Classical Field Theory
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Lecture - 37

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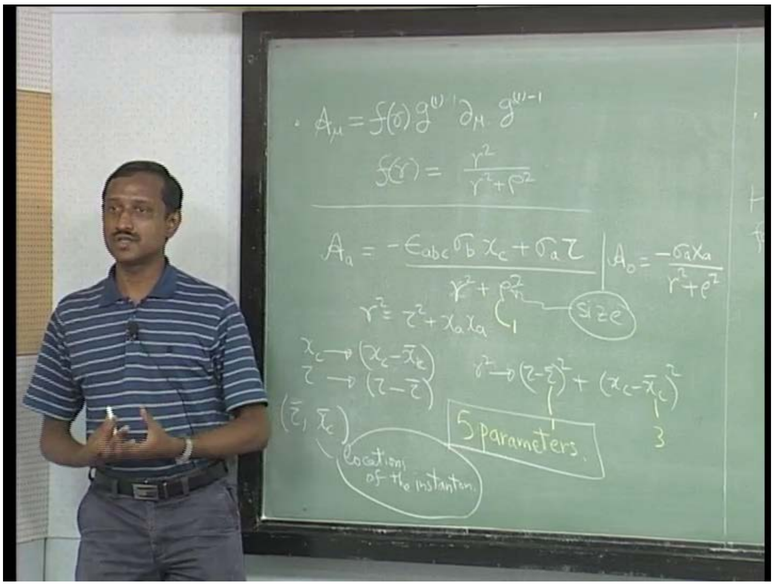

Multi instant on solution. And so, yesterday we saw that we could write an anzat or whatever for this thing by starting with something which g inverse g mu g other way round, and introducing some function only of the radial coordinate in front of this. I did not prove anything, but I said that the following thing is the solution and that was r square by r square plus this thing. And we were doing…

So, this follows from the Euler Lagrange equation and this is something which will be there in your assignments for you to check. But we can look at some, a look at this solution in some detail, and so, so if you look at what this looks like? So, where r square was just tau square plus x a x a. So, the point here is that we could always do the thing following thing. So, this thing can be generalized a bit is to just say wherever we saw x c, we replace it with x c minus some x naught or some x bar c and tau by tau minus some tau bar. And if you and of course, r square will also get shifted appropriately so r square will also become tau minus tau bar whole square plus a.

So, once you do this, I mean its not too hard to see that this will continue to satisfy the equations of motion and these things now there are three parameters. So, rather 5 parameters. So, we start. So, this is one this is three and then just phi which is one. So, we have 5 parameters. So, I did a quick count for what should be for a 2 instant on solution by intuitively saying that fine. So, here the interpretation for x x bar c and tau bar or tau bar. Let me write it in the other way tau bar and x bar c are these are location of the instanton and this is the size.

So, this is how we understand the 5 parameters I should also give you a 0. So, you will be verifying all these things in your assignment. So, this should follow more or less from this kind of an answer sorry and yesterday we missed out a minor factor.

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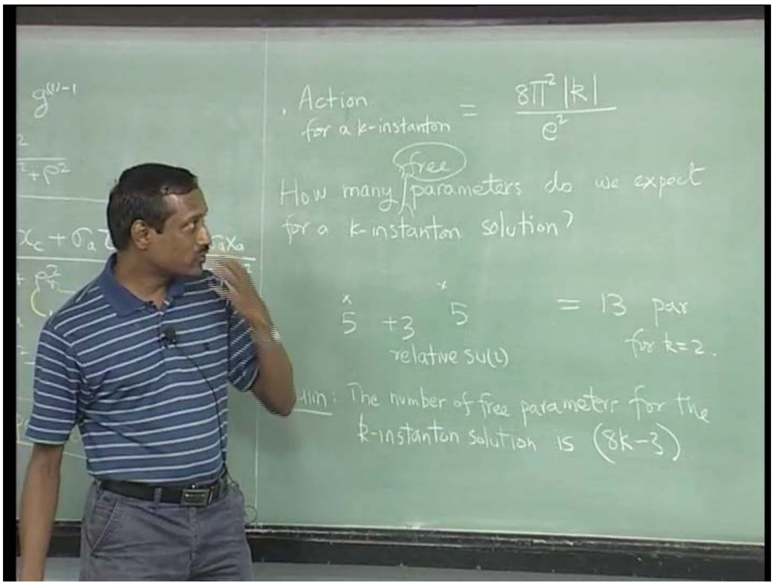

When we worked out we worked out the energy or the action for a k instanton. That means, by this I mean a instanton a number is k or the binding number is k. We got we worked it out to be 8 phi square times mod k. But actually in the action there should have been the coupling constants e square or g square. Whatever we want to call it this is just a overall multiplicative factor. So, it does not change any of our discussion, but this is what it should be, but a. The question is how many parameters do we expect for a k instanton solution.

