Classical Field Theory
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Lecture – 34

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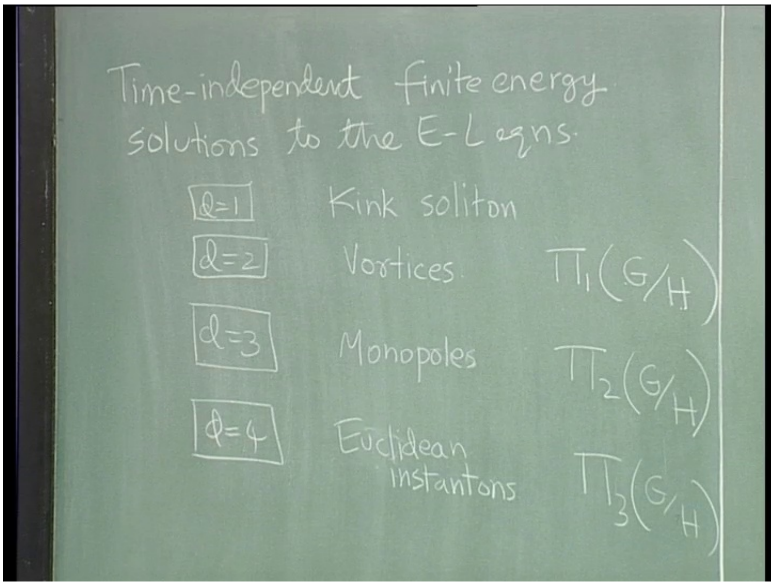

Finite energy solutions to the Euler-Lagrange equation and we saw there was a dimensionality dependent first thing we started at d equal to 1. So, this is special dimension. So, in d equal to 1 we saw kink soliton and once we went d equal to 2, this was the barrier due to Derrick's theorem and we saw that if we add a gauge field in the story we could get vortices and just remind you what we were looking was phi 1 maps from the. So, the boundary conditions were really at this thing and there was some group symmetry G broken to some H and so the example we consider was really trivial we had u 1 and there s an s 1 we got vortices which were quantized and that were carrying flux magnetic flux. The more general thing will be something like this and then we moved on to three dimensions and we saw that we got monopoles solutions. We got monopoles and now this spatial boundary is a 2 sphere.

So, you would be looking from the s 2 which is at infinity to again you have some symmetry G broken down to this, everything locally gauged, etc, etc those details but I am just writing sort of kind of things you would look at. And it is obvious that this less should continue and except one may say look we live in 3 plus 1 dimension, there is no need for us to go beyond that but just for a heck of it lets just look at what you would get at d equal to 4, then things should be classified by the boundary at infinity would be a 3 sphere. So, if you start looking at pi 3 of whatever G mod H or whatever you know something like this should be classifying these things and what you should call these objects, etc. We will see that these objects they have a name, they are called instantons Euclidean instantons; that is one way to get to 4 dimension staying in 3 plus 1 by Wick rotating time.

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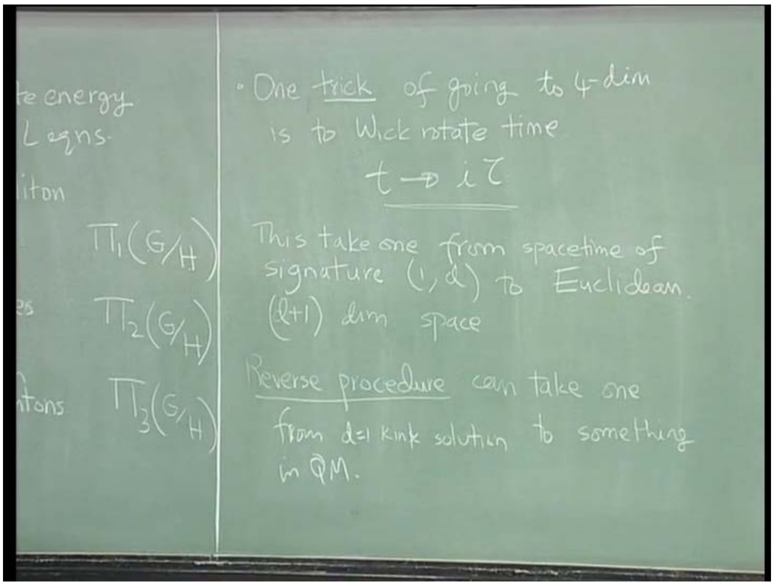

So, one trick of going to is Wick rotate time. So, you take t goes to i tau. In fact, you can see that if you. So, you can even to go back to all these various solutions and look. So, what the idea here is this takes you from take one from d plus 1 dimension to now I should not write it this way. So, space time of signature say 1 comma d to Euclidean d plus 1 dimensional space; this is what it does. So, an example of that would be to take d equal to 3 and you would get something analogues to this, but it is clear that you could go and view these solutions as it is coming from something like that. So, here we started out with a 3 dimensional space, but a solution that we get here, we could do some tinkering with it, some Wick rotation or whatever and maybe we can get a solution. So, we could also go the opposite way; we could take some direction or something and Wick rotate it and make it look like this.

