Classical Field Theory

Prof. Suresh Govindarajan
Department of Physics
Indian Institute of Technology, Madras

Lecture - 35

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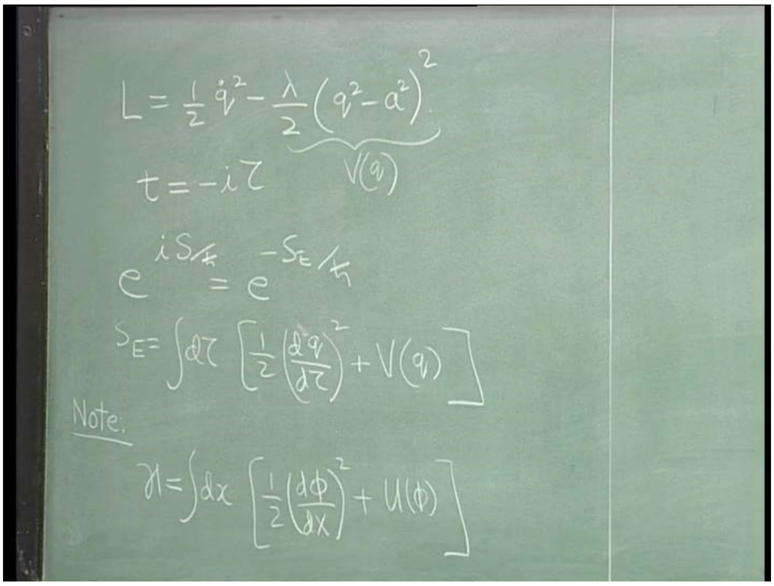

That is means I should work out things in slightly more detail and that is what am going to do now and… So, just remember what we did, we started out with. So, we look a quantum mechanical system, which was let us take mass to be 1 minus… So, whole square. So, all these numbers are tailored to precisely what we had done with the kink soliton. Even these two I just went back and checked so, I choose those numbers to be exactly that we had done the, this thing. So, the idea was to take t and to replace with minus I tau right that is the minus right minus. We fixes the sign. So, the question is so the idea was that e power i S will go to, will become equal to once you make this substitution will give you the Euclidean action.

And so, the Euclidean action was simple to state as Euclidean was just integral of d tau times d q by d tau whole square plus… Lets for completeness just call this quantity be q. Later we use this specific thing, but in general this goes through and just note here is that this is you just can compare this with the Hamilton density which you would have written for one plus one dimension theory when you are looking time.

Independent this is the kind of thing which we are worked out long ago. So, you can see that that if I identify tau with x and q with phi these two actually looks the same. So, the idea was to look at the tunneling solution and the tunneling solution for which these should be the Lagrange. So, you can see that Lagrange is just t minus v in these set up. So, it correspond to a potential which is inverted when it will be speaking.

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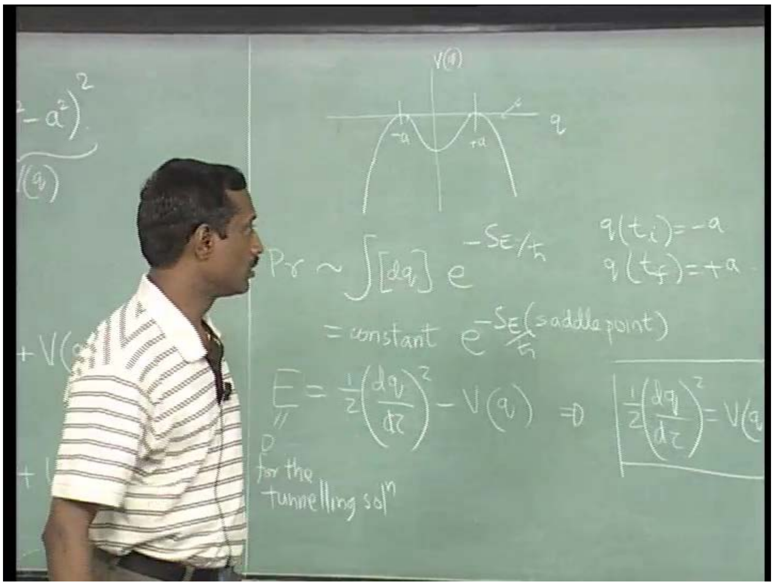

So, let us plot this now whenever plotting v of q. I will plot the one which we are written out there verse q and. So, this would be. So, now, the idea is to evaluate. So, the probability up to some normalization constant would be integral over d q e power minus S Euclidean by h bar and the leading term will be up to will be some constant times e power minus S Euclidean evaluated at the saddle point by h bar.

Now the thing is that if you have many saddle point in principle you should sum over all of them, but am going to act as. If there is just one saddle point even in this example there are many saddle point just we will see and. So, the solution the natural the solution which will take you from minus a to plus a is…

So, the way to understand this is to think as limiting case of the following process you assumed that you have a very very small energy epsilon which is tending to 0 from this side and they are truing points here and here. So, you start your thing here it will go like this. And if you take a limit when the energy goes to 0 these two will be the turning point. And it actually take infinite amount of time, Euclidean time to go from this thing.

But we want to work out the solution with just very very simple solution that is the energy. We can just solve it without one integration comes for free because energy is conserved and that would be just half d q by d tau whole square minus v of q and for r tunneling solution is 0.

So, the American spelling for tunneling has one L in bathers me that is way I write double L, but that is acceptable. So, what we have is that for the solution of interest. We have half d square q by d t square equal to v of q. So, now, the point here is to evaluate this action on this trajectory. And so, that is what I will do now a look at we need to evaluate the action e a at that on that saddle point. But now we can see that v of q is nothing, but half d q by d tau square.

