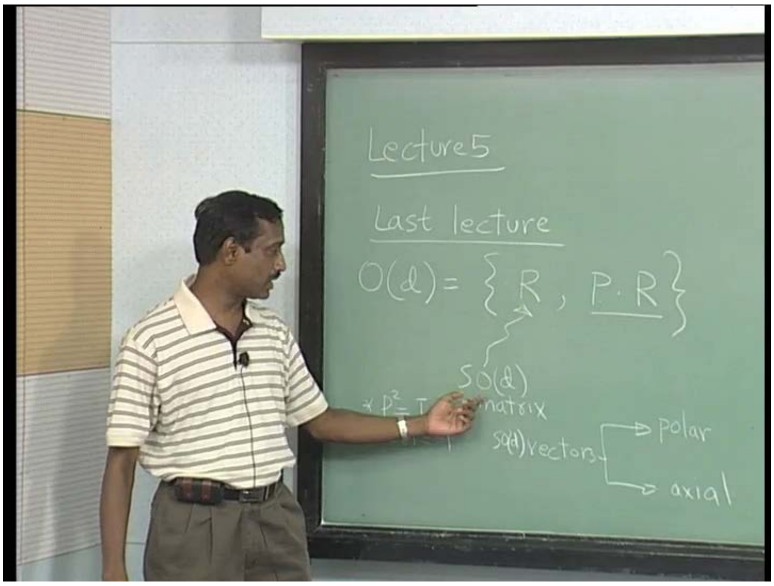

So, last lecture we saw that the orthogonal group, the set of elements in that can be written in the form of, I guess special orthogonal matrices, so let us call them R and P dot R where R was S O d matrix. So, right now you can think of this as a set of O d, O d matrices, and the of course the composition rule is the standard thing. So, it is not hard to see, that if you if you multiply two of these elements, you will end up with an element which is out here, so I will not prove those things, but this is how something, we saw last time we also say that we can specify the action under P, which is parity, P for parity which was p is something which express to identity. And of course, its orthogonal matrix and that p was minus 1, so we saw for instance if you take vectors S O d vectors, you classify them into two kinds; polar and axial depending on how they transform under p. So, quite often, we will see such discrete groups by discrete group, I mean group which has a finite number of element will be discussing it may be later in today's lecture or may be in a next lecture.

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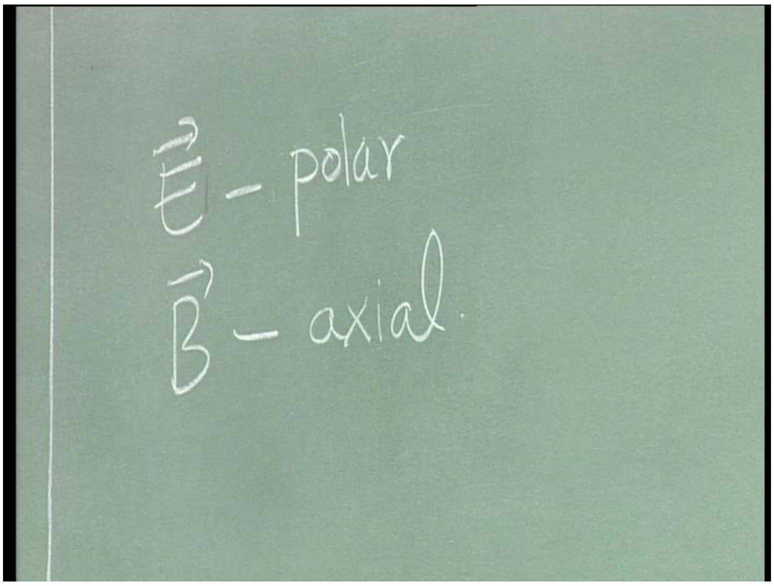

So, so what can you see is that in principle this is, this is the symmetry of the system, but we are only looking at S O d part of it, but really it is there, and hopefully you check that the elector E field is a polar vector field while B is axial, this is the consistent assessment, if you if you write out Maxwell equation, this what you get consistency. So, later we will see that when there are four vectors, there will be more transformation like p, we could include something like time diversely, how things transform under that. So, you will have to specify various things such as what I have given out here.

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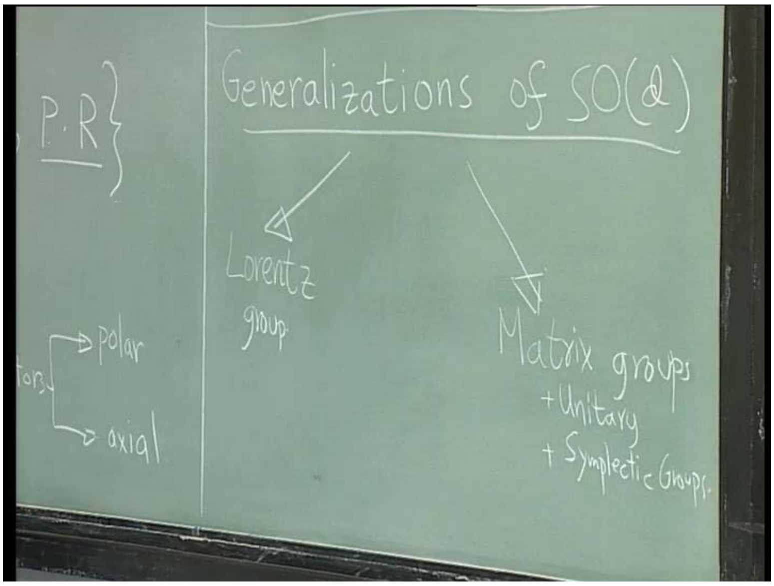

So, now, what we will do in the next lecture in this rest of the lecture is to look at two kind of generalization of O d or S O d may be, fist one will leads us to, what we will call the analog of the Lorentz group that is what we will do next, but the other generalization will lead you to what I will call just matrix group. So, in particular what I have in mind is called the unitary and symplectic. So, let us look at the generalization which leads you to the Lorentz group.