So you may so, but now these solutions it would not be time-independent anymore; they will pick up particular time dependence which is dictated by whatever coordinate you are going to Wick rotate. So, you can go both ways. So, what we will do is to look at something like the kink soliton because that is small lowest dimension and we can draw nice pictures and try to understand, is there any phenomenon which could be capture by this thing. So, what I am saying here is consider the case which is 1 comma 0 dimensional and what would be such a system be? That will be a quantum mechanical system, it is not field theory.

So, if you did this and started out, so in other word you can map, so the reverse procedure; that is reverse of what I am saying out here can take you. See what is happening here is that there is time in this, but we have done is made everything time-independent and we were working with energy density or working with the Hamiltonian really. So, the thing is you could do the opposite and see if you get Lagrangian or whatever. So, let see how that works. So, reverse procedure can take one from one spatial dimension from d equal to 1 kink solution to something in quantum mechanics. So let us look at, lets draw the picture the kink soliton and act as if that were time the x dimension as time; let us just act as if that is correct and see what you get.

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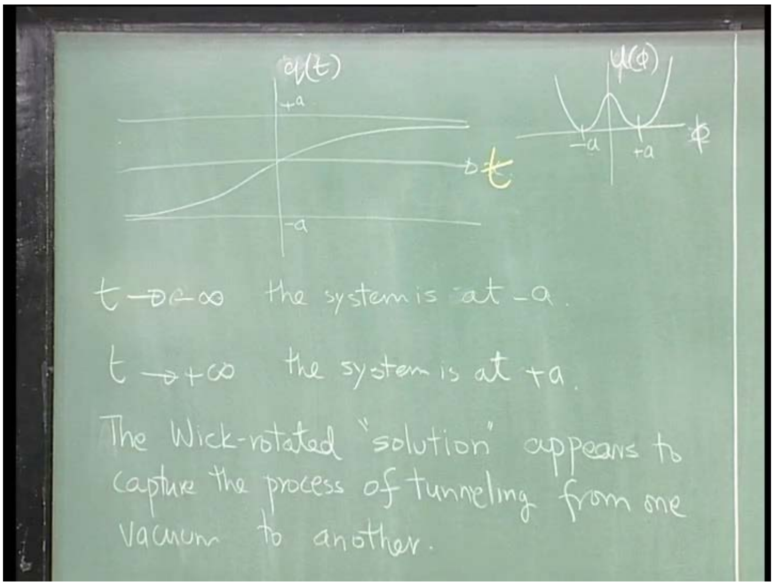

So, if you remember the kink soilton in our example we did 5 4 things there were two. So, the potential was very simple; it was something like this double well plus a minus a phi verses x. This was the potential and so we drew. So, this was the solution; sorry u of phi verses phi and this is phi of x verses x. So, if you consider the kink interpolated from something which went from, this thing pi minus a and plus a. So, now what I am saying is that let us think of this as time. Suppose this were time, let us see what is this process; can someone tell me what would the process.

So, let us look at what happens at t equal to around t tending to minus infinity. This yellow color is very bad; so, I would not use too much. So, as t tends to infinity you are studying out your system in something in this ground state. So, the state system is such that. So, phi becomes now also we should think of this as just some q of t. So, this is the replacement that we need to do, system is, say, at minus a and t equal to plus infinity, say, the system is at plus a. What would you call such a process, tunneling. So, you can see that a kink the Wick-rotated solution appears to capture the process of tunneling from one vacuum to another. So, the question is how do we understand something like this? So, I will write just one formula from quantum mechanics and then we will.