So, roughly here we had an interpretation this breaks translation in space and time and there is also the scaling symmetry which is broken there are other symmetries which might lead to gauge transformation. So, they are not a they are they are they are still a symmetry of the solution even though it might change the form. You can show its gauge transform of the same thing. So, the idea here; let us do the counting for two instantons. So, I said lets think of two guys separated widely and then the thing is then we would have 5 parameters for this guy. 5 for this, but the key point is that a out here there is always a global s u 2 transformation if you want to keep that. So, you can count a.

So, I can go ahead and act on this by an s u 2 transformation and I will get something else. So, that is three more parameters this is a Euclidean this thing. So, a you talk of s o 4 rotations s o 4 rotations is not really broken it is a gauge transformation. So, there are no more I mean there are no more parameters out here. So, the point here is that. So, you would have thought.

So, this thing and you except the point I made is yesterday I said there are three more parameters because there is a relative gauge transformation relative s u 2. So, if you take this the this had an s u 2 global thing which I could have done similarly I could do with this, but then the overall thing can be removed. So, plus 3 for the relative s u 2 freedom. So, this num this counting will give you 13 parameters for k equal to 2.

So, now, you can see that what we have in other way of understanding this as sixteen minus three its simpler to associate instead of saying 5 parameters with a one instanton solution you say that there are 8 including the you treat the s u 2 as extra freedom and overall we subtract ten e. So, you can see that that is kind of fruitful because when we talk of k instantons for every instanton we will count 8 and then you subtract at 3.

So, the claim based on this thing is that the number of parameters by the way these are called sometimes they are called moduli the reason they are called Moduli is because you can change these things without changing. In this case the action in the case of solitons you can think of them as they are not changing the a the energy.

So, the number of free parameters for the k instanon solution is 8 k minus 3. So, now, the thing is that why did I write it as a claim is there a way of proving it the answer is very nice the answer is yes there is a method of proving it. So, what you do is you take the if you have some continuous parameter. Obviously, you can if you choose a solution that corresponds.

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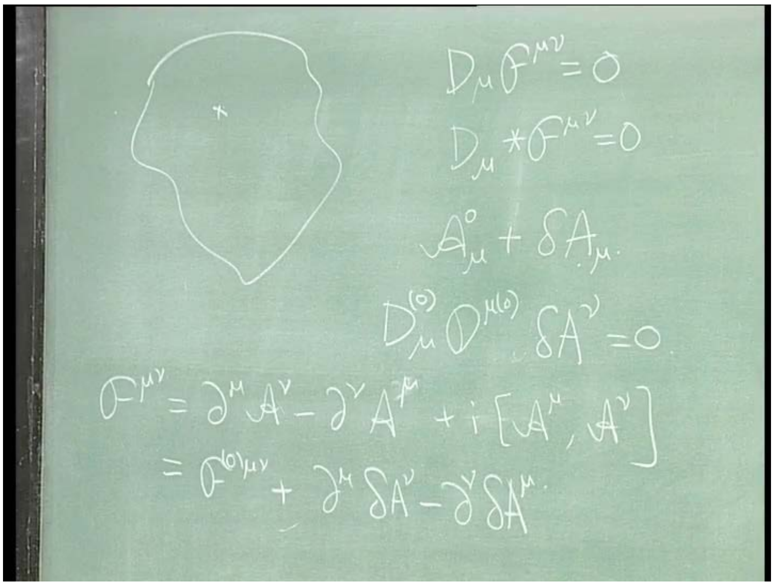

So, let us think of this is your parameter space and you phick your solution at some this thing. So, for instance here I could have said that I have chosen row to be some number and the location to be the origin. So, it does not look that those parameters are visible and how did I here? I was able because I had some geometric insight. I was able to bring these things back into the problem, but that is because this is a one instanton solution, but out here the question is it easy to see these three parameters we do not know.

So, what you want to do is to. So, you if you are given a solution you want to ask how many directions flat directions are there. By that I mean flat in the sense the things which do not change the action. So, what you do is you ask for what are called 0 modes of the equation.

So, you look at the equation of the motion in this case it is just you could write the equation as D mu f mu nu equal to 0 and the Bianchi identity is just D mu star f mu nu equal to 0. So, these are the Euler Lagrange equations and the Bianchi identity. So, what you do is you take. So, you start. So, you are given a solution. So, let us say we are given a solution A mu 0. To this you add and a small perturbation. So, you say that. So, in other words what I have in mind is suppose we did the first.

So, here we start with let us focus for instance on the location. So, what we have out here is that initially we set the location at the origin and I want to shift it, but I want to shift it by a very small amount. So, then I can think of the position x bar c as small and expand it as a power series in x bar of c and keep only the first order phice and that phice whatever is the coefficient of that I can that delta bar a delta of A mu. And so, that should continue to be a solution of this equation to its to all orders, but to count for just getting the number counting its sufficient to go ahead and plug things in here and you can see that a.

And so, we just need to substitute this here and keep terms to order delta mu nu. So, a naught already satisfies the equation. So, the first order phice will look very easy. So, this will just satisfy something like D 0 of mu. So, you take times D mu naught of delta A mu I think something like this. So, what I have in mind is suppose let us look at what I am doing here as F mu nu would have been D mu.

