Classical Field Theory

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Lecture - 36

One thing and adjusting the coefficient in front of it and I think they are quite a few errors in my last lecture. So, I am going to do it again and fix all the errors.

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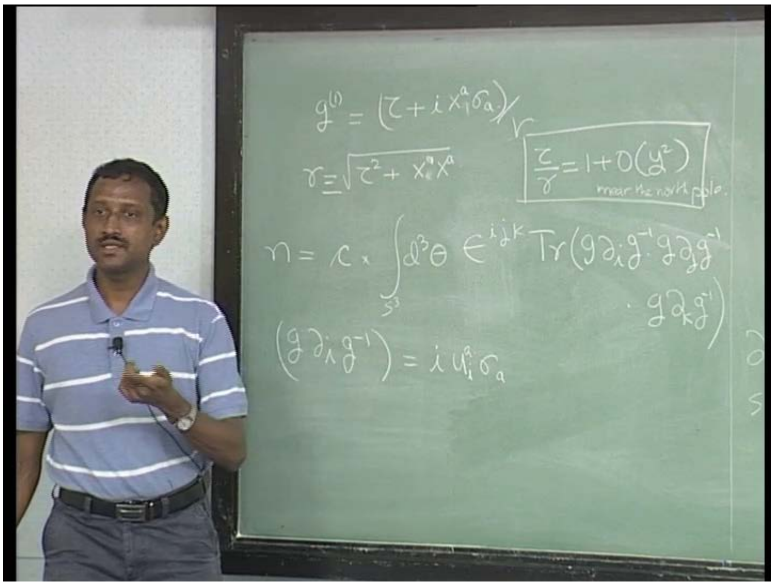

So, what we did was to choose the winding number one. And so, that shows in to be divided by r probably I put sigma a by two. That was one of the error which was pointed out by somebody right after class so, but any way. So, this is the definition of r. And so, the idea was to define n dividing number to be some constant times integral over the three sphere. Some bunch of coordinates thetas.

I gave you an explicit representation of that last time and times trace of. So, here I will not keep putting this super script one. Remind this is going to be. In general this would be what we would take g d j g inverse and this is the constant we want to fix and we will fix this constant by plugging in g one and choosing the c such that you get n equal to 1.

I am not proving that this is a topological variant etcetera, etcetera, but that is not every hard to prove and. So, we just going to do this computation and so, we needed to compute what g d i g inverse should be equal to. This what you wanted to compute, but without any computation. I was able to tell you the this had to be of this form this. ah This is a you can see that if I just take a d i is like some translation operator. So, you just its asking how g differs when move little bit on the group manifold. So, this is like a le algebra value elements. So, this has to be expanded in terms of le algebra value its staff.

So, I just. So, it should be writable in terms of something like this and. So, they are it can now to computing this u of a, and then we can plug things into this and we can get we will get the answer. And so now we… So, the thing is we use a nice trick out here to compute this basically to compute this number and let us see how that goes. So, what should is lets think of the analog of the north. So, we on three sphere and we pick any point on this. It its fine but we will just chose what we will called a north pole.

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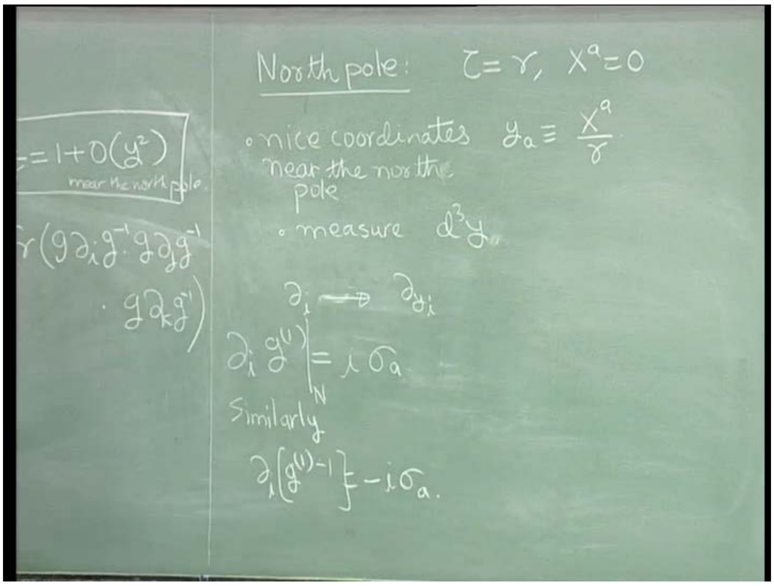

So, the north pole will be this point toe equal to r and all the X a s being 0. Its will be hard for you to imagine this thing. So, it is better to think of thing in right hand side. The reason I am calling at the north pole is because if you think of this the north pole in 3 phase.

In two sphere would be precisely you would have said z equal to the radius of the sphere and x and y coordinates being zero, they just one higher dimensional things. So, this a is three coordinate 0 one of them being non zero. So, I am going to chose this thing and we want to chose what the idea here is that we can coordinates near the north pole are very easy I do not need to use those coordinates which valid everywhere coordinate. So, nice coordinates. So, in a first order neighborhood because in some things only one things in after first order out here and in computing taking a derivative.

And then… So, nice coordinates are actually y a s or y a s which I will just defined to be X a upon r near the north pole, so so the measure. So, the measure for instance would be nice will be just d cubed y the volume measure on s three in for instance so. So, now, the thing is we want o evaluate. So, these I's are suppose to be coordinates on this sphere. So, now, what I will to do here d i. So, what I will do instead of evaluating any I will just evaluate d of y i. So, will just choose this as my coordinate. So, d i I will just say that this can be represented by d of y i. And so, we just go head now we can look at this and you see that g one in this coordinate system would be tau by r. But tau by tau by are in this tau is where were I given its here. So, you can see that tau by r will be equal to one plus order y square in the north. Near the north pole is this clear.