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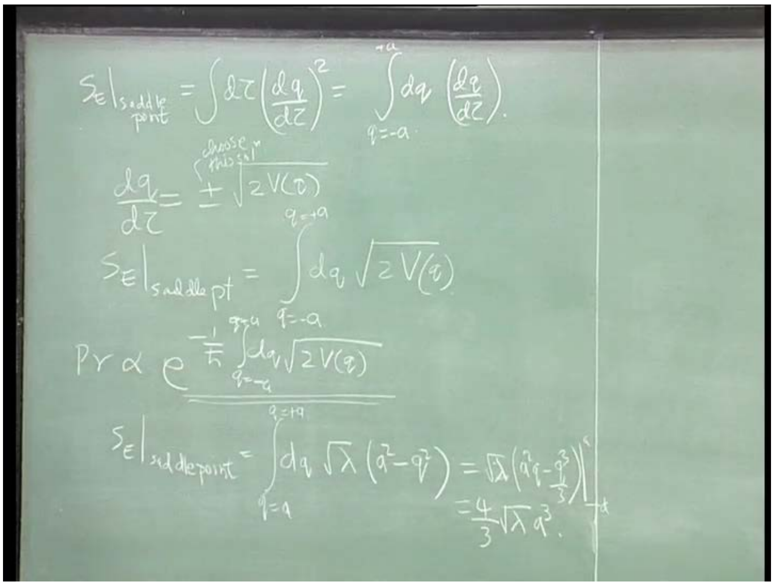

So, I can plug that back in. So, S Euclidean on this saddle point one just one point here is that the in this path integral at the boundary conditions are at that at. So, that at that some x q at some t initial equal to minus a in q and some t final equal to plus a.

I could consider the opposite one, but that would just exchange this rows. So, S q of saddle point would be integral over d tau of. So, that box quantity tells you v of q is same as half d q by d tau whole square. So, I can just go head and substitute for this thing would just became this would be just d q by d tau whole square because half plus half is one.

So, I can just write it in this fashion I will write it slightly and. So, I just read it one this thing now you can see that the tau dependences. So, there is a one question about boundary conditions etcetera you can see that it sort of gone. I mean all we really need is that at initial time it is here and out here. And now we can we can go back to this and try to write it in terms of there is still at the residual time dependences through tau which we can get rid of by using this, but we need to figure out which square root to take.

So, this square root should be. So, you just take at some point it should be slightly positive it should be moving d q by d tau should be positive. If you want the other solution it goes from plus a to minus a; you would choose the negative square root. So, d q by d tau will be equal to plus or minus square root of 2 v q. So, we will choose this.

So, we let us call this will call this an instanton and the opposite guy we will call it anti-instanton. So, this tells you we just have to choose this. So, we get S Euclidean evaluated at saddle point is equal to integral from q equal to minus a to q equal to plus a d q of square root of 2 v of q. So, you can go back and compare. So, what this all tells you is the probability is proportional to e power minus 1 by h bar integral q equal to minus a to q equal to plus a d q.

So, this matches the saddle point answer which we had because that had an e in it and you just said equal to 0 because that our condition and that is what you get. So, this agrees with the saddle point takes for this problem. This is the way of understanding who I mean there are prefectures etc ete. But you can fix all those things correctly. And if you want to you can use you can look up some of the literature for ancients goldsman lecture.

So, that is what we get out here, but we can do more we can actually evaluate what this guys is for an answer. So, am going to be more explicit . So, the saddle point where what is v of q in an example its just lambda over to. So, two times that. So, that just becomes. So, what I get 2 v what I want. So, that the these two will go away and left with square root of lambda times square root of this thing. But I have to make sure that it is positive in the region. I should choose a positive. I should choose a square minus q square.

So, this will just became d q square root of lambda into a square minus q square. We can just work out what this is the simple integration. I can just do this it is a root of a square q minus q cube upon three. Evaluate between minus a and plus a both these limits will give the same thing. So, I just two times evaluate this. A cube minus one third a cube which will be two third a cube that is the two. So, it just 4 by 3 root lambda a cube. If you remember the kink solution the kink soliton had the same mass.

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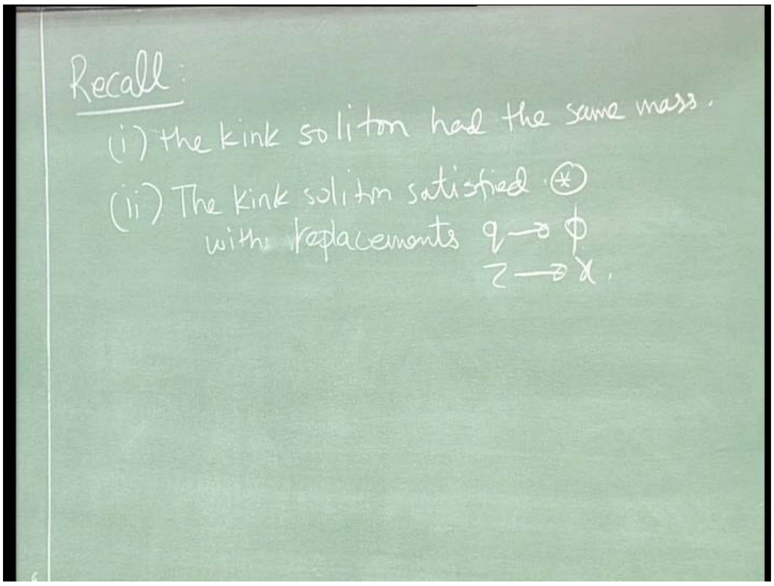

So, recall several thing first the kink soliton because that kink soliton this not Hamilton density. It was energy. It was the mass of this what we call the mass of the soliton. That is what we got. In fact, we also saw in that discuses the x thermo that the that the kink soliton actually satisfy this equation. So, you can see that the solution which we wrote in x space actually became Euclidean soliton the same.