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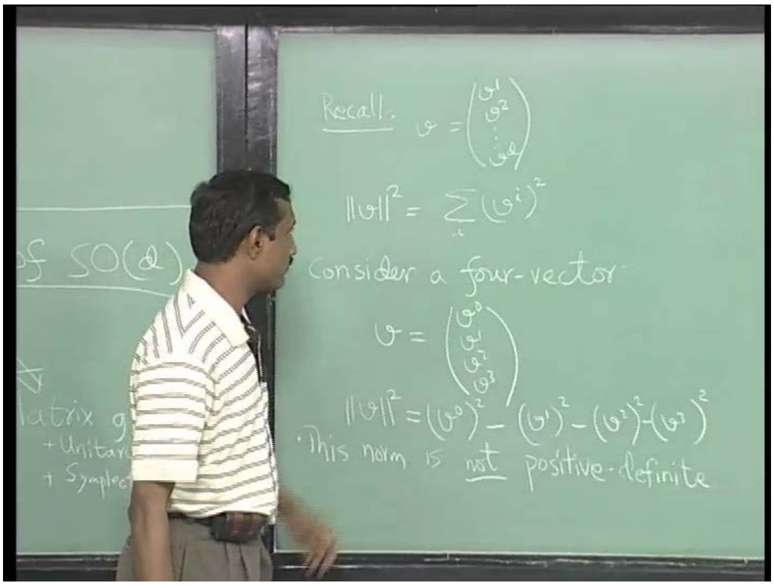

So, so recall that we defined S O d by or O d by looking at vectors, taking some vectors v 1, v 2, v d and defining it to be the set of linear transformation which preserve the length of the this vector and the length or the lets you notation v for this short form. So, mod of v square was sum over i, So what we will do, now and this norm has special properties for instance, if the only vector has 0 norm, is the 0 vector that follows, because this was sum of squares and the only way it this object can be 0 is, if every term is 0 of course, implicitly my think is these are all real number, but even they are complex right now these are real number, so what will the generalization which will gives us a Lorentz group is the following, consider so let me be very specific here and let us consider four vector, I use the same notation, but you will we already saw examples of four vector and I will define a norm which is slightly different from…

So, this is the norm which I will attach to a four vector, so you can see that here it was sum of four, so even if you take d to b four, it would look like in terms of vector, it would like I can write my vector going 1, 2, 3, 4 or 0, 1, 2, 3, but what we have out here is a different norm, which has minus sign out here. So, so you can see that already, this norm is not positive definite, in particular if you say that the norm of some vector four vector is 0, it does not follow that it is the 0 vector, 0 vector satisfies that, but you could have situation, where v 0 square is cancelled precisely cancels this this quantity, so but but form the view point of defining so.

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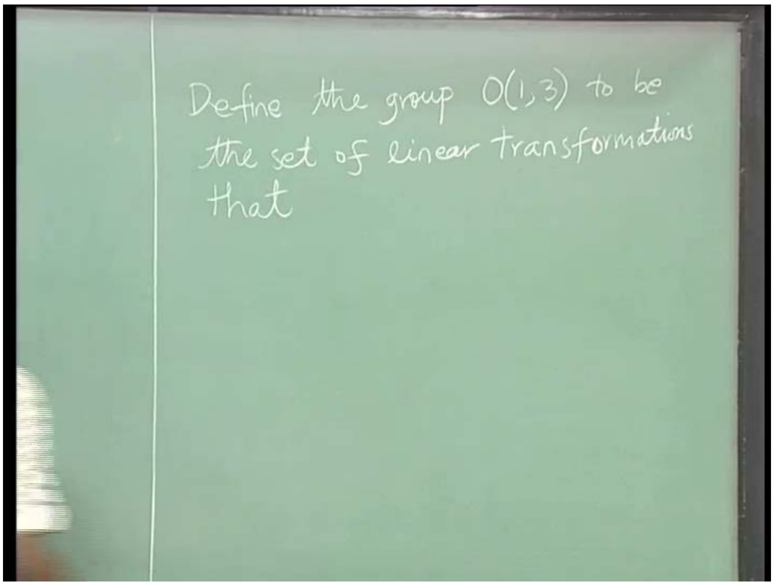

So, we will define the group O 1 comma 3 to be the set of linear transformation that.

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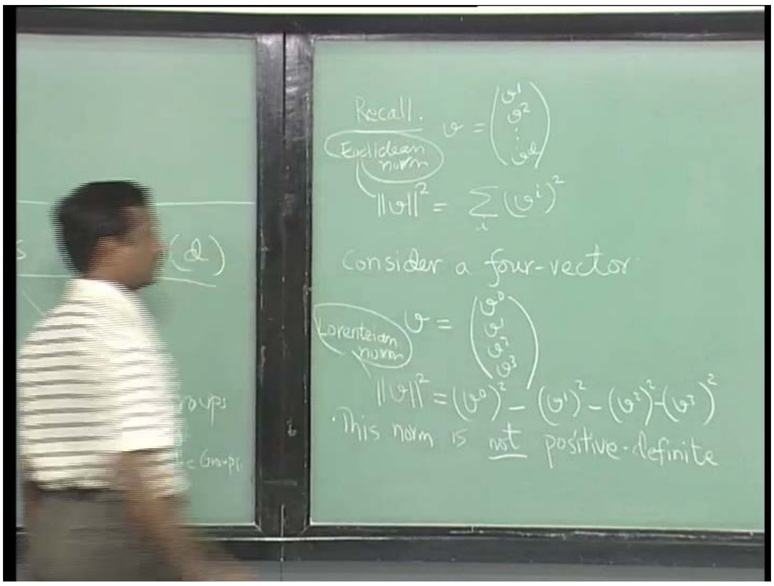

Let us give this, let us give this norm a name, let us call it a Lorentzian norm as a oppose to an Euclidean norm.

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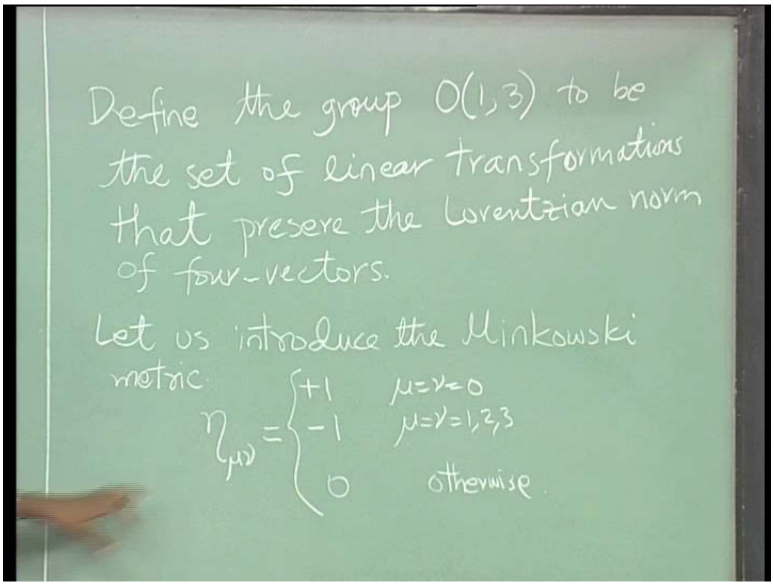

So, what we have it is a Lorentzian norm, and this 1 and 3 tells you that there are 1 pluses and 3 minus, but if little bit of thought, we will show you that actually I could have written 1 minus and 3 pluses that will really change the story. So, I can change redefine my norm with a overall minus sign, because now that we have given a positivity of norm that is no special thing about this being, but in this course we will always work with this kind of a norm and that preserve, the Lorentzian norm of four vectors and let us, let us introduce the Minkowski matrix. So, in principle like recall this Minkowski norm also, the Minkowski matrix in keeping with our earlier things, I notice that I have defined this to have two lower indices. So, this is the Greek symbol eta so its mu. So, it is a diagonal made, if you think of it as 3 by 4 by 4 matrix, it has plus 1 in the 0 0 entry, and minus 1 in the 1 1 2 2 and 3 3 entries.