So, I have a whole bunch of terms. So, let me write out this term plus there is the other the quadratic phice, but just look at this. So, if you are going to ask how does this look you can see that to leading order. So, this will just become F naught mu nu plus this phice will give d mu of delta A nu minus D nu of delta A mu now what about the other phice the other phice would have been something like two a's. So, it will be a mu A nu kind of phice maybe there is an I this is the phice.

So, here now you can see that if I want two linear order the linear order here would just become replacing a a one of these with a naught and the other with this thing delta a. So, all that will add up and give you some sort of term which will look like this. So, you have to find solutions to this kind of an equation. So, this looks like some adjoined valued scalar kind of thing with some equation which is coming from the given Euler Lagrange equation. So, you want to solve it and you ask how many independent it becomes a linear equation.

So, you can ask how many independent solutions it has and that is the way you go about and you can you can convert this into a mathematical problem. If this is a nice well posed problem you can you can actually hmm you can do the counting in some cases. It gets mapped to what are called index theorems and you can do the counting. And then you can prove that this is correct. I am just giving you an idea how you go about proving this because this is just hand waving. I could have missed something, but it is like at least you can go back to do the fancy math and then come back and see if you can physically understand it and this is a nice picture. So, but now the question is how do we go about writing a solution which has all these 8 k minus 3 parameters?

That is not something which we will do, but what we will do at least is to write a slightly different simpler solution where we will get instead of 8 n we will get 5 k instead of 8 k. We will get 5 k kind of this thing minus 3 etc etc is not. So, is not a big deal. So, let us. So, for that I will re write this particular thing in a slightly form.

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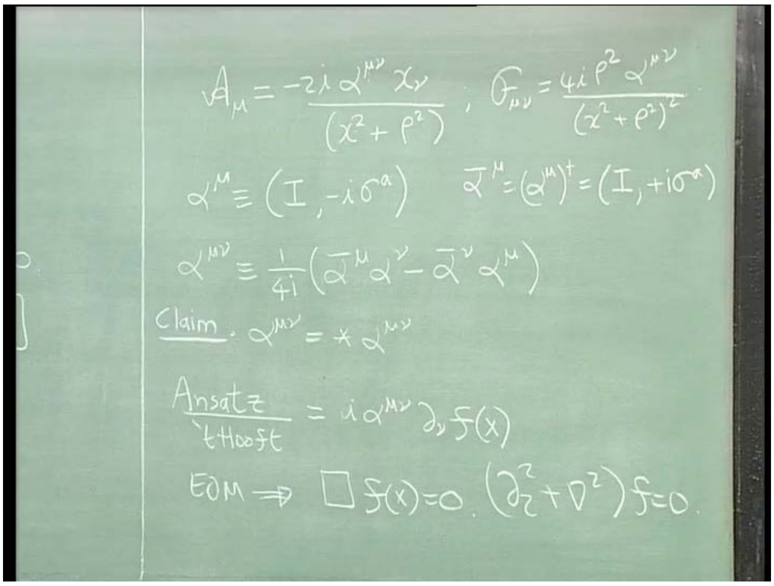

So, I will define something called alpha mu nu. So, x square is what I called r square there maybe I should call it r square no. I will call it x square you will see why because I want to shift in things like that. Then it x thinking of it as x square is sensible and of course, mu nu goes from 0 1 2 3 4 1 2 3 and f mu nu for this looks like 4 i row square. So, alpha mu is just identity alpha 0 is identity and we have seen this combination quite often. For instance when we wrote x a we combined it with the Pauli and the identity matrices to write the most generative two element. So, you can see that this is has that thing and define alpha bar to be alpha mu dagger. So, that will be identity dagger is this, but just change the sign here. So, define alpha mu nu.

So, the claim here is that alpha mu nu is equal to star alpha mu nu. So, it is self dual. So, a it does not take a lot of work to just plug these things in and see that you get something which looks exactly like this. I will not swear that the signs are exactly the same or whatever, but this is what you get. So, so the idea here is to. So, the anzats I guess this is pettuce anzats for multi instanton. If you look at this thing and say lets do the following lets choose a mu to be i alpha mu nu times d nu of some function. Some function of x. So, this is the anzats and then you can ask what is the equation of motion for f?

And that is very nice the equation of motions imply box f box f of x equal to 0. This is a 4 dimensional box. So, by that I mean this is kind of very nice it is very interesting also you started out your actual equations of motion are non-linear, but what is coming out of this is this is after this anzat is that you are getting something linear. It tells you that super position works. And so, now, you can see that I can go ahead and just sort of roughly add one instanton solutions and here is what one does and.