So, what I am going to do? This I am going to evaluate g g i g inverse at the north pole that is why I am going to do. So, for that thing you can see that I am looking for derivatives acting on tau it could act on a term which is y square. But I am going to at the north pole y square. If you take the derivative of it will give y some derivation you will get a y which vanishes. So, I do not need to. So, it is really very simple all I need to do is to just take derivative of the y's out here.

So, that you can see that d i of g one is just I y a. This is x a by r. Its just I sigma x at the north pole. And what about? So, we have to, but we want to get g inverse. So, it not again in hard to see that the inverse of that will give a minus. It will come with a minus sign similarly. So, it tells you that u i a. It is very simple. It just one. It is a conical delta.

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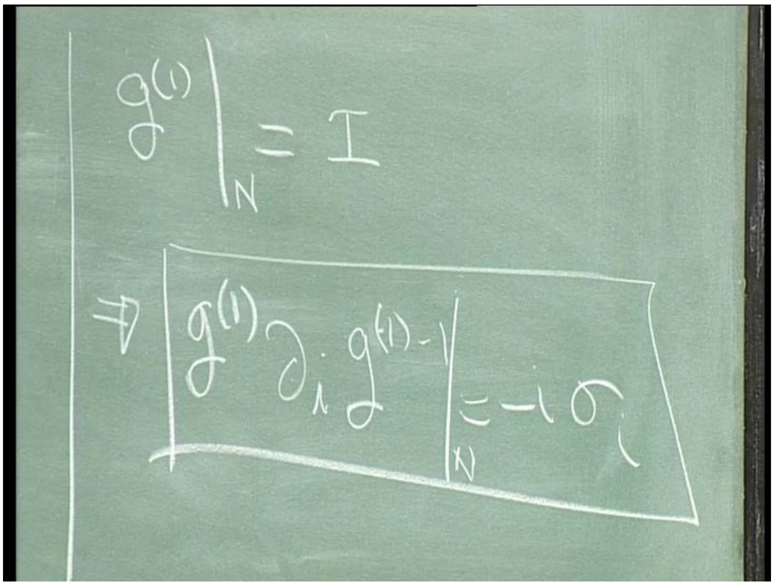

So, this implies. In fact, now you may ask what about g? One at the north pole is again what is that equal to y 0 and it is just identity matrix at the north pole. And we have done this calculation yesterday put putting this thing here and we found that we were getting determinant of u. And what is that? Tell you that determinate of u is equal to 1. That is really the relevant thing. And so, there will be no coordinate dependence coming on this things.

So, this whole computation which you need to do just corresponds to doing some number coming from here times the integral over this sphere. I have not written out the exact measure, but we know what is the measure is and had worked out the volume of the three sphere last time. What was the answer? It was two phi square.

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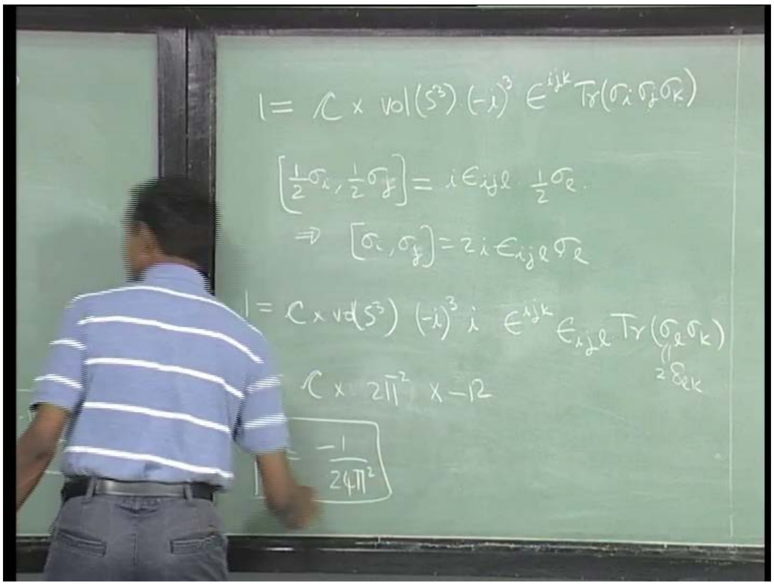

So, what we see that and 1 should be equal to C times volume of s three times. This trace of there will be a minus i. So, let me pull out the I cubed epsilon i j k. That is all that remains right and I have done some this computation last time. But I think I there was some errors what I have done. So, I will just repeat it you just take me a minute.

So, the thing is first thing I did yesterday was to cover this into a commutator and there I think I messed up on a factor of two and this is what I should have remembered that its half sigma I which satisfies the s u 2 le algebra. And we know the structure constant for that that is i epsilon i j l half sigma l. So, now, I can just. So, I want sigma i sigma j commutator. So, this implies that sigma i sigma j commutator is q i epsilon i j l. So, the way the trick the way to do this take sigma i sigma j because there is an epsilon out here. I can write this as half of the commutator of sigma i sigma j then half will cancel this. So, now, I can just do this. So, 1 equal to C times volume of s 3 times. Then you can get I out of here from this.

So, its minus this thing in to I times trace and this is equal to 2 delta l k. So, you get 2 and then you get epsilon i j k epsilon i j k. If you contact with the delta l k, but that is three factorial which is 6 and you will get two from here and this gives you minus, but we have already seen that the volume of s 3 is 2 phi square. So, that I can just write that out here. So, we get c equal to minus 1 by 12 into 2 24 phi square, all this hard work. Just to compute this and this we done only for s u 2, but just to show you how it goes this is just minus 1 by. So, this is how this winding number is...