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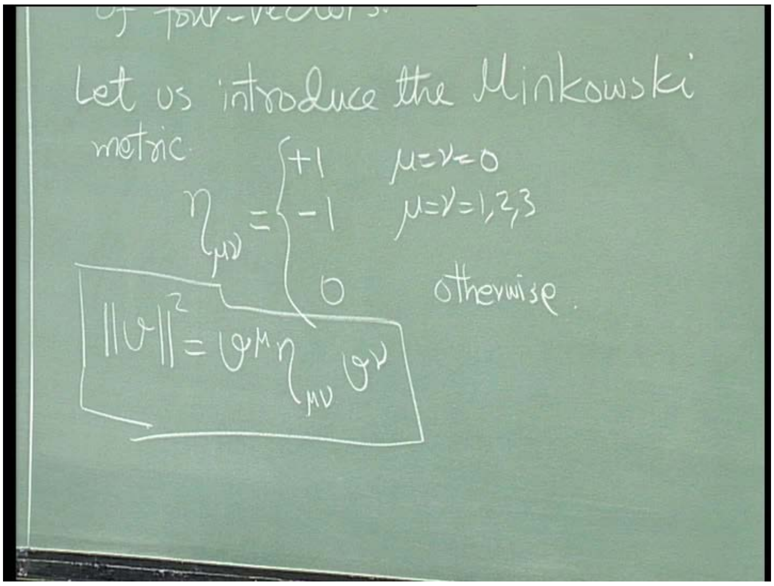

So, the advantage of this is, that I will be, I will rewrite the norm now in, in the following way, at this point it again the reason, I put this with two lower, two with two lower indices is that, I want to satisfy the the rule which I have introduced long ago which says that we can only repeated index one lower and one upper should be sum over. So, if you look here, I am very careful, I am writing out this summation out here, because I am implicitly violating that, that is why, I write this summation out here, but here I am not violating anything like this, again you can see that you can think of this entry as a column vector and this as a row vector and this multiplication again will give you exactly this, so this very concise way. So, now, let us go ahead and write out the linear transformations which preserve that matrix. So, you we will see that it is no longer or transpose or equal to identity, but it be a variation there also.

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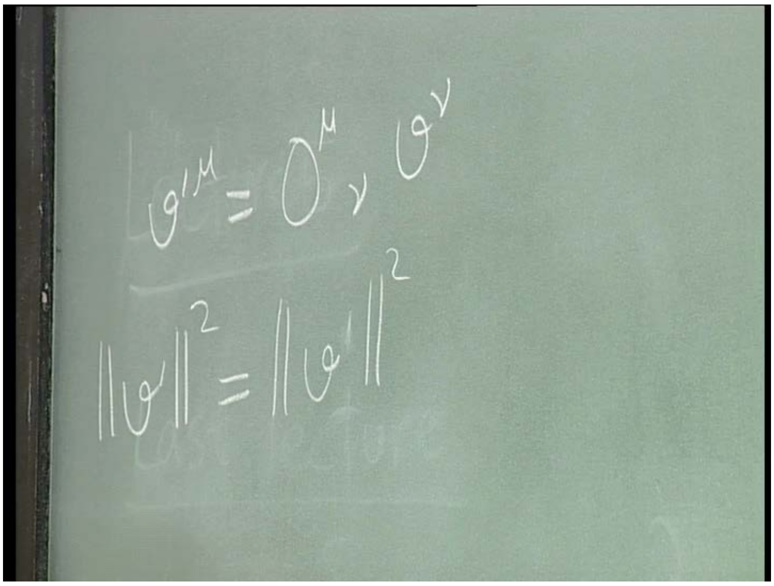

Let us consider, so some linear transformation would just tell you that it has to be of this form, and now we what we want is, so we look for linear transformation which preserve this norm, so let us go ahead.

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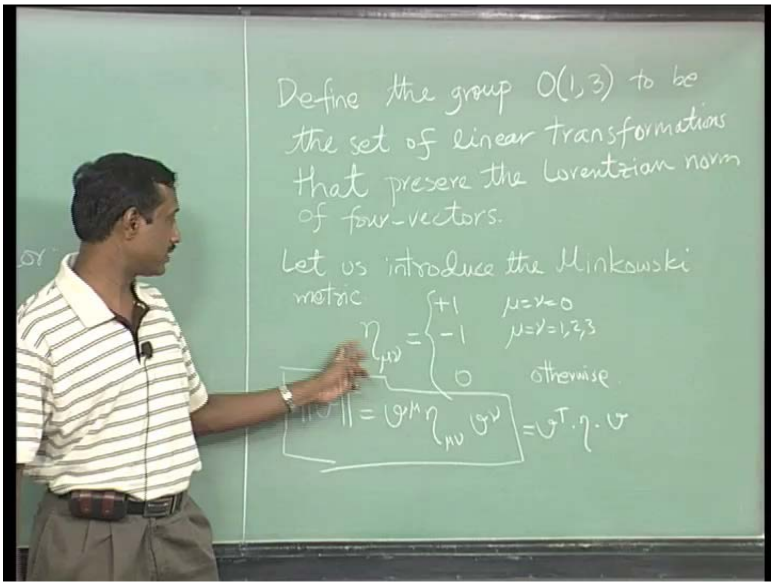

And now I will write it symbolically this thing, I can write as v transpose dot eta dot, this is same thing as this is the same as this, but I have hidden all the structure, but more or less, you know what it is because you know eta has defined right now, it is a lower index.