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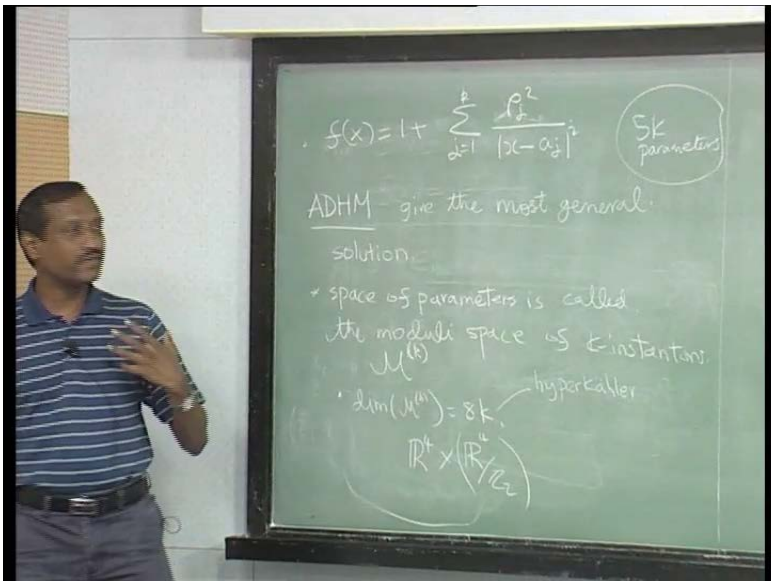

So, you can we can choose f of x to be say 1 plus this 1 x minus. So, straight away you can see that there are these 5. There are 5 parameters for out here one for the size location four coming from the location and this is the size. So, you get 5 k's. So, this has at least 5 k parameters. So, the nice thing is that it does not capture all these things, but it at least captures some of the I mean and you can see what are the missing ones in some sense it is a relative gauge stuff. But you can also see that it is not. So, simple to it is clear that this kind of anzats will not. I mean there's not much tinkering you could do with this to get the other the other ones the relative ones it is not.

So, simple no there is some underlying integrable structure in the s l mills theory equations. So, it has related to that I wouldn't say that I mean that is why its much deeper than that so, but it is quite of its kind of nice that there is a simple way of understanding the solution and you should know that we are looking at the solution.

So, this is lap lasses equation in 4 dimensions you know that the its 1 upon R in its 1 upon R the green function is 1 upon R in 3 it is 1 upon R square. So obviously, this thing has singularities. You can see if I look at 1 by its some by R square. So, it will have singularities at those locations of the instantons, but beyond that it is it satisfies that equation. So, this is a solution due to a t Hooft and the more general thing I was thinking about it I feel that it’s little bit complicated, but when I talk.

About dualities there is a nice geometric way of understanding the ADHM construction. So, the ADHM construction gives the full 8 k minus three parameters. So, this is just a remark. So, all these things you will be working out in your next assignment. So, you that know how these. The algebra works out give the most general a solution. And so, this. So, coming back to this there is this funny space which is 8 k minus 3 dimensional space.

So, in some sense let us forget about the overall three and think of it a s 8 k dimensional space. And so, this space is called the such a the space of parameters is called the moduli space of in this case k instantons. So, we can just write something like this M k and remember now I have gotten rid of that minus 3. So, this dimension of the space M k equal to 8 k this space turns out to have beautiful properties. Again it follows nicely from this construction it is a special space it is called hyper Kahler.

So, a Kahler manifold is some first thing is you can go. So, this is counting in reals. So, real manifold you can go to you can put some you can work in complex coordinates there are some consistency conditions. So, a space from goes from real being a real space to a complex space, but Kahler is very special it is a it is a space which is both complex Hamesian in some and it is also has a simplistic structure.

So, it can also be like a free space hyper Kahler is has much more structure. So, this is a very very special kind of this thing the hyper Kahler manifolds necessarily have dimension which is multiples of 4 just as complex manifolds necessarily have dimensions which are multiples of two. Obviously, example of a non complex manifold is r. Nothing to complexify. Even R 2 yes you can make it into one complex R 3 again you cannot. So, that way hyper Kahler spaces are. So, again there is one more reason to this thing, but this. So, the 8 dimensional one is actually very easy to understand. So, 8 dimension what were the parameters there was an R 4 and then we s u 2 is just an s 3 and the size is like a radial direction. So, you can see that the s 3 and the radial direction can combine.

And become a an R 4 with a with if you wish the. So, it becomes actually a second R 4, but actually there is some there is little bit of discrete subtlety that space becomes R 4 with a z 2. The center of s u 2 which is a z 2 does acts trivially on this solution. So, the moduli space is. So, it is do not worry about this is a technicality, but all I am saying is that the moduli space is 8 dimensional. It looks like two cophis of this thing. This z 2 means that there is a identification. Beyond this I will not.

So, the. In fact, there is should I say more I should not I will just stop here. By the way I got the number of pages wrong. In ADHM paper I said one and a half its actually two with a bit on the other side references coming on the other side. So, it was two pages which all these things were. So, we sort of motivated instantons by looking at tunneling in quantum mechanics and so we looked at the double well example.