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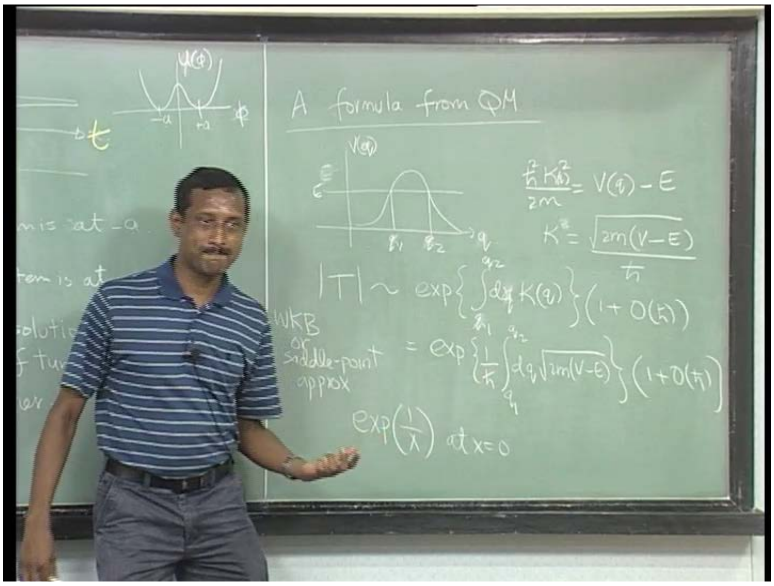

So, a formula from Q M, so the picture is as follows. What we have in mind is suppose we have a barrier and you are tunneling through that. So, we have some barrier potential v of x verses x or q, should I call it q may be. So, this is you have some energy E and so you have classical turning points which is x 1 and x 2 and you must have done some scattering problems and worked out what is the tunneling probability. So the probability, magnitude of the tunneling probability goes as exponential. So, this is the energy which we are choosing. So, let me just define h cross square kappa square upon 2 m. So, it is just a thing a particle with mass m and this thing should be equal to v of q minus. So, kappa is some q we have v of q minus E. So kappa square, kappa will be root 2 m into v minus E divided by h cross. So, this is just proportional to integral of x 1 to x 2 d x of d q; I should use q's right. So, q 1 q 2 q 1 to q 2 d q kappa q which I want to put back in terms of this because I remember it this way and which is equal to exponential of integral of 1 by h bar q 1 by q 2 d q.

So, let us look at this a bit. We know that the classical limit is h bar being small but look the h bar dependences of this thing. So, for instances this formula you can check for the easy one to do is to take the square potentials or rectangular potentials where you can solve it and do matching and stuff like that and you can work out this thing, but this formula derived that way. This way you can see u agree with that in the case of such potentials, but this is derived by something called the WKB approximation or saddle point. So, the point here is. So, whenever you have a small parameter usually you think you can do a computation such that you know the answer classical answer which is h bar equal to 0 and then you start doing corrections, but look at the behavior of this term it is not analytic at h bar equal to 0; hopefully all of you know that this function exponential of 1 by x at x equal to 0 is lousy. It is not analytic; there is no Taylor series expansion because it blows up at that point.

So, that is exactly what is going on out here. So, there is not any perturbative way of computing this thing. So, the two methods which normally one uses in quantum mechanics; one of them is w k b or another way is to show that. So, for instance the way I would show that the two ground state; the easy way to see that two ground state is to choose following things. Choose something which is Gaussian here centered here, another Gaussian centered here and estimate the energy for the correct Hamiltonian and then take liner combinations and you can show at least in certain approximate computations you can show that the plus combination the symmetric combination has much lower energy than any one of these two guys to show you that the two ground state due to tunneling etcetera is this thing.

But you could ask suppose what is the probability of starting with a solution at x equal to minus a and asking how long; that is a perfectly allowed initial configuration and such that you are in this other vacuum at infinite time which is the kind of thing which we are talking about here. So, what one is trying to say is that you cannot do any simple perturbative computation to get these things. So, such things are usually called non-perturbative. So, the thing is now how do we go about. So, the question would be is there a way of understanding the same WKB kind of answer in another method and the answer is yes, you could use path integrals.

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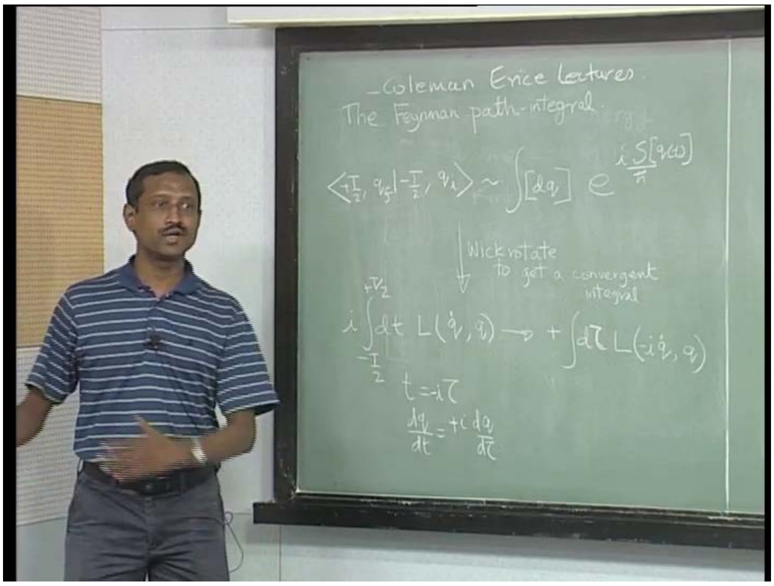