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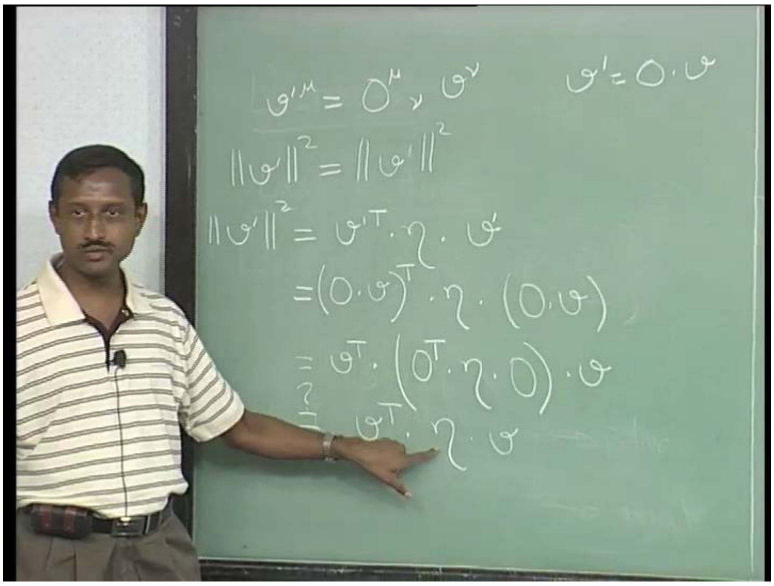

So, so let look at start form here and see what and so this again, let me write it by hiding the indexes v prime equal to O dot v. so here I need to, so this would be equal to O dot v transpose dot eta dot O dot v. So, now under transpose this things exchange, so you get O this equal to v transpose time O transpose and the question is, is this equal to v square, so let us put this question mark.

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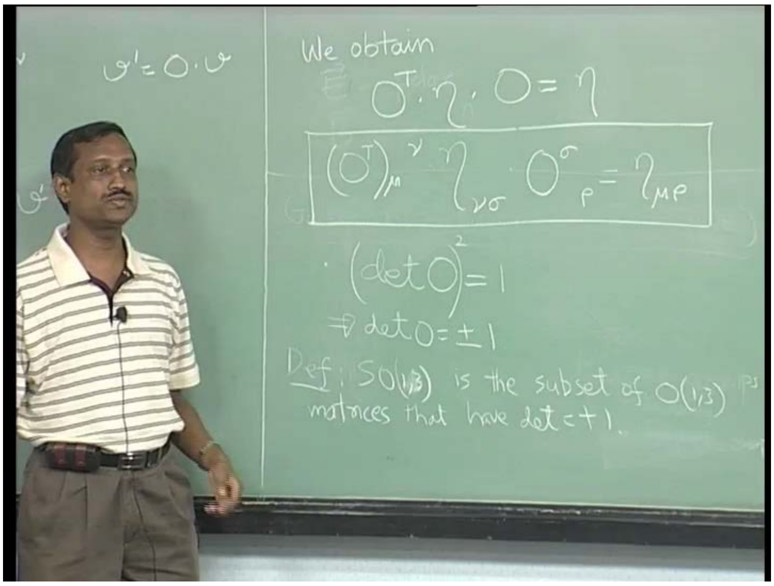

So, you can see that, so this has to hold for any v, so you can see that that can happen only if O transpose eta O is equal to eta. So, now we have that definition is made much more concrete and we see that, so the condition is, so we obtained O transpose dot eta O is equal to O equal to eta. So, now, it is a good thing to go back and put put back in this is and we should remember O is such that, O the first row in the x is an upper index while the column index is a lower index, but for O transpose this two will get rows and column get exchanged, but you do not bring this down. So, let us let us write out, so O transpose will be something which has its first index is lower and the second index is higher, upper and then it of course, two lower indices, so let us call it nu and sigma, sigma and there should be some free indices, so let us call it rho.

So, that is I am just putting in in this indices to this various things see have, I have taken care that I am not violating any of the rule which I have mention and so now, new index is repeated. So, you can it is a dummy index, you can forget it sigma index is this. So, only the free in this is mu and rho, you may worry should I write mu rho or rho mu, but coming back to the original definition that is a symmetric matrix. So, what we will do is all norms which we will look at will be symmetric we do not violet that. So, this is a definition of an orthogonal matrix.

So, now it again, it looks very complicated and but we can, we can still after few statements about this, what about determinant of O. So, like before, if you check the determinant of the left hand side, it is equal to determinant of O transpose times determinant of eta times determinant of O equal to determinant of O eta. the determinant of eta, its non degenerate its non-zero, its equal to minus 1 in in in my convention. So, that is minus 1 cancel away and you can see that again, the condition is determinant of O is equal to rather square is equal to 1 as the same as before. So, this implies.

So, obviously, we can define S O d is the sub set rather S O 1 comma 3 is, the subset of O 1 comma 3 matrices that have determinant 1. So, are there any question with regard to this? It is very easy to go back and check at the composition of two such orthogonal matrices, these are still call orthogonal, orthogonal matrices satisfies the group properties in particular product of such things will, will give you another, so go home and check the following thing take O 1 and O 2; assume both O 1, O 2 satisfy, satisfy this condition and check that O 1 dot O 2 and O 2 dot O 1 both satisfy the same condition. It is not very hard, it is very, very simple, but it is useful to check these things keeps axioms next to yourself check each one of these things and work them out.

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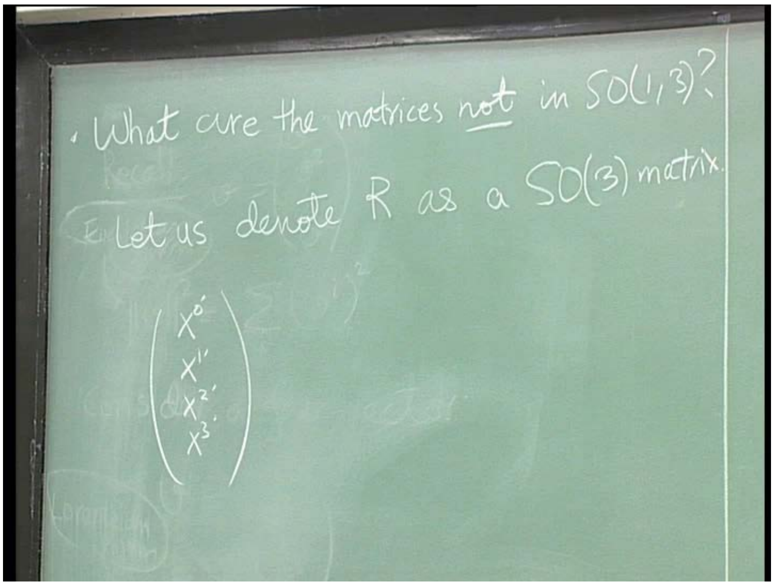

So, now the interesting thing is what are, what are the matrices not in in S O 1 comma 3, actually before we answer this question, let us ask what are the kind of matrix which would be in S O 1 comma 3. So, so let us denote as S O 3 matrix. So, now, I have a question for you, so these have these would be in this case, because they acted this would be these are linear transformation acting on a vector of length 4. So, this would be 4 by 4 matrices. So, R here is just a normal rotation, it acts on 3 vectors, now the question is can we convert this given such a matrix, can we actually rewrite as a O S O 1 comma 3 matrix? So, the question is how would one do it? Anybody, I want to write 4 by 4 matrix, so under rotation, so let us let us go back and look at what happens in a rotation. So, we look at, simpler to just work take the example of the position 4 vector and ask how it transform?