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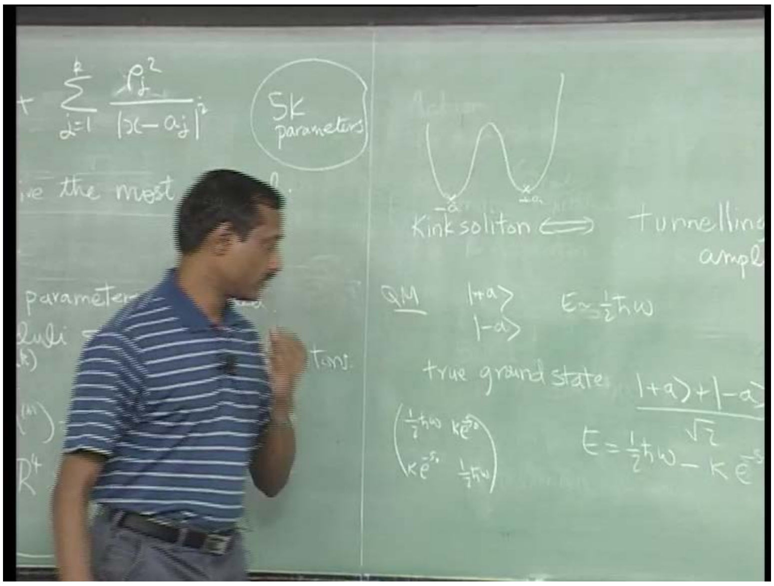

And that is where we started and we saw that the kink the kink soliton was related to gives the tunneling amplitude if you wish. So, there was one at minus a and one at plus a there were two vacua, but of course, we know quantum mechanically because of tunneling the true ground state is neither is neither of these two, but some linear combination. And so, just let us call the. So, in quantum mechanics let us just call plus and minus as the two solutions no leave it should I call this no no I will just call it plus a and minus a. So, this is a solution which is localized at plus a and this thing the other one is localized at minus a and we know that the true ground state. So, classically this is this is sort of capturing what we would have done. Classically it has two ground states 1 is x l when it is at x equal to minus a.

So, I am just saying let us think of some harmonic approximation and you can think of this as a Gaussian centered here as an approximation and this as a Gaussian centered. As I told you the other way round this is centered here this one is centered here, but the true ground state is neither is the even guy. And so, let us say the harmonic approximation. Let us say that the energy would be some half h cross omega where omega is determined by a the curvature of the potential. Out here the second derivative that would be the harmonic approximation, but the true ground state would have would be a linear combination.

So, it would be something in this case because of symmetry it will just be this a good guess would be something like this and the energy would be a minus something and this is due to tunneling. So, one way of saying this is that suppose. So, the energy would be would be something like half h cross omega and the off diagonal terms would correspond to the tunneling contribution.

So, and because of symmetry this will be some number and this will be actually determined by the action. So, it its typically of the form of some constant times e power s naught and this is also the same constant e power s not and. So, this thing will be. So, there will be two this things. So, e plus minus which will be minus or plus. Let us stick to the ground state some constants times e power minus s naught where s naught is the action evaluated for this guy for the king soliton for instance that would be the semi classical estimate for this thing, but suppose we have several infinite suppose we have a periodic potential.

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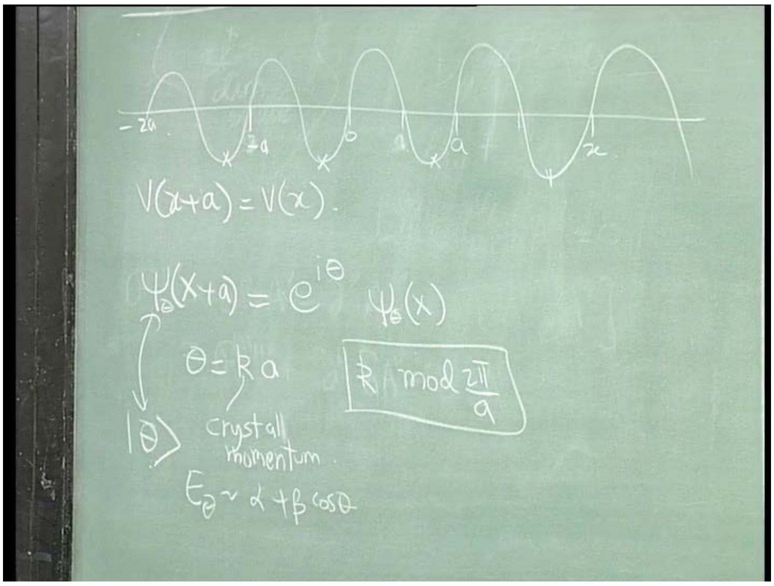

So, then the number of ground states are infinite and so; obviously, you can tunnel from from one ground state to any one of the this thing. So, there will be many different kind of instantons here there are only two kinds of instantons instanton and anti instanton. But here which would be off from one and taking you from minus a to plus a if you have a periodic potential say of a sign for instance and let us say that this is 0. A one full thing should be an a 2 a minus a so on so forth. So, as I have lattice spacing.