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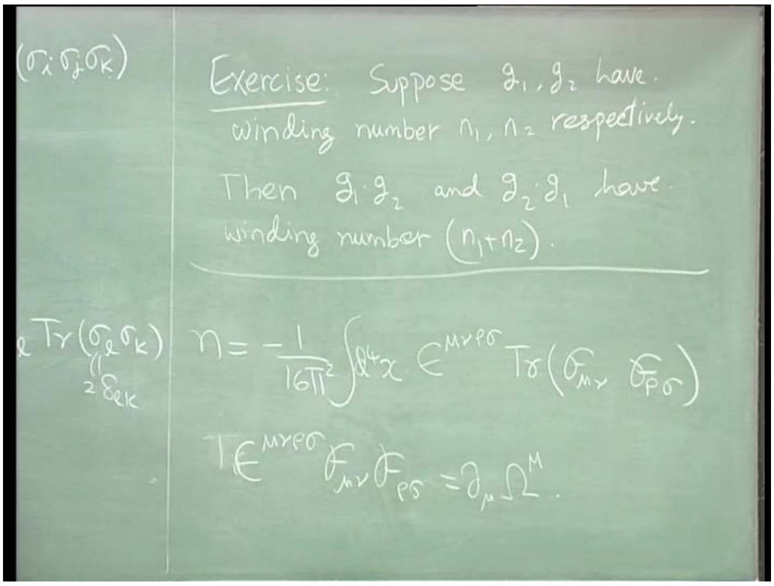

So, now, there is some exercises which you should exercise is the following of winding number which intuitively is clear. Suppose there are suppose g 1 and g 2 have winding number. I keep calling winding numbers is. You understand what I mean? n 1 n 2 respectively then g 1 dot g 2 and g 2 dot g 1 have winding number. So, it came more or less follow from this definition you have to plug in separate out things and see and see how it looks k.

So, that is the claim of the additivity of this thing. For instance it is easy you see that g inverse g inverse has opposite winding number because if you just look at the there is a standard formula. If you replace g by g inverse you will get g g inverse g i g inverse, but there is identity which says minus of that thing. But each of them will change sign last check minus one cubed this was minus 1.

So, it flips the thing that is one. So, this is this thing, but now we will like to connect this up with yet another quantity which we will see and what will show is that n has the another expression. And that is this is something which you have seen in your assignment in your assignment. You had found this quantity. You had shown that this quantity was a derivate total derivative. So, all this formula I am writing out the numbers are correct only for s u 2. So, what you what we showed was trace rather epsilon mu nu row sigma f mu nu f row sigma was writable. As some d mu of I think the symbol are used in the assignment of omega mu. Any way asked to find out what is omega mu.

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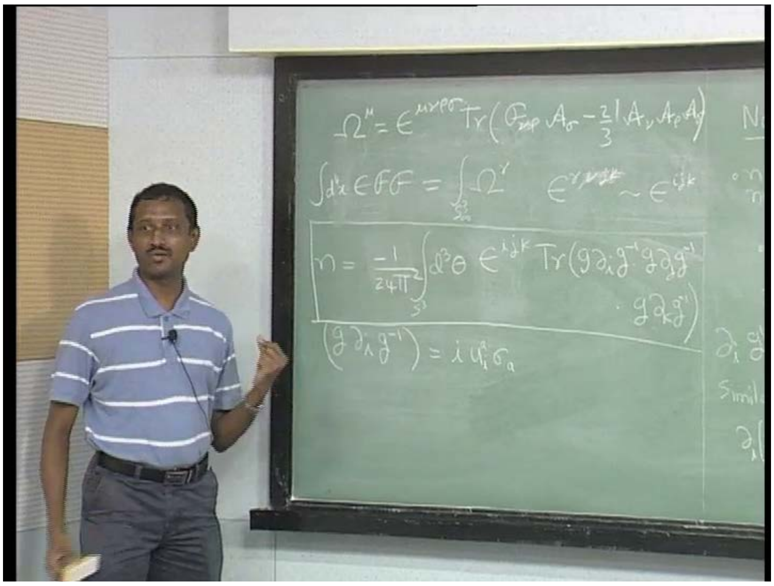

And the answer is kind of nice mu nu row sigma trace f nu row a sigma minus 2 thirds. And you can also write it. If you substitute for what f is and you do things you will find that that corresponds to flipping this sign. So, that is another formula you would see which would have d nu a row a sigma minus plus 2 thirds of this thing. And the easy way to check this is to just write I mean in terms of derivatives etcetera, etcetera.

You look at this term this has precisely two derivatives and you pulling out one derivative. So, omega mu should be constructible from just one derivative and gauge fields. So, this is the term which has one derivative and two gauge fields and this has three gauge fields.

So, there will be I depending on I think Euclidean (( )), I am not which one. So, you want in your assignment you got an I. Good. Right now I will this is this numbers a are important, but what is more important is the structure out here which is that it takes this form it has this thing, and so now because it is total derivative. This thing this an integral all space. This reduces to an integral.

So, what that tells you is that integral of that quantity. So, I will just write it as epsilon f f just to since epsilon f f over 4 space will be equal to an integral over the three sphere at infinity times omega in the radial direction. This is what you should get. And so, if you pick this thing to be the radial direction this nu row sigma will necessarily become along the spar along the s 3. So, epsilon r mu nu row sigma is should be identified with. So, these things should take values it should be i j k. So, what I am trying to show you is that really it is coming down to this so, but omega r is the function of this a's and but the key point here is we are choosing boundary condition where the gauge field is pure gauge.

So, now, a prior it looks like these two these two are different terms, but for the pure gauge they give the same things and it only change some coefficient and that is what they have put out there. So, you can see that it is easy to see that this gives that term this kind of term, but again here if you look at this f also will give a same term it cannot give something else because there is a anti symmetric guy working. So, in... So, what I am saying is when it is pure gauge it comes back to this form.

So, you can see that this object can be written in terms of complete fort derivative of this kind I am just writing out the correct answer and since we are. So, the other difference what we did earlier and now is that we are in Euclidian space we will see why we need to be in Euclidian space a little bit better again we will later but. So, now hopefully at least you can see that there are two different expressions, but they will do they will match. So, that was the reason for me to work this detail, but I have not shown you that this coefficient becomes sixteen phi square that is why you to do. So, that just some sitting there and putting out looking at looking at the various term seeing out how they work out. So, the question is now what we would like to do is to ask can we find the general solution do.