So, number two is that the kink soliton satisfy lets what is the equation let us call the equation satisfies star plus sign was kink can be minus sing was antikink can be here. We will be as call it instanton an anti instanton.

Satisfies with start of course, the replacements with the replacements called q phi and then tau should became just space. So, now, you can see that if I ask if I tried mean there is no issue. I could with things around I could say that I could choose it like this. But that is not same flipping the limits. We also need to choose the correct solution and that will also give you a minus sign.

So, the answer for the same you get the same answer whether it just instanton or not the anti instanton the sign work out. I hope this address you question which were asked me. So, it is a one thing. So, at least you can see here in a very concrete example what we have seen is that in some sense or time independent solution which we look at what where soliton in some d dimensions could be sort of Euclidean thing in one less dimension in some sort of.

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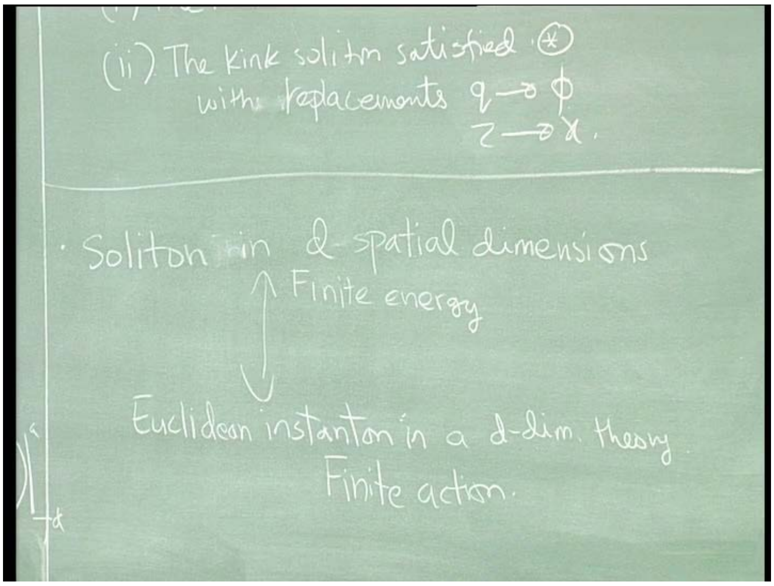

So, soliton in d dimensions d special dimensions. So, by this should have said us time independent in d plus one this kind of has seems appear as like it has some relationship with an Euclidean instanton. In some other theory right, in d dimension theory this is the kind of thing. So, the condition the key thing was that when you go from here. The finite energy becomes in this case finite energy gets snap to finite action. But the quantity you are measuring is really the same. If you wish its just call something else.

So, what we are going to do is a like you said we could gone in a. So, we followed a step right we went from the 1 plus 1 to 2 plus 1 to 3 plus 1 and. So, in some sense you would say that fine with this end of story because we are will leave in three plus one dimension we are not in interested in higher dimension. That is one way of thinking about it, but the point here this tells you that maybe we should thing about four dimensional object.

So, one more thing what is this instanton do? It tells you that you are actually ground state of your system its some coherent superb position of two different vacuum and so, similar things even instantons which we.

Will look at now in three plus one dimensions will be of that kind. So, lets go back to. So, we return to we return to three plus one because in some séances now if you give a vertex solution or monopole solution I think you can just follow the rank. It just really nothing to do just a think that you are just be think of this think solution in a theory. So, so let us get back to this think. So, the thing is that one in interested point is that you could ask, can kink, can we disuses something like analog or derricks thermo for pure young mills. Its an interacting theory can we find finite time independent.

So, finite energy time independent solutions in say 3 plus 1 young men's and here. So, that the answer is no again there is some scaling argument similar to that of what we did for x thermo where we did for scalar field. Here we do it for vector field and this due to coalmen and you can show that really you cannot get finite energy solutions. So, of course, you can get finite energy solution where by adding other fields in the story like we did in the scalar fields we added them. So, we are talking about 3 plus 1 dimensions.

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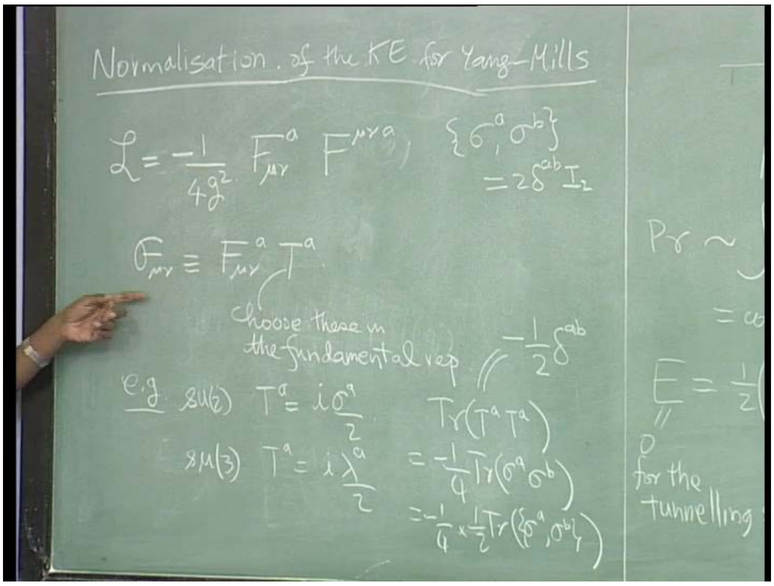

So, before that lets just fix some normalization young mills field non-abealin guy's felids. So, I keep using young mills and non-Abelian guy's field interchangeably. So, the natural thing. So, the Lagrange we have written something like this minus 1 by 4 some g square f mu nu a F mu nu b, but we can also ask how to write this in terms of traces over. So, we can we know that we can define something like this f mu nu we can mean valued object as F mu nu a times T a where T a is a other generator of the le algebra. And lets choose this in the fundamental representation.