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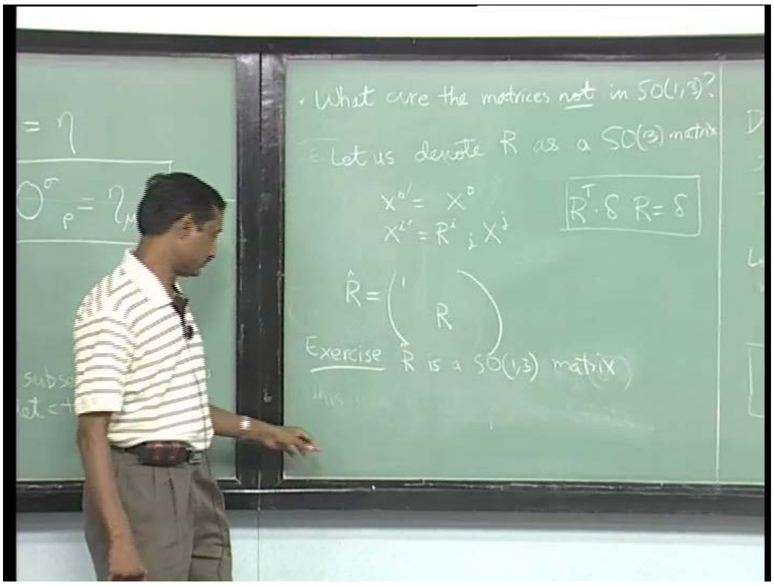

So, what we know is that, let me erase this first and under such a transformation x 0 prime goes to x 0 and x i prime goes to R i j x j, just remember there are convention that the roman in this is 1 2 3 this top there. So, now, you can see that this is 3 by 3 matrix, but we need to combine these things into a 4 vector and it is a kind of very easy to see how this happens. So, we can write may be just put a hat to indicate this, it is 1 R and there is zero's in principle there should be 3 zero's here, 3 zero's here, I do not indicate the 0 elements, this is a four by four matrix, you can go back and I I claim that this is a R hat satisfies this condition.

So, exercises you might think that you know, the problem would be because of this minus sign, but actually it would not be a problem, because it will be there will be minus sign on both sides which cancels away, so really the the basic property which you have for R which by the way you can write as R dot delta dot R equal to, so it is useful to remember that (( )) it is not a rotation, I would not call it rotation if it involves x 0, I would call it boost if there is something which involves that, so there so; obviously, this is that is why it is an S O 3 rotation matrix, but to show you that your norm may rotation matrices are indeed example of this thing, this something which we have already saw when we work out the in variance, it was by inspection there was nothing to check. So, the only question is what about Lorentz boost?

So, let us simplify things. Sir Yes (( )) delta being the matrix call equally in space and eta call in clause base, would you say that any equation in the form or matrix transpose matrix that is definition of rotation, of orthogonal matrices. So, the general statement is that let, let replace eta by any symmetric matrix and then you can define the orthogonal group to be of this form, but you can always bring eta to be if you given a symmetric matrix, it can always make, it look diagonal and you can even use some scaling and you can bring it down to simple form and all that will matter is actually is the number of things which have plus and minus sign.

So, we have I wrote 1 comma 3 which was 1 plus sign and 3 minus signs, in principle I can define something called O p comma q which is p plus sign and q minus sing. What is the restricts to interfere to certain matric yes symmetric any symmetric yeah yeah (( )) yeah that is right. So, so if you replace this by some anti symmetric matrix, you will get symplectic matrices. So, that is that was the generalization which I was just going to talk about, but I just wanted to complete the definition of what we would mean by the Lorentz group in an abstract setting. So, this is it and so but we still need to work out Lorentz boost.

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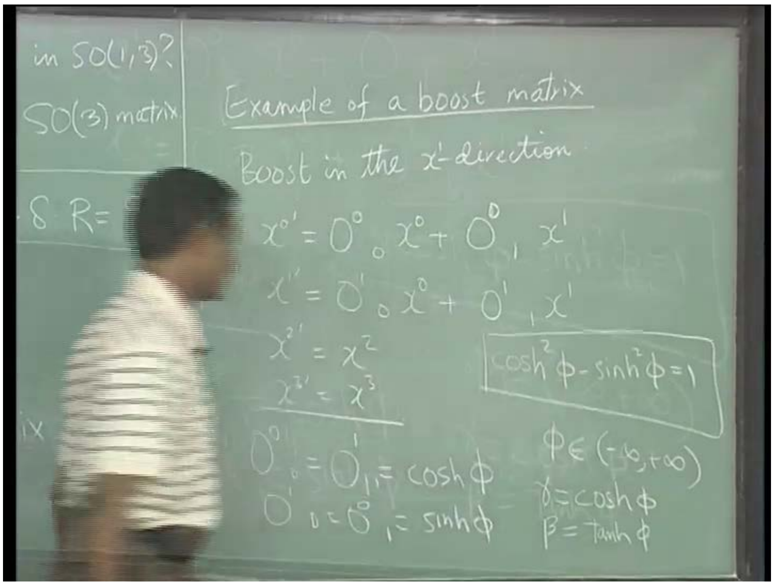

So, an example of Lorentz boost, so what I will do now is to write it in a slightly different form and so let us look at a boost. So, let say a boost in the x direction or in x 1 direction. So, so another words what we have what have in mind is x 0 prime should be equal to O 0 0 x 0 plus O 1 1 x 1 and it has nothing to x 2 and x 3, and x 1 prime is equal to O 1 0 x 0 plus O 1 1 x 1 and of course, x 2 prime equal to x 2, x 3 prime equal to x 3.

so this is the transformation which would like to plug into this and we would like to check. So, so one thing you can see here, is that this whatever you get, this should be this will be an equation which is a quadratic in O, so and its important, another important point here is that eta has a minus sign. So, from the sub space which we have there is nothing to check in this in the 2 3 space. So, it reduces the problem 2 by 2 problem which I will not work out, I will write out the solution for you what will you find is that O 0 0 equal to O 1 1 should be equal to O 1 0 equal to O 0 1 should be equal to sine hyperbolic phi. So, this is very similar to the rotation matrices except that what is happening out here is that sine, sine and cosine function get replaced by hyperbolic functions and, and the hyperopic function have crucial have a minus sign in this simple identity. So, this minus sign has its origin in the in minus sign in in out out here.