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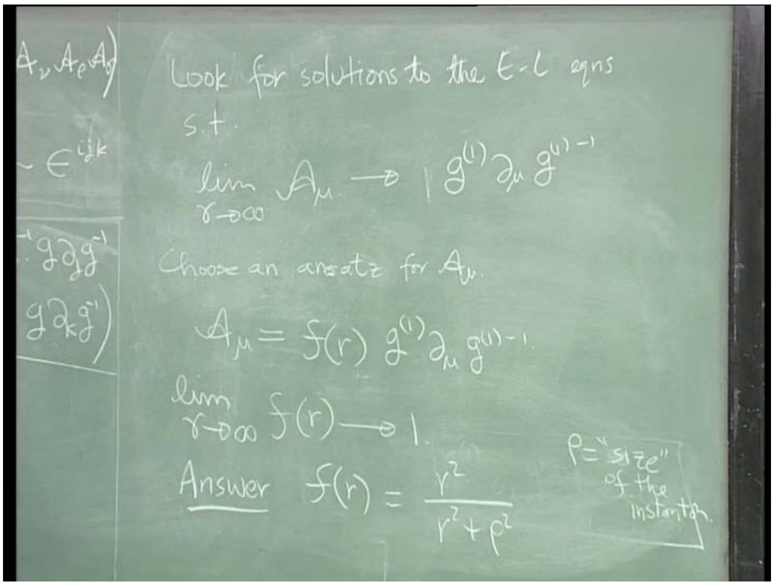

So, what we want to do is to look for solutions Euclidean Lagrange equations of motion such that a mu limit r tends to infinity of ah a mu goes to pure gauge. So, I am just choosing for con concreteness the winding number one. So, this is what we want to do and you just go ahead and put an ansatz for chose and ansatz for and the ansatz its kind of nice some function of r times and limit r tends to infinity f of r should be one and you can go head and solve langrage.

So, you will get nice differential equation for f of r and I would recommended working out. And so, the answer turns out to be something like this where row is some parameter. Row is a parameter which will say is the size you can see that this first thing is length scale row length scale and the reason to call it a size its roughly gives you an estimate of how broad this thing is. Because when r is less than row you can see the or much less than row this thing will be will be close to 0 and that is where its deviating from the asymptotic this thing. So, that naturally the size of this thing question to ask is what happens if you consider various symmetric that you have in the theory.

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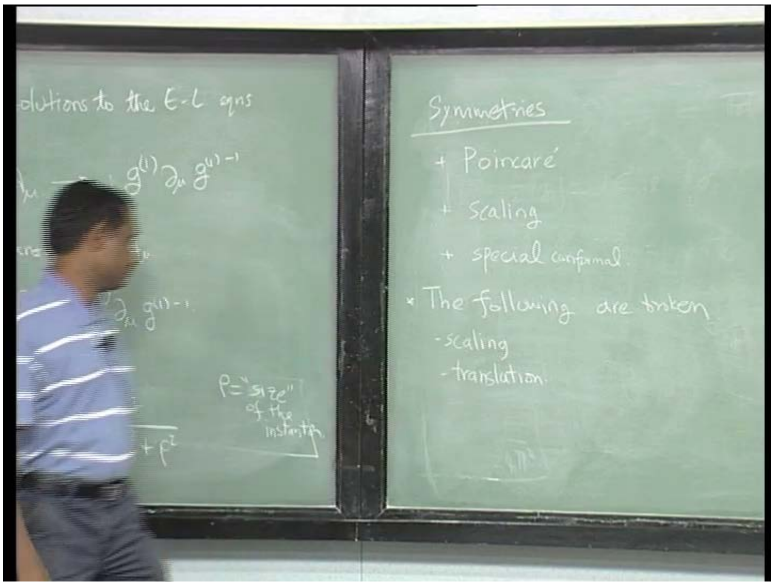

So, you have what are the symmetries. So, we have poincare, but classically the young miles has a extra symmetry which is scaling and conformal we already seen some of that. So, that is also scaling and also there is what is called special conformal.

So, now, the question is here if you had given any solution to here let assume that we are checked this is solution to the add equations of motions. And now we want to ask how do this symmetries act on this thing some symmetries will act trivially. But now we are in a gauge theory. So, for instance suppose I ask what happens under rotations or special conformal? You can it is not if your transformation is such that it takes a mu to a gauge transformer still say that it is a same solution you will not say it is a different solution. So, it is really this is the only stealthy or difference between when we looked at scalar thing we looked to gory detail. Here this always modular gauge transformation's. So, what you find here is that the following symmetries are broken. One is scaling its very obvious if I scale everything row changes.

So, scaling is broken also translation in space time these two are broken all other things are not. So, for instance hard exercise would be to actually to take the special conformal transformation put that into the solution and show that it is a gauge. It is not it is a gauge transform of what you are started off that can be done, but I am not going to do it. So, I am just saying that that is what is happening. So, again here r equal to 0 that is nothing special you can shift origins.

So, in some sense this is an object which is localized in space and time at some point in this way here it is the origin, but there is nothing special about that point you can always move it. So, you can. So, it is easy to shift I mean right solution when it shifted. So, this thing has only 4 free parameters. If you 5 parameters free parameters one is four from coming from translation in space and time and one from scaling. So, these... So, that all that is about the solution, but the nice thing about this that we need not solve these equations. We solve the set of first order equations there much easier to solve and that is what I am going to do. So, it is very similar to what we did earlier expect. Now what we looking is at finite action.