So, example would be for s u 2 a lee lee algebra s u 2 T a s would be i sigma a by 2 where going to follow the physic convention which is to put the required things to be. This is correct is this what we want to do. I we want to do this way. So, it has to be q summation perfect good for s u 2 it would be this s u 3 it would be T a’s a. Again the same thing. I wouldn’t lose sleep if somewhere else had define minus of this. That is not I just want to i’s correct. So, this what you want to get. And so, this the fundamental. So, this is 2 by 2 this 3 by 3 representation. So, we also know for all this cases trace T a T b. So, let me do it for one of the cases you get i square which will make it minus and there will be half square which would be one fourth trace sigma a sigma b. I am just doing it for polymatrices. And so, this where give you we can rewrite this as follows the standard take is write as this as sigma a sigma b plus sigma b sigma a and, but that is equal to 2 time delta a b. So, let me follow the thing twice. So, just became into half trace and you should know this sigma a and identity into by 2 matrix.

So, if you take a trace of this you see that you get trace of this will give you two. So, it became 4 delta a b which will cancel this 4, but you left with a half. So, this is equal to minus half delta a b. So, the idea here is that what I want to do here is to write this same action in terms of this kind the matrix valued of this object. So, now. So, let us look at what we get for trace.

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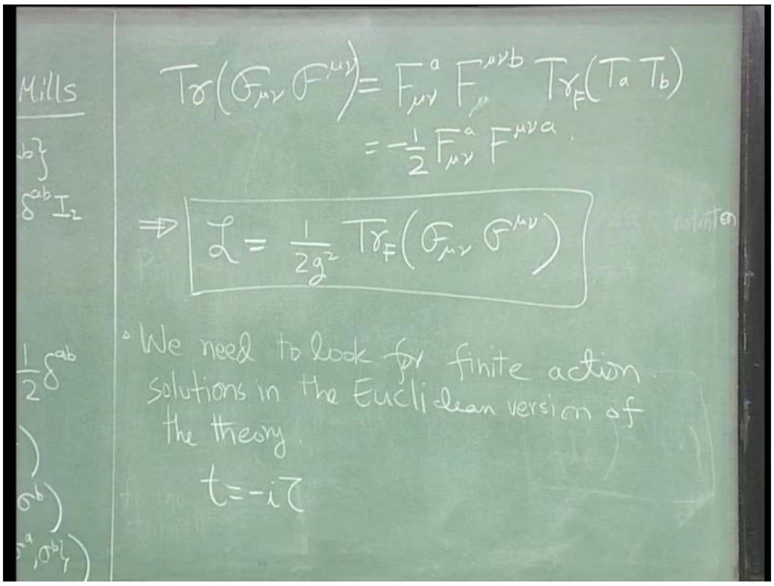

So, this is just pull out there mu nu a F mu nu b trace and we are going to work out in the fundamental representation. So. But this is going this knowledgeation at least for these thing minus half. So, you can see this is equal to. So, we can see that this same Lagrange. So, this implies that the Lagrange can be written in terms of matrix valued object as follows trace in the fundamental representation. So, it goes back to the same thing. So, it can mention to write the action like this in terms of in the fundamental representation because if I choose it in some other representation you will still get delta a b. Even the sign will work, but this constant half will became something else. So, then the number prefecture will change.

So, prefecture can be larger etc etc. So, I just want to fix this because I do not think I fixed it anywhere. So, far in the course I just want to do once for all. So, now, we also the idea is now we are going to look for. So, we need to need to look for Euclidean also solutions finite action solutions and the Euclidean version of the theory.

So, we need to do the same thing which we did and. So, so let us do that. So, first step is to take t goes to minus t t equal to i tau minus i tau and. So, for simplicity am going to do now is to do it for the abelian case. And then we just the non abelian case. We just work out in some chance. Let us do that.

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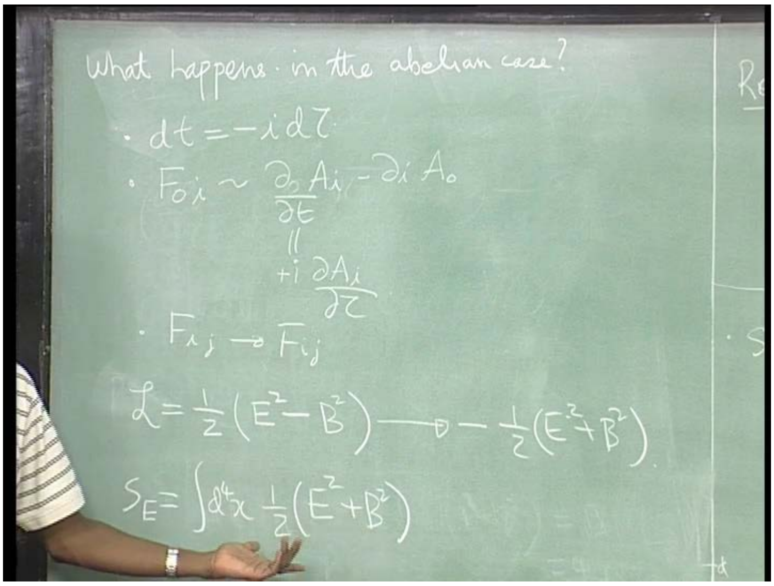

So, let us just consider the Abelian. So, the action is still the same, but there are no none of this a thing is no traces nothing. So, let us see what we need to know. So, first thing to realize that. So, d t becomes equal to minus i d tau that is good now what happens to F mu nu there F mu new can be broken up into 2 parts F 0 I which is the electric field and. So, that has derivative. So, F 0 I is d 0 A i minus d i a 0. So, what I what we need to do is to make these two go the same way. So, A 0 also should be taken to.