So, you can see the this the unique solution, but unlike, unlike when when, when it was a trigonometric function theta was identify with theta plus 2 pi, this has no periodicity, phi can take values, phi can take values form minus infinity to plus infinity.

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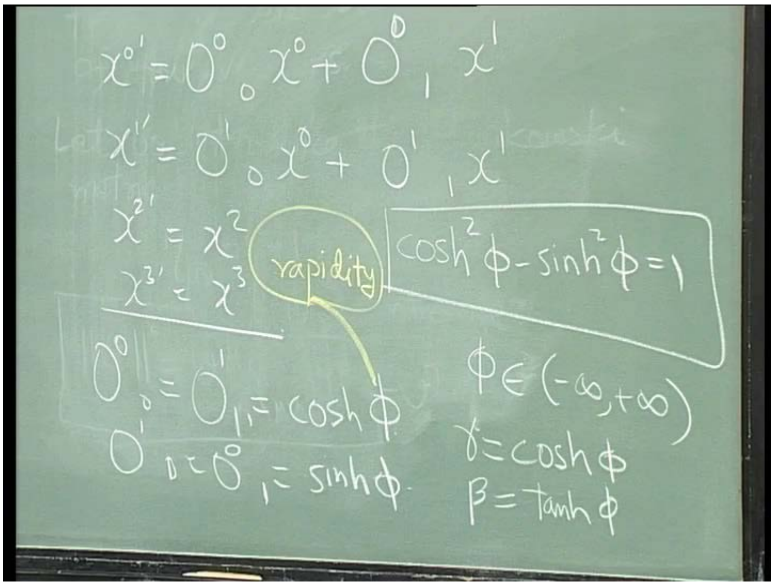

So, it is an also an exercise check that what we had written the same boost that you could rewrite things in terms of some gamma and stuff like that, you will check you can check that gamma is equal to cos phi and beta equal to tan hyperbolic phi, this quantity phi has a name it is called rapidity.

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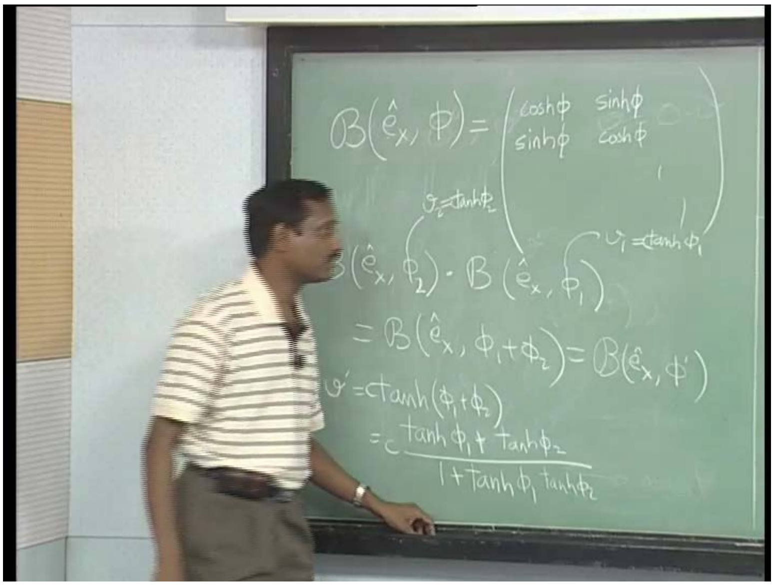

So, what I will do now is to write this particular matrix out in 4 by 4 form, so that you can see for yourself how it looks. So, let us, let me call this matrix, it is a boost in the x direction with rapidity phi, I am not writing all the 0s again. So, these are 1 1 out here, these would be 0s here. So, this is what you get this is the boost, but there is a nice, nice thing which happen if you compose two boost along the same direction, second does not it matter, you can show its equal length to a boost in, so what this tells you is that at least a life not, so band mean if you take two boost in the same direction and composite it, it works like exactly like angle added except this what is the adding is rapidity, but now comes its stuff beta was v upon c.

So, you have a addition rule which you get which is, so in the non logistic setting, we know that if you take two boost in the same direction then they add; the velocity is add, but what this is telling you is that, that this is not true, what adds is the phi 1 or phi 2, phi 1 and phi 2 which is rapidity, so the velocity, so prime. So, let say that this is related to, so v v 2 is equal to tan hyperbolic phi 2 and v 1 equal to tan hyperbolic phi 1, I should divide by c, but I will said c equal to 1, because it is I mean you can always put it back.

So, now what this tells you is that the addition rule will tell you that v prime would be tan hyperbolic after you compose them together, so let us call it v. Now; hopefully, you know your tan hyperbolic addition from let us almost the same as the tan thing with some minus sign here and there. So, what is it I do not, let me write it out, I will fixed the sign, you can always fixed it is, this correct plus 1 and minus only. Now comes the, so already you can see out here that v 2 equal to tan hyperbolic of phi 2, it you can see the following thing tan hyperbolic function is bounded from above and below, by plus one in minus one and so this is, so it tells you that v 2 equal to actually, let me put this c back.

So, you can see that that an upper bound on speeds which is c, this an algebra way of saying as an implication of various thing, and so you can also see another thing, you may think find what happens if I have two things which come with velocity c by 2, c by 2 facing each other. So, we should adopt and you should get something, you think naively it should adopt to give you c or then you can check that whatever you do out here that would not work anyway in, in principal you can see if one of them one of the tan hyperbolic becomes 1 then this should also become 1, now that s tells me that, there is a sign error, this should also be, I was using that the check thing so but this is standard and so now, you can see that if if tan phi 1 2 phi 2 become 1, this cancels away and this also becomes 1 or c now there I have put back. So, this is short of an algebra way of seeing, seeing these things, but meet comes because this is the invariance of Maxwell equation which has a scale which has a velocity, natural velocity built into it.