So, my potential v of x plus a in one dimension is equal to v of x. So, in quantum mechanics you must have heard of blocks theorem which we can view again from a different viewpoint this is system where the potential breaks translation symmetry right. So, momentum is not conserved, but it is broken down to this discreet group. And so, momentum is conserved modulo this thing. So, blocks theorem tells you that you could write the way function psi of x plus a will not go to itself, but it will go to something up to some phase times psi of x. That is your symmetry in and this theta is usually written theta is usually identified with something called k times a and k is called the crystal momentum, but k is only defined modulo 2 phi over a. But the key point here is that there is a whole. So, this is a characteristic of the wave function. So, you can say that we can say that it carries psi of theta this thing.

So, theta going from 0 to two phi over a or minus phi by a to plus phi by a which is called the first bill one zone. So, k will take values within that then you would not have this mod 2 phi over a is taken care of. And but the point is that you started out with a discreet set of ground states. So, what this would tell you is that rather there is a theta set of vaccua in some sense. So, I just identified this wave function with some theta it is a continuum out here we just got two two vacua we got.

And it was just plus or minus and here the energy of this thing goes like some alpha plus some beta of some theta of course, n theta. I do not know something like this depending on the details of the potential etcetera etcetera. So, now, let us come back to our instanton solutions. So, situation is exactly like this we have infinite number of classical solutions all which have 0 value and how is that?

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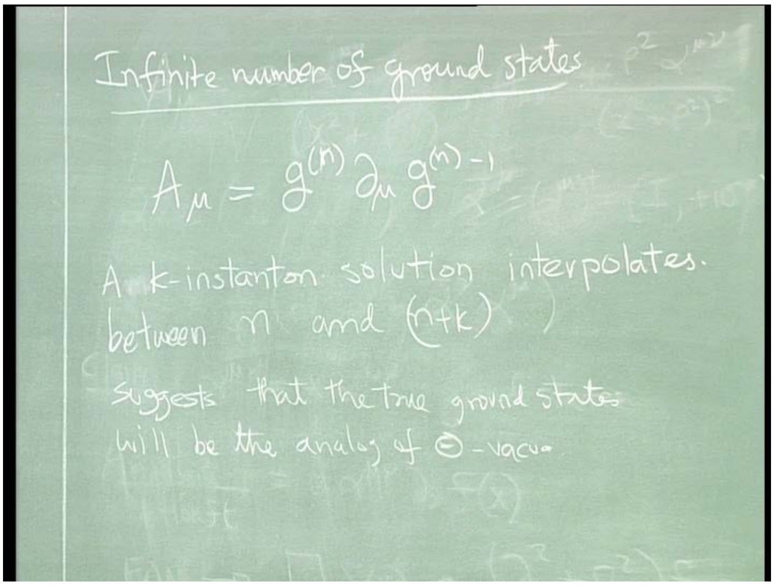

So just take a mu to be pure gauge, but this binding number n like we did last time each one of these things will give its pure gauge. So, f is 0 and the action is 0 e b everything is 0. So, these are these are really. So, so the point here is that pure gauge solutions are indeed ground states true ground states of the theory and there are infinite of them. And so, instantons which we have constructed you can think of them as. So, an instanton solution k k instanton solution interpolates between ground states n and n plus k right it need not be 0 to k it can change it should it should only change.

So, its exactly like this I could two instanton solution can take you from this vacuum to here, but it could also do this it could take you from this to here. So, in that sense it parallels exactly this thing and it tells you that you have theta vacua. So, it suggests that the true ground states or the through true states if you wish states or whatever yeah true ground states will be the analog of the theta what we will call theta vacua something like this. So, is this clear now a related fact actually related to this is a fact that if you go back to the action.

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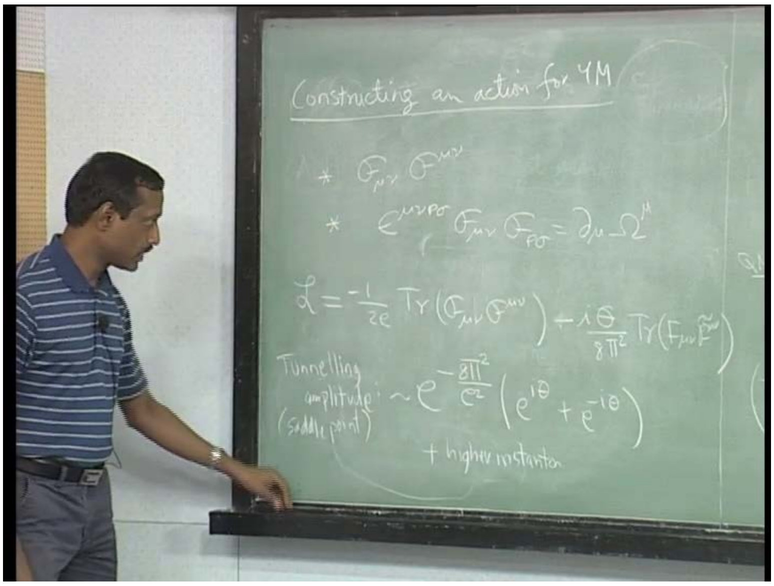

So, if you were looking for constructing an action for young mills in your assignment, I asked you to write out the most general two derivative terms involving the gauge field. So, the key point is that there are exactly two kinds of terms one is a normal term which we have written, but we could have written something like this and what did we say this. So, the argument will be that this is a total derivative and. So, it does not. So, usually it does not change the equations of motion it does not affect the equations of motion. So, you can add such a phice it will not change the equations of motion.