The one way of understand 0 comes with the wrong kinetic energy. So, we need to flip the guy’s around, but also there also lets look at how d by b t goes it just b d by d t. So, this should became. So, this goes to i plus i and I you can figure out how the other one works out. So, I just need to track of this. So, what this tells you that E i E is E goes to i e.

So, and nothing happens to b is this clear because there is no this thing. So, the only thing which picks up funny factor of I is this thing. And so, so you can see that the Lagrange density verse half E square minus B square. And so, what happens is and no no no this was correct half E square minus B square right. And so, now, you can see that if I under the wick rotation his goes to this change sign. But became overall minus half of E square plus B square.

So, the Euclidean action would look like x d 4 x half of E square plus B square. But this we should guessed it should look like Hamilton density in one high. We are work out for general dimension. We are work out the answer was more or less I mean it should it should look like this. What you get?

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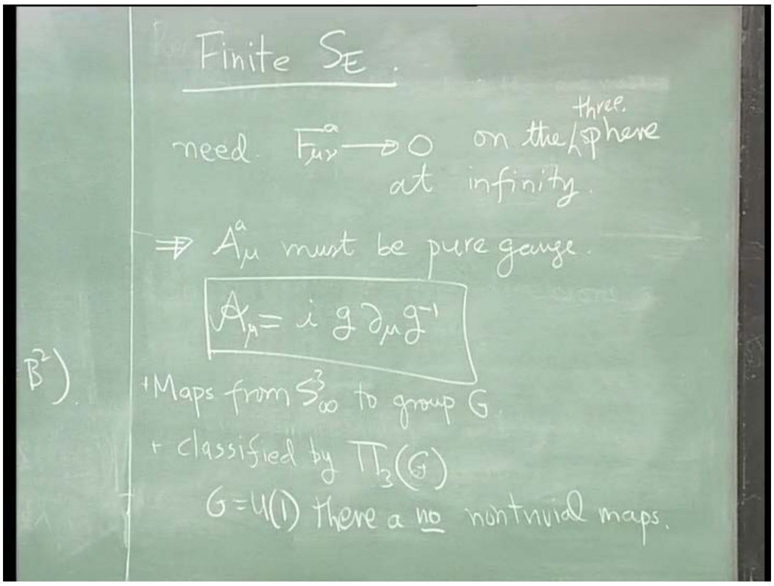

So, thus now we have Euclidean theory and what we want is finite action and last time we saw that. So, and now I return this is. So, before I proceed here I did for abelian case, but you can see it goes through in non abelian case as well it just extract pieces here. And it goes through nothing different about the whole thing.

So, now, the thing is we have look for finite S E and. So, the point here is that. So, you need F mu nu to go to 0 as on this sphere. The three sphere at special infinite at infinite. No need to call it and we saw that this implies that a mu must be pure guy's. You just go ahead and write mu as at least what should happen for larger. And we are in.

It is a three sphere because this is in 4 dimensions this what. So, there is nothing deep about the whole thing and except now. I return in terms of the le algebras valued element. You see that is why easier to write. Otherwise if I have to write interims of m u a is a headache. That is one reason to go back to this is what we need to do.

So, in other words configuration. So, what we have is we have study maps from s 3 special s 3 let us call it s three at infinite to the group g because g is the group elements and. So, this is classified by 3 of group g so.

Somebody has to come and compute this or we have to see this what this look like. So, the first example I should take Abelian case which is u one you can show that there are no nondurable facts which goes from s three to this. This thing everything be easily define smoothly deform without any discontinued or whatever because this is smoothly deform to that privies one.

So, g equal to u 1 no nondurable maths in otherworld. No configuration you cannot come up with the finite action configuration some. So, there is no nothing which is. So, the next one is s u 2, but for any non Abelian group this will go through will see for s u 2 its s 3, but the key point is any non Abelian group has always has a s u 2 sub group. So, you could get inequality s u 2 which each of which.

You could you know has you could have a many s u 2 and you could generate different instanton or different sort of not different instanton different maps which go the group. So, you can find non trivial elements quite few non trivial elements. So, if you find something for s u 2; you are more sure to find it for any non Abelian compact group.

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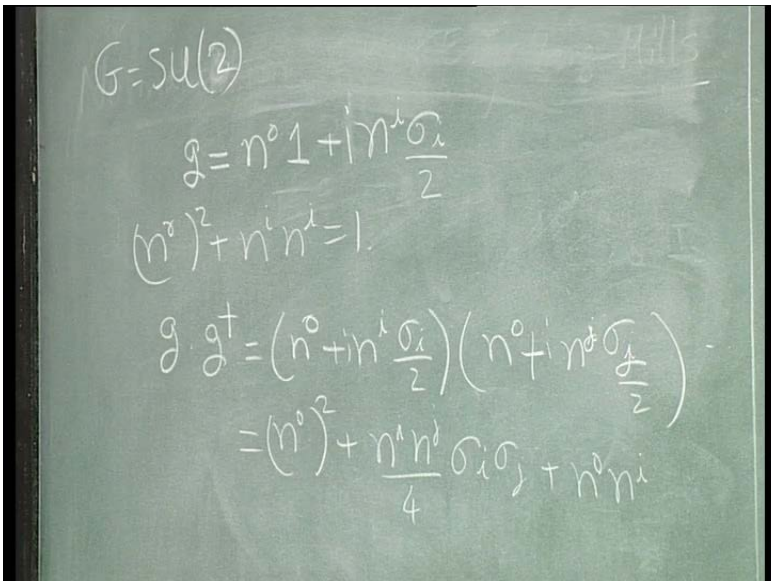

So, we stick to only G equal to s u 2, but we already see that s u 2 group man form is you could write any group elements as some I remember something n 0 identity plus n i sigma. I was it like this was there is a half or not there is no two it was this and we say that this orbitary group elements and it was just n 0 square plus n i n i was equals to 1.