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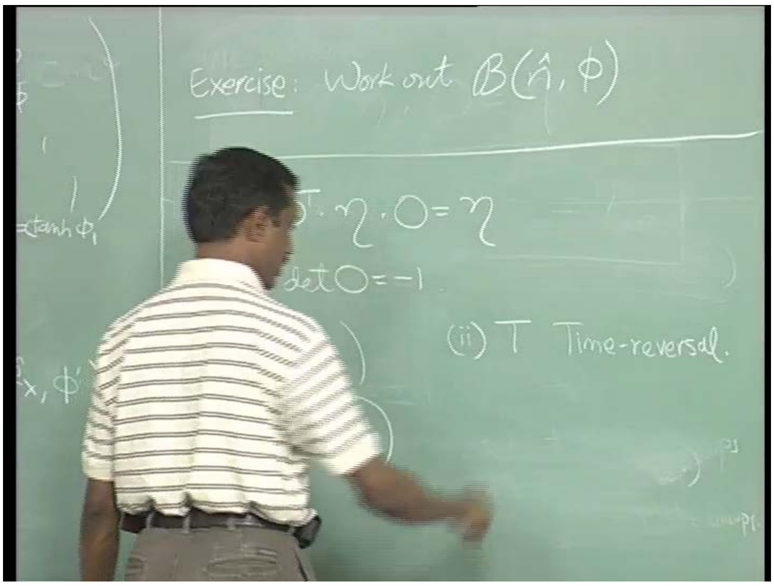

So, one of the exercise which I have given you in your next assignment is to work out a boost matrix in some arbitrary direction not x 1 not x 2 not x 3 each of these thing we know how to write, you can do that and one, one more, one more thing you can see is that the idea of a Lorentz boost, the reducing through Galilean boost is also easy to see out here, it is just a limit when the velocities are small; v over c is very small magnitude of that is small compare to 1 then we can approximate tan hyperbolic function, but its argument. So, v 2 would become form small velocities; v 2 is actually c times phi 2. So, what you can see is that, what we have go what at low velocities which we look at things what we perceive that addition of velocities actually comes you can trace its origin to addition of rapidity.

So, you have to work out exercise, this exercise it can be little bit painful, but there is actually simple way of doing. So, which I will not tell you, now I want you to swept over it and next week when we will discuss it sometime I will work it out. So, yes now, we I have more or less I have shown you that at least this group, which we started writing out has some, some, some balance of what we wrote out as Lorentz transformation Lorentz boost. So, so now, but there are still what are the matrices which are not in a S O 1 3, let us let us look at that, so we need matrices, so what we are doing is we are looking it for matrices with transform like O transpose O equal to eta, and we are already know 1, determinant minus 1 matrix what was that here parity, but that was written as a 3 by 3 matrix, so we need it to upgrade it to a 4 by 4 matrix and how would we write that. So, we had chosen let me remind you, the p 3 by 3 matrix was...

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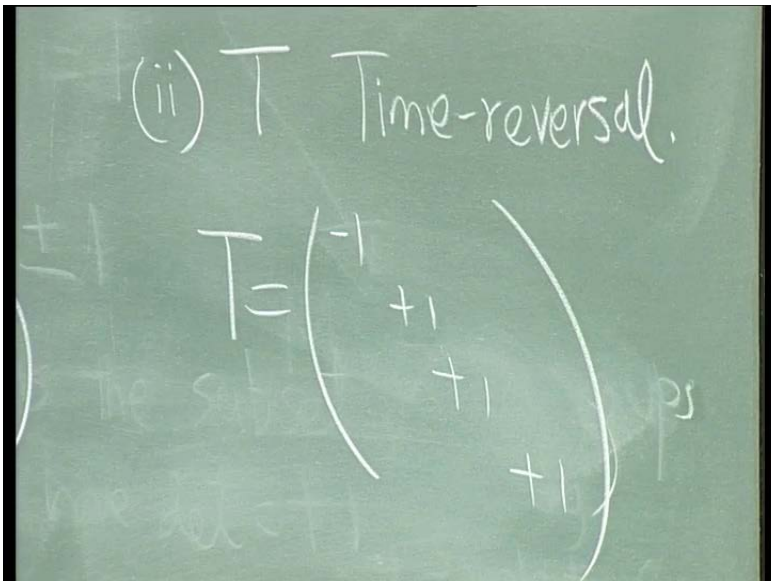

I did not choose it to be all minus 1, because I was writing formulae for arbitrary dimension, so which in even dimension, we have saw that if you invert all of them, you get something else. So, what would be p hat under parity what happens to time, you just ask that question noting, so nothing is put by putting the same way like that and; obviously, that determinant of this has minus 1. So, this is the first example, the second example is what we will call T, which is time reversal and T would be it reverse inverse, so again its determinant is minus 1. So, I leave it as exercise for you to think, if there are any more. So, now, in the next ten minutes or so what I will do is I will discuss the other generalization, so before I do that, do you have any question with regard to what I have done so far, I have given you a proper definition of what we mean by the Lorentz group and we have I have given some evidence has to how it relates to, what we did few lecture ago couple of lecture ago.

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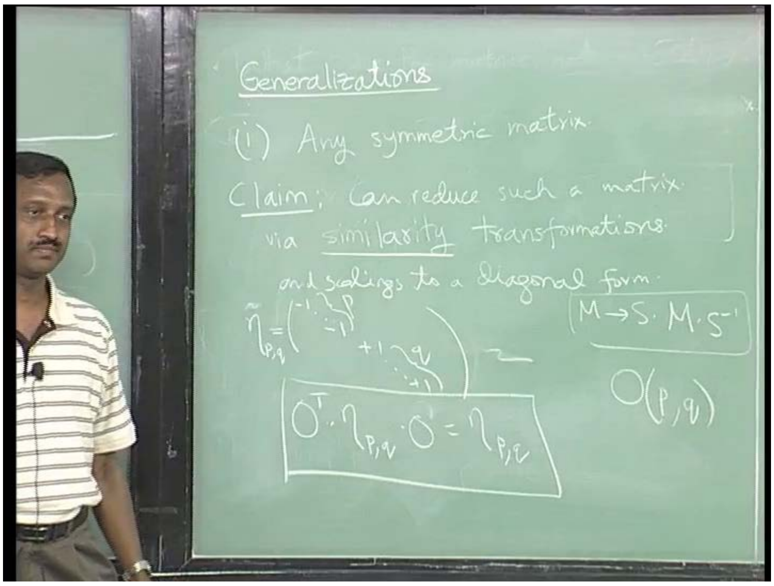

So, now the generalization already somebody asked the question so. So, so what we saw here was we replace the delta with eta, but in principle like in just replace it with any symmetric matrix real matrix and that will give you a norm, but this is just a claim which I will not, you can reduce such a matrix by, by a similarity transformation and scaling to a diagonal form, you go around and play with this, with say sort of number of minus 1. Let say p of them then plus 1 such a group we will called so any O transpose. So, let us call, let us put, let us call this also eta, but eta p q to keep this call the signature, O transpose, so this will give you the group O p comma q. So, it is just if you give me any matrix like a changes of basis, so it is a any matrix m determinant of s of course, should be non-zero. So, m goes to I will call such a thing a similarity transformation. So, if you think in terms of vectors elements of linear vector spaces, similarity to transformation is nothing, but a change of basis.

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So, so the second generalization is following, so for we have assume that v 1 to v d belong to real, but suppose each of these elements belong to complex numbers and let us have a norm which is real, so I just define with the following and define I do not put minus sign here, so define, so you go ahead and look for linear transformation which preserve this norm and you get a condition which I will just write out for you u dagger dot u equal to identity, because what we have here is v 1 star v 1. So, u dagger is u transpose star and use may not have real entries and more they can have complex entries.