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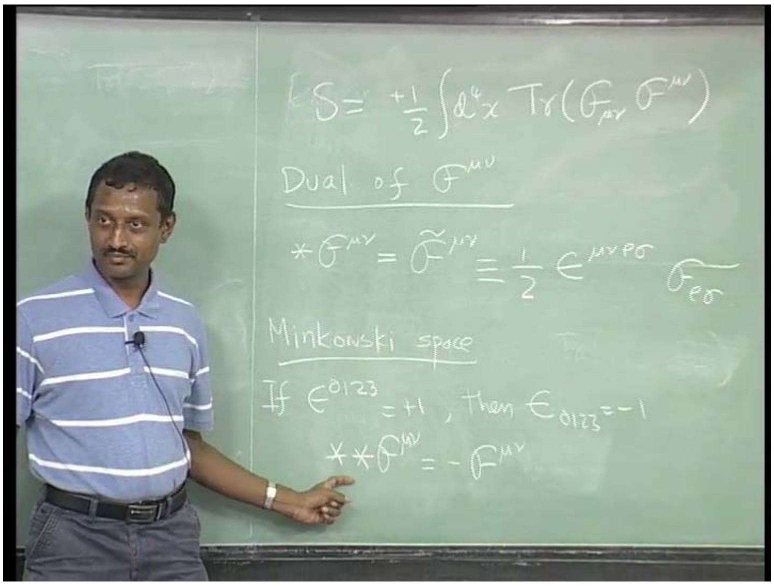

We looking at the finite action and so, lets we had worked out the normalization with the traces. What is the normalization? Can you look up your notes and tell me was it one fourth half or half? Just last lecture we had worked it out I think plus 1 by 2. Is it? So, the point here is that we can define the dual. Dual of f this is just a generalization of what we saw in electric in the case of electric magnetism where we saw that there is the symmetry which exchanges electric and magnetic field. So, the same thing we can do in general out here define star f mu nu or sometimes this is also called just f tilde mu. I keep switching back and forth. So, this is nice people are comfortable with the form notation. And so this just define to be half epsilon mu nu row sigma f row sigma. This is definition and you can see easily that if I take.

For instance electric field would be f 0 I component and because of this that will go to i j of the and there is the difference. One difference important difference between in Minkowski space if we define epsilon 0 1 2 3 to be plus 1; the lower guy would be minus 1. If then why is that because I have to lower these things and irrespective of your convention either 3 3 minus sign or 1 minus sign. So, this happens in Minkowski space. So, now, you can what will happen if I do this twice? So, Minkowski space basically because of this thing if you do the star operation twice. So, you would have thought I take electric field to magnetic field and then I do it. Once more magnetic field comes back to electric field, but if you remember there is some minus sign there is important minus sign. So, what it does? If you do it two times it does not go exactly to itself. It goes to minus itself and you can trace the origin to this.

If you wish yeah so, but you can see that if you are working strange space time where they were two positive and two negative of signature two. This sign would not have appear and for 0 also Euclidian. So, since we are going to be we are working in Euclidian space we will not have this minus sign.

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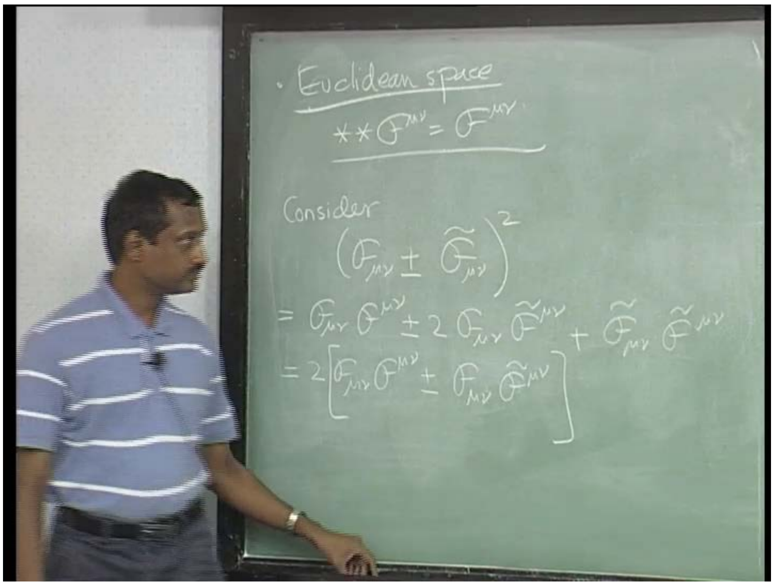

So, in Euclidian space there’s the big difference in Euclidian space and. So, let us now do something in Euclidian space let us consider the following object plus or minus square. So, if I expand this by this I mean you know writing the other one with upper etcetera, etcetera. So, you can see that then I get this is equal to f mu nu f mu nu and then plus or minus 2 plus. But there is an important point here if you look at f tilde tilde. This is actually the same as this think about what happened at this quantity was. E square minus b square in Minkowski, but it became anyway it was became e square plus b square.

So, you can see that exchanging e and b even minus sign and all it does not matter which these two will give you the same thing. So, I can rewrite this as two, but this object is positive definite.

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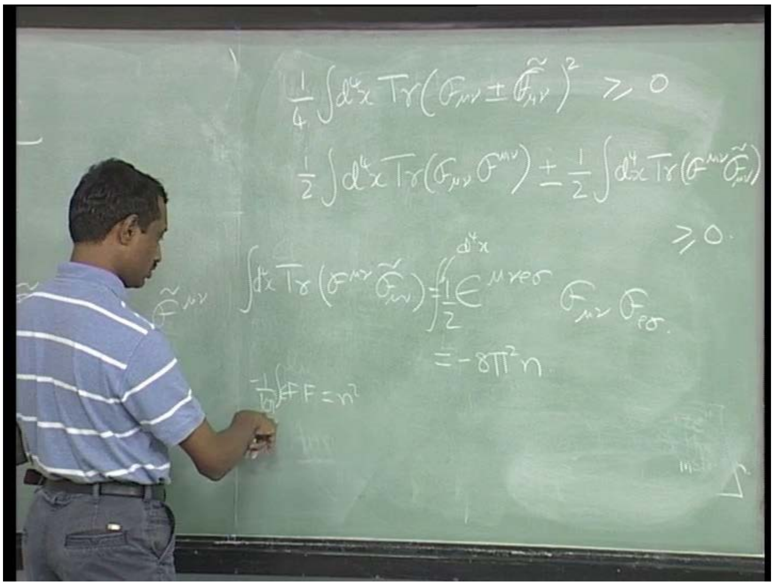