So, the group man form is s 3 was the 3 sphere is these correct. Or which tow you are referring?

This two or this two that should be a two here. So, this easy way from to fix this is just look at e power i theta a t a we just expand it right we just kept 1 plus i theta a and t a should be some i sigma a by 2. So, there is no I there is no I here correct because t a will have an I and this. So, it goes off. So, that helps. So, I get something like thing there is no I here. So, now, we are in good by actually very easy to check you just says that this unit rewrite just g g this identity it should.

So, and this should be the condition for that it should check it correct let me do that so I get n 0 square plus n i n j this term am writing. So, s 4 plus the other term which is n 0 n i. So, this does not cancel I do need a n i I do need a n i. We are doing at double to many doubles that is a n i. So, there is a n i. So, this clears everything. So, I just want to write it right correctly ones.

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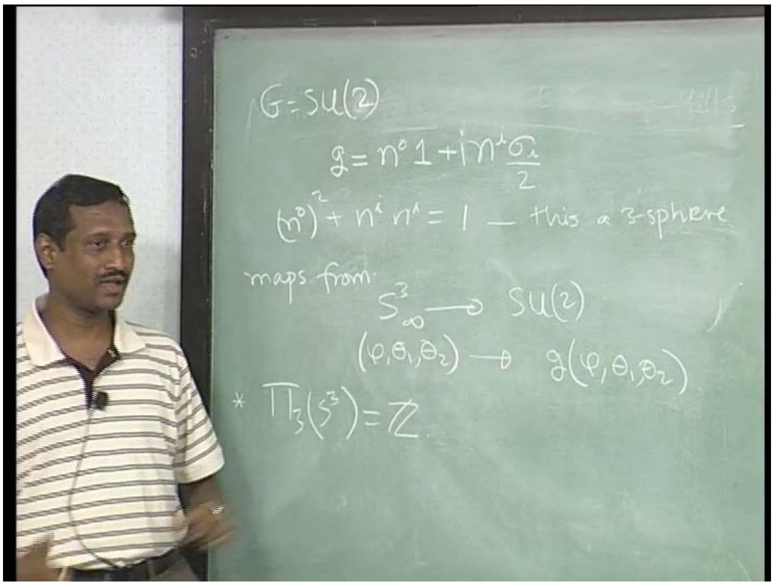

So, once we see is that n 0 square plus n i n i will be equal to 1. So, this 3 sphere. So, we have maps. So, what we get here. So, we have to look at maps from s 3 special infinite to the group to s u 2. And so, lets choose some coordinates on s 3. So, I just choose I write something and then there should a 3 coordinates. I call them phi theta 1, theta 2.

You see why am choosing this would go to group elements which would be some g of phi theta 1 theta 2 and I will just take the results. So, the phi 3 of s 3 is at we said that there is for every integer you gave me I will give you the some kind of winding number this is just statement. And so, now, I will I have to construct for you g which has this property. So, the easy thing here is to just construct something. So, roughly speaking what you want this also a 3 sphere. You want all this angle should get map to this angles in some way and I just write out the group elements.

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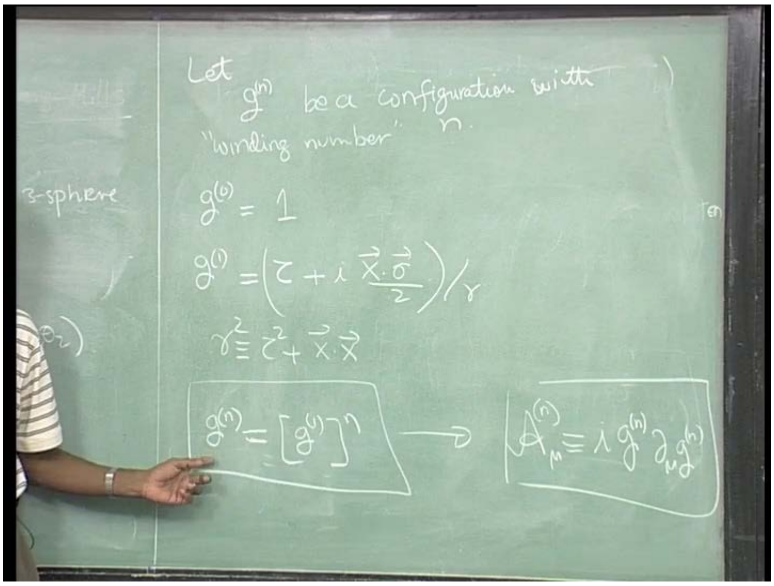

So, let us write something like this g of some n be a configuration with we will we have to define. This thing I just call it find in number n. So, whatever is this integer n. So, I have to write for you yes exactly that is where shouldering comes. So, so what I will do for you is to of course, g of 0 is a easiest one I just one choose it to be one. So, I have give g of 1 and this where you shoulder the thinks. I just I identify with tau plus I times special and divide by r where r equal to. So, this is little quantity in 4 dimension.