So, these are called unitary matrices, so this will give you another group call u d. and I must tell you if you that is a different between the math and physic book; physics book will use dagger, but math book will use star and never ever tell you that there is a transformation, but it is there it is just a different convention, in physics we use a different symbol for complex conjugation in this thing its now. So, so it depends if you look at math book, kindly accepts star to b equal to dagger, in a physics book it is not, so this is just important notation. So, this is the second generalization and I have there some something found out here, so any complex number can be written as as sum of two real numbers with an I have put in there, so if you look at that thing this each of this term will be the sums of the squares of the real part and the imaginary part. So, you could you could rewrite this things as two d dimensional vector, but there is some extra structure out here.

So, what you find this exercise show that u d is a sub group of S O 2 d, there are elements of, so two d which are not u d, so we again will one use set theoretic notation, so what this means is that as a set, this is sub set, but it also a group, it preserve group, so whatever is inside things, so same symbol is used no need to invent new symbol, because something is a group. So, whenever I mean this is standard in math that whenever you have a set with some extra properties and you expect I mean under two similar properties when you say subset it means it preserve that property, in this case its group composition, so this is fun exercise.

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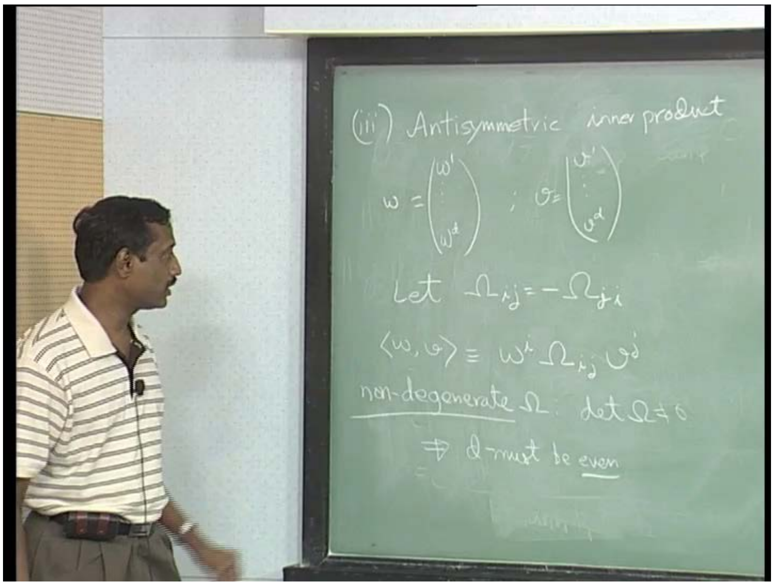

The last one is by the way that there is similar generalization like putting some minus sign certain number of minus sign etcetera. So, you can define u p comma q. Now, the third one is instead of looking it norms. So, you look at anti symmetric objects, I mean I will call it a norm. So, the key is that it is a norm, I just call it in a product suppose, I have let call this w and v 2 such guys I will an omega, let omega, it is anti symmetric then I can define some sort of inner product, if I change this two, it is anti symmetric. So, clearly if I take same two guys, I get 0 this is anti symmetric and lets us assume, this is non-degenerate; that means, determinant of omega is not equal to 0. For anti symmetric matrices, one can prove that if if you have a odd dimensional matrix, it necessarily has 1 0 Eigen value, so so this implies the d must be even. So, what I stared out with look like d could be odd, if I put it non-degeneracy condition, you can, you will get a condition and again we can look at set of linear transformation that preserve such an anti symmetric object and it will be a similar formula.

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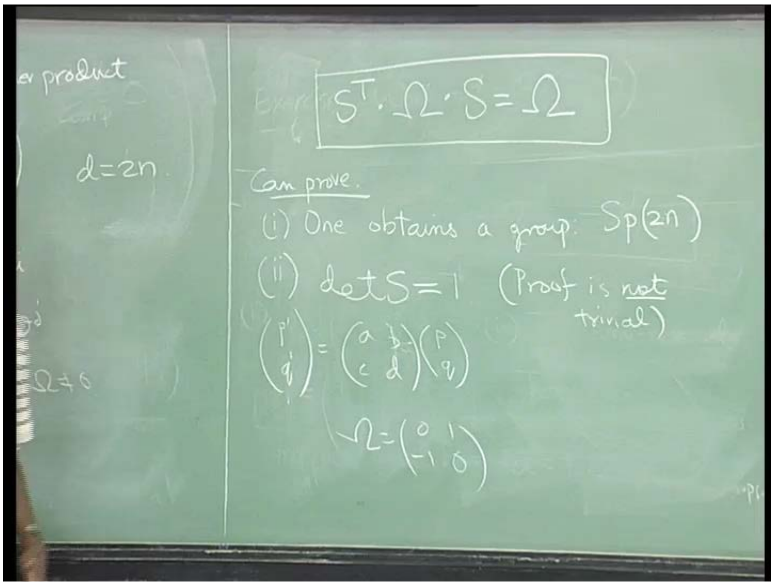

So, I will just call it, so let me use a symbol s, so the same story as before such again, you can prove the set of such s, the set of… One obtains a group. So, let us go ahead and say let makes d even, so let us write d equal to 2 n and its called the symplectic group s p 2 n, so namely again you would think, determinant of s should be, if you take a, if you go ahead and take a determinant of this expression, you would get determinant of s times determinant of omega and determinant of s equal to determinant of omega, and you would decide determinant of s whole square equal to 1. So, again you would think that the determinant of s should be plus or minus 1, it turns out that prove its not, so simple, not trivial somewhat tedious, there are different proves and it is a hard exercise you have to play with it and use, certain definition you can prove, there are this is strong enough, to actually imply that determinant of s equal to 1.

So, there is no minus 1 in this thing, so now this look like an obscure group, but actually we have all seen this things you take p q, consider the following set of transformation. So, this is when d equal to 2, you ask the condition under which possum bracket p prime q, p prime, this is a canonical transformation you can show that this is exactly this, where omega would be the matrix 0 1 minus 1 0 has to that.

So, these are actually what I call, so we have see three examples, we have the orthogonal, we have the unitary and this implicative. So, these are actually infinities families of of groups which we have seen for every, such even integer you get a symplectic group for every positive any real number, you get orthogonal and unitary matrix. So, you have a whole infinite class of groups and the need stuff is that discover all most, just about all most all of certain family of group call lee groups and we will see aspects of this during this course and, but right now I think this is all, I want to say about this matrix group, what I call matrix group and we will move on into the next lecture where we will do finite groups, because we can be little bit more explicit about certain things, and I can introduces certain concepts which you can actually work out for yourself, so that we do in the next lecture.