So, this is a total derivative we know this is equal to some d mu of omega mu it is a total derivative, but the point is. So, we could try to write an action which has such a term and let’s say we did something like this. So, we choose the Lagrangian the Lagrangian will be 1 upon 2 e square plus i theta times the instanton number I will put that. So, I will just define things theta by 16 phi square right theres a half in the definition of is that correct. So, I will write this you could write as I put an I out here.

So, this is the kind of action you could put in and you can ask if you have an instanton background. How would the action change and what you will see is that suppose. So, you ask for a transition amplitude in the presence. Without this term it would have been just the instanton action. But. So, so for this we have to. So, tunneling or transition tunneling amplitude if you want to work out the tunneling amplitude would have been just s minus instanton 8 phi square times the instanton number upon e square. This is the contribution; that we would have got by from the one instanton. This thing and the higher instanton terms will have further suppression.

So, this is the one instanton contribution, but there are two such contributions if you add such a term one of them will give e power i theta and the other will give plus terms which are suppressed. So, higher instanton terms, so this is the saddle point. So, in general I should point out that when you have when you have when you are doing this saddle point evaluation you have to sum over all saddle points and you should remember that every multiple instanton solutions also give you other saddle points. They are as good solutions to the equations of motion. So, in principle you have to sum over all of them. So, here I am doing the Euclidian thing. So, you can see that the two instanton guy will come with a two out here which is exponentially suppressed relative to this. So, I am writing out the most dominant term and this. So, this will give you like a co sine theta which will be a which is a contribution from this thing.

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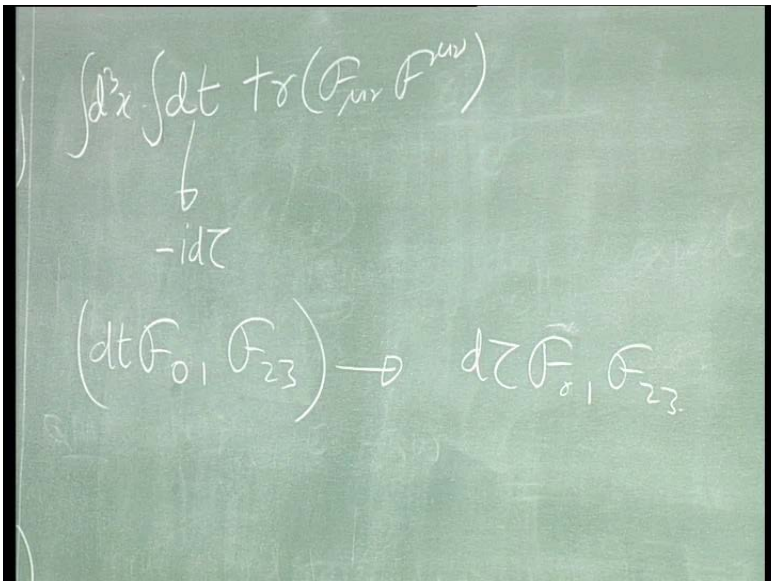

This term this term is actually interesting this factor of, I does not go away when you go to Euclidian space the reason is the following is that. So, how would you? What would happen is that you would have some d t and. So, lets look at. So, there will be a d cubed x first let us do the normal term f mu nu f mu nu. So, what happened last time when we did the weak rotation out here we saw that this was nice and this took the it was e square minus b square and the. So, what happened is that the b term flipped sign, but the I which came from here remained. So, d t went to minus I d tau, but if you look here a typical term the term here would be f 0 1 f 2 3 this kind of term. So, there will be an I coming from this f 0 which will remove get rid of this i. So, such a term.

So, what I am trying to say here is d t of this thing will just go to d tau of the modified f 0 1 f 2 3 because the I coming from this gets exactly cancelled this has a d b yd t if you remember. So, this goes away. So, it just goes it remains the same. So, this factor, so this with when I put it in Euclidian space. It just gives you this kind of a phase, but when you go back even to minkowksksi this remains this factor of I does not go away. So, there is a pure imaginary it gives you a phase. So, in some sense this captures this kind of phase.

So, the so the upshot of this is that most general vacuum Lagrangian you would write would be something like this, but like I mentioned this will not change the equations of motion it is a total derivative. So, equations of motion will be exactly whatever we wrote earlier and, but there is also a nice complexification it causes. So, 1 upon 2 e square.