So, I have to dived by this because otherwise it would be square of this. If you wish this think has an analog of finding number one, but the nice thing is if you give wading number one. Now, I can construct for you for wading number n any idea how act on it n times right. So, if this is doing wading number one of course, there are finding I can just see that includes minus 1 which means take the inverse. So, we can. So, now, we can see that we are able to write out configurations with which one something have that wading number I have not written for you.

What is the wading number is you will sort that out. We will I give you definition for wading number. And so, so that will. So, what we will see that given this thing we can constructs for you a mu with the same wading number n define to be i by the way. This not the only way of writing this there might be other way of doing that. You are grantee that this gauge this has wading number one. This will have wadding number n its just inverse I put inverse sign.

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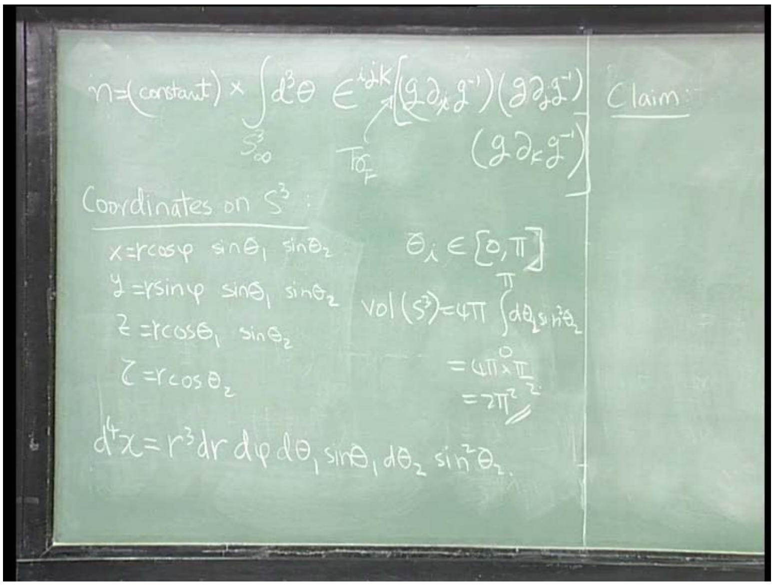

So, let us see what this wading number would be this. n will become constant which will determine in a movement times the following object which is an integral over the three sphere. At infinite choose some coordinates. I just call it has some angles this d q theta is any set of this angle. What is the natural measure of times? Something like this. So, I just define this, but we could have actually gone and done in similar fashion to what we could written. This is coming from some integral over 4 sphere some conserved topological charge. We can do that, but I will leave that as an exercise for you. We will come back to this. Right now what I want to do is to show you how to fix this constant and we will do that by choosing this guan, this solution and working out the integral. That is what we will do.

So, first thing is let choose coordinates that is the coordinate independent way of doing thing, but let me just choose coordinate s three for a movement. So, what we will do is you start with usually x y and z coordinates and then does last guy's is tau. So, what you usually do is define an angle phi through this and then we define an second thing which will call cosine theta one sine theta one. So, theta one and phi would be a normal circular polar coordinates, but now the natural way.

To go would be in the hyper sphere or three sphere is to add one more angle and just do the same thing I just do this cosine theta 2. I only thing I need to do work out the measure is I need to work out what the measure would be and that you can do by showing that d 4 of x i just call this as four variables. So, from dimension ground there will be one radial coordinates. I just have put that r also here and d 4 x would one dimension ground would be r cubed d of r. And so, for we are written out something which is normal thing. The other one I just need to write there will be a d theta 2.

So, we are work out the Jacobin and this what you get if you go to a next f 4 dimension sphere. You have to introduce one more theta kind of angle thing and that would just give you d theta 3 and sin cube theta 3. So, usually thing to remember, but coming back to is the theta we have define them you have to check what is there limits the limits of this thetas. All the theta I go from belong to 0 and phi. It belong to the interval 0 to phi and phi is the only real angle it is a goes to 0 to 2 phi. It is a circle valued object. That way use this different notation. Its worth remembering this thing and once you give this thing. You can ask what is the volume one of the nice computation to do is to do the volume of a 3 sphere.

So, the analog of. So, volume of s three. So, I will not do the r integration we will just forget about that what I mean is volume of the 2 sphere is 4 phi. What is the analog of 4 phi for this by volume I mean in that case this surface area. So, that for one third and all came from the r integration there. That is not sphere. That is not what am doing just want to do the integration over this things. So, this two integrations will give me it is like omega integration will give me a 4 phi. If I need to do only one integration which is 0 to phi b theta 1 b theta 2 sine square theta 2. This is very easy to do because I can use the identity writes 1 minus some casein two theta. That is an integral over the full this thing that average is to 0 it is just gives me a half.

So, I will be half into a phi. So, it will give me a phi by 2. So, useful result is that 4 phi into phi by 2. So, this is 2 phi square that is one more need point which is suppose I make coordination transformation. I have chosen this coordinate this is not only set of coordinate. So, on the 3 sphere I can choose any thing I want go head suppose you make another set of three coordinates and you work out how this transforms.

What you will see is that because its an epsilon on you get Jacobean. Each one will give you one thing it will combine this thing to give you a Jacobin factor precisely which is the inverse of the measure. So, in other word its the answer is coordinate variant, but this is to be accepted because like I said this you can think of raising from some topological charge which haven't shown. So, now, we are left to we need to real go back and evaluate this thing for this answers. Accept now we you may thing I need to know lot of details about this.