So I can. So, what we can see is that integral of one fourth of d 4 x trace of must be greater than or equal to 0 not a deep statement, but now we can use this. And so, this implies that half I am taking care of this 2 half integral d 4 x trace f mu nu plus or minus half. And let us look at this quantity. This is exactly the winding number guy because what is this. So, let us work that out trace f mu nu f tilde mu nu equal to there's a half in the definition epsilon mu nu row sigma. And so, the integral of this d four x is equal to and what is this equal to? It was, is equal to the. So, integral of that thing is equal to minus 16. So, and there's a half here. So, this becomes minus 8 phi square n it goes to 32 wait. So, what did we show we showed that integral 1 by 16 phi square f f epsilon f f was minus of that was equal to n, so integral epsilon f f by 2 how. So, this is equal to integral epsilon f f equal to minus 16 phi square n half of that is 8 is that clear is that. So, then I have to…

So, what you can see is that this thing is now minus of 8 phi square n. So, now, we can see something nice happening what we see is that and this quantities are action is are action l. So, this is s Euclidean.

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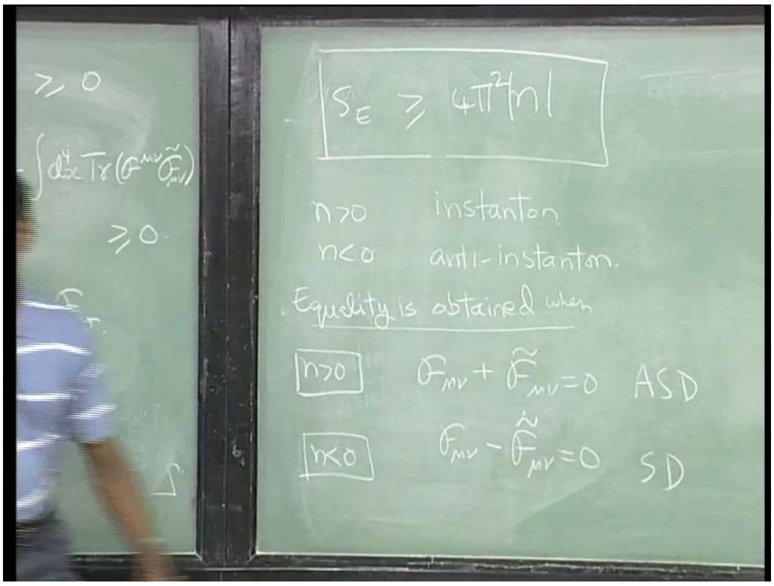

So, what we get is s Euclidean plus or minus or minus or plus or I will take it to the other side is greater than or equal to 4 phi square and because there's a half further half out there hopefully there aren't no I mean. So, at least with self consistent I am not saying this is the correct answer or whatever assuming my 16 phi square was correct, but at the point here is not that. The point here is that you can see that it is bounded the Euclidean action is bounded from below.

So, you can rewrite this as. So, this implies that it is strictly you can replace it by mod n. So, depending on your convention n equal to plus 1 is called an instanton positive is called instanton. And n less than 0 is called an anti instanton and now you can ask when is the thing saturated right.

So, the equality is obtained when this quantity is 0 and now for when n is positive we chose the upper sign according to this. So, now... So, when n is greater than 0 when n is greater than 0 we be this will imply f mu nu equal plus f tilde mu nu equal to 0 and n less than 0. So, these are first order equations. So, here this thing is when f equal to f dual. So, this is called self dual instanton and this will be called anti self dual. So, these equations are a lot easier to solve because they are first order so.

In fact, you can check if the binding number 1 solution that we wrote solves this equation. You should check that you go compute this thing check if it satisfies that. So, it is a lot easier because it tells you all you need to do is go you have given the anzat for mu. You plug the thing in you get a differential equation for f and check if that f satisfies a lot easier. I just have a sort of a remark to make. If you remember when we did representation theory of the Lawrence group we saw there were two s u two's.

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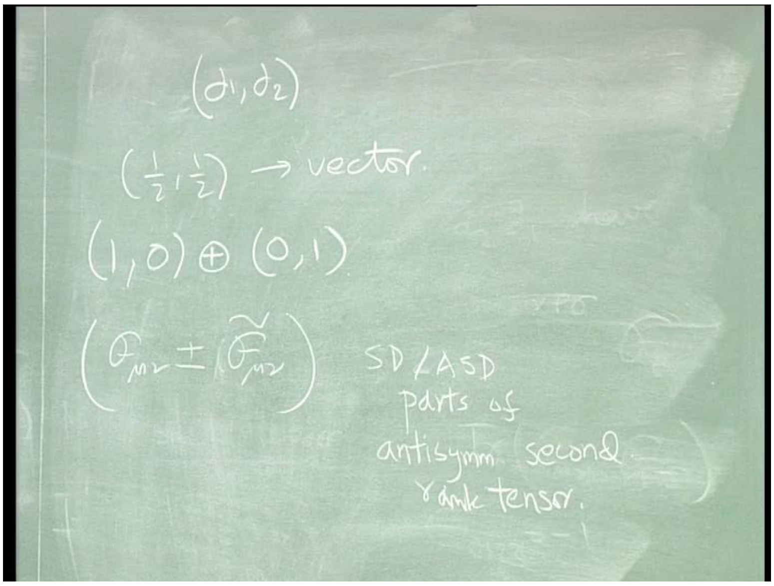

And we saw that any object would be specified by which transform in a nice representation of the Lawrence group would be given by 2 numbers it will j 1 and j 2 and. So, for instance we saw that half was the vector and but I told you that the field strength has its absolutely reducible and. So, it is a anti symmetry. So, that actually comes like this. So, f mu nu has 6 components.