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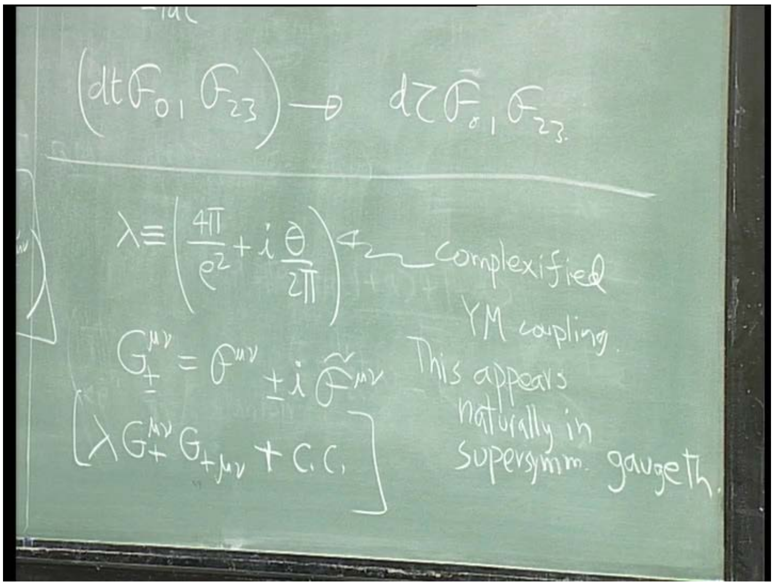

So, you can you can define I am going I have a tau now, this is normally called tau. What should I call it? You define lambda to be 1 upon 4 phi you put here 4 phi upon e square plus i theta over 2 phi. You just define it this way and so this will give you a, this is a complexified. Then this action has a can be written as a real part of a complexified f. So, how does that work. So, how so so you define something which will lets define f mu nu. So, let me define something called G mu nu to be f mu nu plus i times I am working in minkowkski here. So, this plus i comes. So, I am taking the analog of the self dual combination and g lets call these things I guess g plus minus to be this thing. So, upto factors you can just write this as lambda g mu nu plus now I am writing this from memory. So, I think this is the thing plus complex conjugate. If you write this out g mu nu plus this thing plus complex conjugate then what you will get is this action with a complexified coupling.

And so, this in fact combination is very natural in super symmetric gauge theories this is how you get this that is how. So, this appears in super symmetric something which we have not discussed. But. So, I just want to conclude. So, one thing today to mention something about u 1 we saw that phi 3 of u 1 was trivial. So, there are no instantons for u 1. So, this is something we know.

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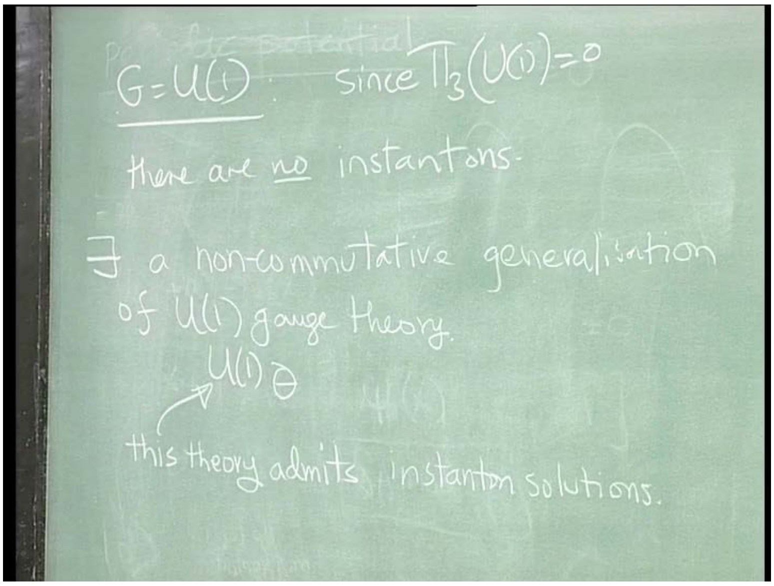

So, when G equal to u 1 since phi three of u 1 is 0 there are no instantons. However, there is a deformation. However, there exists what is called a non commutative generalization of u 1 gauge theory which basically says the basic fields do not commute there is a theta parameter which measures non commutativity. So, its like an h bar kind of thing. So, I do not I do not want to get into details, but all I am saying is that there is a parameter called theta. So, there is a family of if you wish non commutative gauge theories with this parameter theta theta equal to 0 is the commutative limit. But if you switch on at theta; this actually this theory admits instanton solutions theta is a real parameter.

So, this is actually very this might seem like a very right observation. Why do we care about these kind of theories? But like I said that the ADHM construction has a very nice realization geometric realization in string theory. And that there was a puzzle which came up which in the geometric picture. There should have been it sort of implied that there should be an instanton for u 1 gauge theories. Now there are two ways to think about it that that is total nonsense and the person who proposed it were given quite a bit of flak. They took quite a bit of flak until it was realized that actually there is a slight deformation.

And so, once you do that thing. So, that theta theory has instantons and it resolved a very important problem. So, it is because I might be discussing a picture a pictorial way of a of a understanding instantons and that time we will see this comeback. So, I just I am finishing with this comment that there is u 1 theory which is a slight deformation of normal which admits instantons. So, by instantons I mean finite action solutions etcetera.