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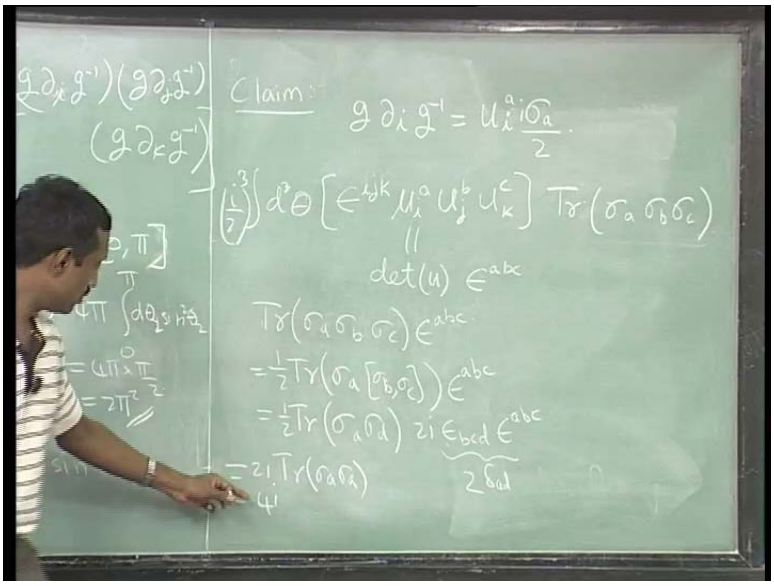

So, the thing is, but let us look at this thing and. So, the claimed here is that this object here I need to put it this a le algebra valued elements. I should putted trace. I forgot to put that. So, and we are in the fundamental representation. So, the clam here is that this guy'ss g d i g inverse. Whatever happens here this thing because of the epsilon i g k multiplying its.

So, this is like the answer will be only proportional to this ah epsilon i j k time some constant factor which we need to fix. So, one thing we could do first thing is to look at the trace of this thing and this are all le algebra valued elements. So, I could write whatever you do g i d i g inverse should be some u i a sigma a. So, its le algebra valued I by two this is what it is should be. And so, now, we lets plug this is not a deep statement this just saying that this is le algebra valued element. And we need to evaluate that we need to fix what this u i's are.

But for now so the thing is now we can plunge this and into this and ask what this looks like. So, this i by 2 whole square cube this what we have to evaluate. But what is this equal to? You can think of this as a 3 by 3. Its take three values this take 3 values. So, if you think of u i a has a 3 by 3 matrix. What does it mean? What is this object? Its determinant. This is equal to determinant of u times epsilon a b c right because I can just go ahead and take 1 2 and put a b c to be 1 2 and 3 and then I get the determent.

That is the determinant because a b c run all those thing. I this two are equal any one of this two are equal the answer will be 0. So, its forced on you this is what you get. So, you can see that now we need to just evaluate trace epsilon a b c, but the. So,. So, that just that is sigma a sigma b sigma c epsilon a b c. And this I can rewrite this as half trace sigma a commentator of sigma b sigma c what is the commentator of polymatrices of sigma a and sigma b. Its 2. I this equal to 2 i sigma 2 i epsilon b c d epsilon a b c. This is what we get d and what is this is very easy to evaluate in the following sentence.

I can just by permutation I can make it b c a this will just be delta a delta a b a d with some factor and that that factor is juts. So, we can do that by just picking some value of c take c to be 1 and then this can be 2 and 3. So, this just two times you can use that delta delta formula. But you can just evaluate truly it is just two times delta a d. So, you get four it cancels with this u get 2 i trace sigma a sigma a, but sigma a square is just identity it gives you a 4 i I will done. So, you can am going to erase this part all we need to remember this is epsilon with this gives me 4 i.

So, this I cube will give you a minus i which will cancel with this which 3 sigma for a.

Thank you. There is a 3. I missed out 3. Thank. That is the 12.

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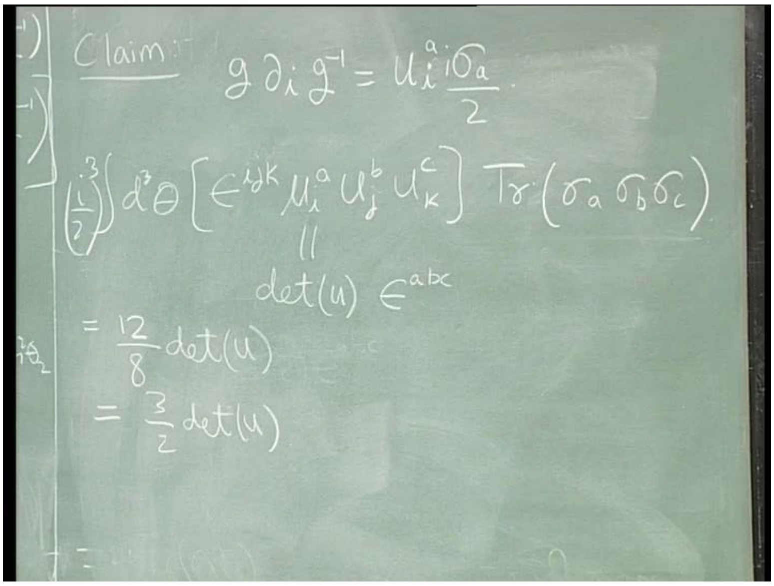

So, this is equal to. So, 12 i, but this I cubed is minus i into I which is plus i dived by h into determinate of u we stop here hope full there is no more errors. So, this is just four thirds what am doing its three half determent u. So, the problem is just left to us. Evaluating what u of a. Its what? This I will it as excise.

So, you, but I will also work out next time. But right now we just leave it. So, you can see that once we get this we can actually work out. We will what we want is to choose this constant such that we get one for this configuration. So, this am not proving that this is the configuration. I am just using this as c to fix this thing is that clear. So, I stop here we continue in the next lecture. And next lecture we will see how this how to relies a an explicit configuration which leads which has this has dot x. That is the very simple answer. You get a very simple solution.