So, this object will ah this is phi one which has three components this has this. So, coming back to this thing this set up here is actually these conditions correspond to setting one of these combinations to 0 and this is a Lawrence invariant condition. So, depending on your convention you can call this self dual part and this is the anti self dual part.

So, if you are g i if you give me a field strength which is neither self dual or this thing you can just take linear combinations. And so, the way to do that is exactly what we saw here. So, you f mu nu plus or minus f tilde mu nu will give you the self dual and the self dual parts and just to in you can do this. Even in Minkowski except you have to put an I because tau square was minus 1. So, its Eigen values are plus or minus i. So, this I would be there in this thing. So, these are the self dual and anti self dual parts of a of any anti symmetric second rank tensor. Are there any questions with?

So, these equations are actually very, very important and very nice one of the things which one can show is that for instance let us look at the monopole equation which we the Bogomolny equation which we saw which we got by. So, in some sense the self dual yang. So, these are. So, if you are looking at configurations like this we would call them self dual yang mills configurations.

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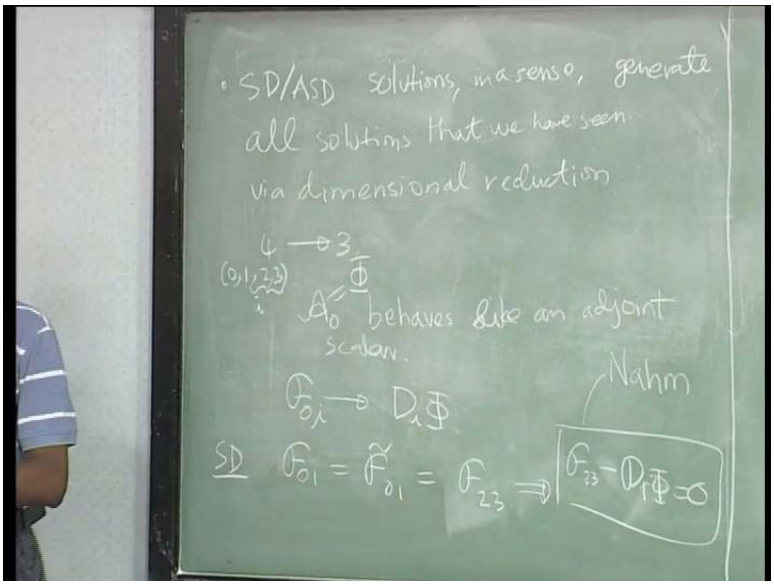

So, in some sense these self dual and anti self dual solutions. In a sense generate all solutions that we have seen in this course via some via dimensional reduction. So, I will just illustrate what I mean by this. I what by that I mean whenever we be square intrigue work we got some we always had some equation which was first order. So, la I will show for instance how we can get the monopole equation. So, let... So, the idea of dimensional reduction is. So, we starting in 4 dimensions Euclidean mine due. So, we are going from 4 to 3 and. So, I just take a. So, we chose our 4 to be 0 1 2 and 3 and we will just say that we will look for configurations usually what people do is when they wick rotate they call 0 as 4, but I just did not do that does not matter.

So, I just say that things are we've already seen this idea we say that things are independent of this coordinate and. So, this a... So, a 0 behaves like what like a like an adjoint scalar. So, let us look at the self duality equations and just look at the thing is pick say let us say that lets chose this and say mu is 0. So, we have to look at f. So, this will also we saw that f 0 i. So, we will use I for these 3 f 0. I will just become the co variant derivative of this is something which you have already seen and f i j nothing will happen because. So, and we… So, what we what that tells you is that f 0 i for self dual equations f 0 i is equal to f tilde 0 I which is equal to half.

So, let us take f 0 1 just to be concrete others will just follow what we will get is half of epsilon 0 1 2 3. So, this will be half of no half will go away this is just will become f 2 three, but what is f 0 1 it is this things was this equation just becomes. So, this just becomes f 2 3 plus hmm plus or minus yeah minus d 0 i d 1 of a 0 which we will call what did I write here. I wrote something completely d i of. So, we will just call this we will call it as scalar phi. So, it just becomes the, i of phi terribly. So, that is becomes d 1 of phi. So, this would be the d field the analog of the b. So, this is the equation remember which hmm in the Bogomolni limit. This is exactly what we got.

Now, it gets more interesting you can ask for instance what happens if I if I take go one further dimension below just say that look at 0 and three keep 1 and 2 and make things independent of 0 and 3. So, now, there will be many pieces. So, so what it tells you is that suppose you are able to write instanton. If you are able to solve these equations you can you can. In fact, get some solutions of this kind also you could write a different things. You can solve these things in the context of monopoles. This was this is due to Nahm. So, these are some time called Nahm's equation and if you lets go now from 4 to 2.

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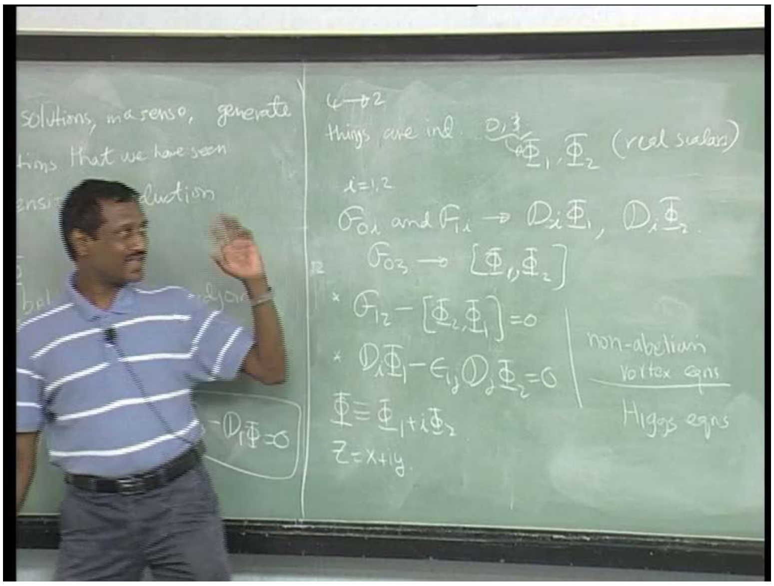

So, things are independent of 0 1 coordinate and what I will 0 and 3 and just let us just call these two scalars as phi 1 and phi 2. So, I end up with 2 adjoint scalars and I need to work out what. So, I runs only from 1 and 2. So, you have f o i which will go to f 0 i and F 1 i this will just go to D 0 D i of phi 1 and D i of phi 2 and F i j will remain F 1 2. There is nothing so, but there is now the other bit which we saw already f 0 3 will go to the commutator phi 1 phi 2 I am very lose here with signs are not. So, important I mean the structure is much more important. So, now, let us look at this equation and chose f 1 2 out here for instance.

So, the… So, chose the same equation I chose f 1 2. So, I get f 1 2 equal to tilde f 1 2 which will become the same as f 0 3. So, up to a sign again one two wait. So, we started out 0 1 2 3. So, f 1 f tilde 1 2 will be f 3 0. So, minus and but there is still one more equation. I could have considered I could have considered f 0, I here. Then I will get the other guy. So, what we will get is something like this f 0 1 which would become D i of phi 1 will become equal.

So, the I of phi one will become will get the other guy right f. So, this will just become minus some epsilon again sign. I am I am little bit lose with the sign epsilon i j of d j of phi 2 equal to 0. This equation looks little bit messy, but it will become very simple. So, let us just do since we are in two dimensions.

So, these are these are called the vortex equations these are the non Abelian vortex equations that also called Higgs equations. So, this I will write it in a nicer way. This epsilon a j k makes it look little bit weird, but actually suppose we went ahead and defined a complex phi. So, these were real scalars. So, I can complexify it I can define phi to d phi1 plus i phi 2 and we are in two dimensional space time. I can take the coordinates and work with we can just define z to be x plus i y. Then this equation takes a nice form it becomes up to again factors.

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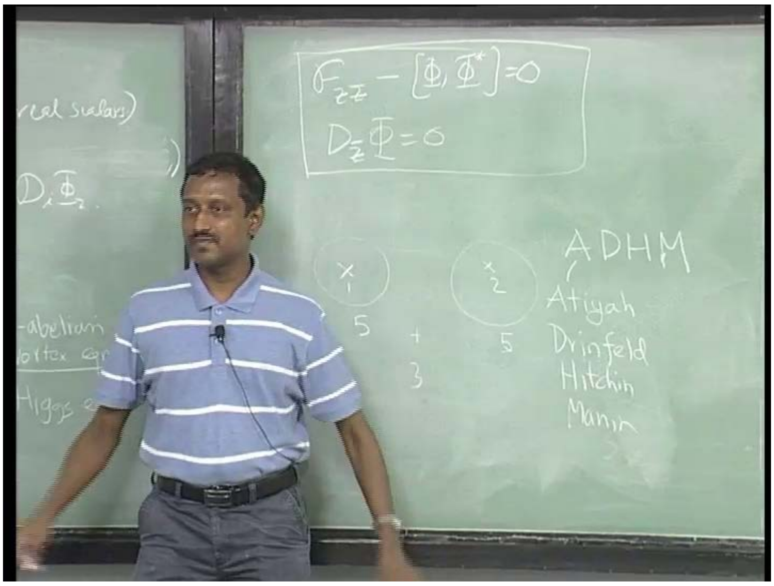

So, this like a gauged version of the of a Cauchy-Riemann condition. So, this says that. So, this has a nice meaning. So, this is written in a sort of compact form this is what you would get, but of course, you can ask what happens. What is the difference anti self dual and anti self dual? One would correspond to holomorphic. That means it is a Deezee. The other one will correspond to d z of phi being 0. So, that that is what will happen. So, So, there is a sense in which the self dual system is special and all everything else comes from that. And so, the thing is now the question is can we find all the solutions to. So, if you give me winding number n. You can ask.

Can you a give me a solution first of that equation second bit is? Can you get all the amount like we like explained here? The instanton carried phi numbers 1 was this k's scale and the other one was the translation modes. Suppose we have we saw one more nice thing. We saw that we could get. We let us say we have instanton number one solutions here one localized here. One localized here of this size this guy has five parameters. This guy also has 5 parameters. Well actually there's a relative s u 2 also. So, that will give you a another 3 parameters plus some parameter. I think it is three. Let us just keep unless there is further u 1 or whatever.

So, what that tells you is that if you are looking for something which has instanton number two that will have many more parameters than just those far away then you take. So, there will. So, there's also. So, the point here is that as you there will be a solution where these come closer and then there will be a full ah proper winding number two solution. So, question is how do you go about writing these solutions and depending on how much time remains in this course. We will discuss the ADHM construction, but I just want to point out that there is this construction due to these guys and there's a beautiful understanding which came much later in the context of string theory where we actually understand where these equations came from.

So, what they do is to solve some auxiliary linear equation in some this thing and they show that the consistency condition. For those set of equation is the self dual yang mills what Hooft has done was. He wrote ma he wrote for instance. Two instanton solution explicitely, but it dint have all the parameters and you know that there has to be a solution just based on symmetry arguments. There must be a solution which has this many parameters in my counting it is 13.

But I am not sure about that I am just doing it top of my head. There might be a reduction which might come, and so these four authors. They for once I will give their names Michael, Atiyah, Drinfeld, Hitchin and Manin. So, two Russians and two English or British authors they gave this miraculous constructions which had all the parameters and its and their solution will work for any group any non Abelian group. It is not just s q 2. So, it is really I mean it is amazing in I think two pages or one and half of physics. Let us b this was completed and it took us many, many… So, this was I do not remember when, but I think the first nice understanding which I have seen for this is due to Douglas which came around 96.

I think in a paper I think titled brains within brains or whatever and it explains sort of a dual prospective. And this is something we will discuss dualities at that time may be time permitting I will discuss that.