So, the Feynman path integral, so Feynman path integral actually gives you suppose you have at some initial time. So, you start out with your system being in q i and at time. So, usually it is done in the following way; you take it to be at some initial time minus t by 2, say, q initial and you want to know, what is the transition of this thing? So, overlap of this thing with the system at some later time t by 2 and goes to some final q f and later what you have in mind is you take capital t to infinity. So, it is like you start out with this sort of a thing and this just is given by the path integral and that is just some more all paths which you write something like this; some more all paths will start at q i and this initial time and go to q f and this thing times e power i times the action divided by h bar. This is something which we discussed I think in the very first lecture.

So, what we would do first is to take this action and we do; the problem with this sort of a thing is this is an oscillating integral. So, one trick is to do a Wick rotation and then you get a convergent integral and you go backwards, but Feynman does not do that, but let us say that we do a Wick rotation on this. So, so let us look at this action. So, what I would like is to do choose a Wick rotation Wick rotate. So, that I get something which is e power minus s kind of thing which is convergent, wick rotate to get a convergent integral. So, let us look what is. So, what we want is to look at this i integral this thing L d t from. So, initial minus T by 2 to plus T by 2 d t of some Lagrangian of q dot and q. So, the Wick rotation I have suggested that we have to check the sigh that it works out.

So, let us take t and say that is equal to i tau and then what else do I do. So, let us look at q dot is d q by d t and that just becomes minus i times d q by d tau. So, we can see that now this will go to once we make this substitution let not worry about the limits, but the key point here is this will became minus d tau times let us look at his Lagrangian which is wherever we saw q dot I should put minus i q dot, but on the right hand side the dot will imply maybe I should can I use dot for it; is it okay, dot now will represent on this side tau derivative. So, I keep i's explicit and nothing happens here. So this is what the Wick rotation is doing; let us follow our no's and see what this implies.

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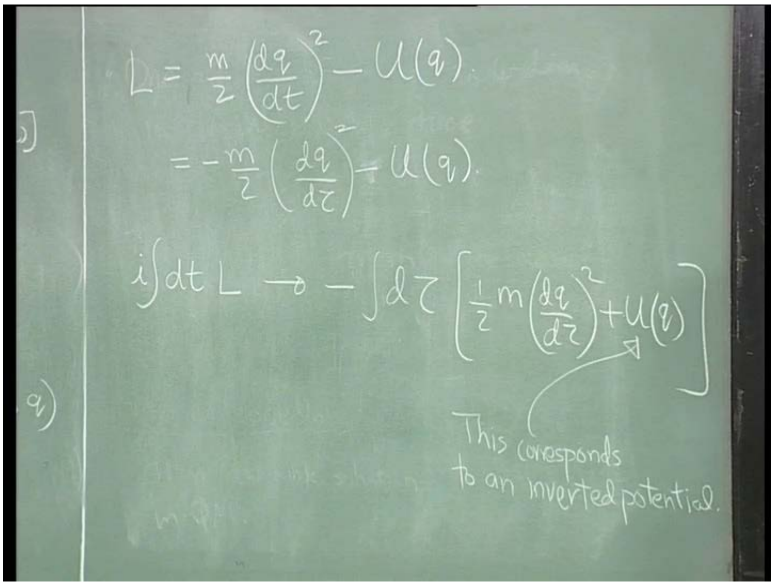

So, let us take L if it is half m v square minus u of q. So, you can see after we do this thing this just goes to after we do this substitution you can see that just this becomes minus m by 2; I am bit unhappy because the reason I am unhappy is because now I mean I wanted convergence. So, I need to I think take tau to minus i tau. The limits also we have to take. Which limits? Minus d by 2 d by 2, I think that will be i equal to minus i.

Yeah yeah yeah, I have not written the limits here; that is why I said I jumped the limits, but I am more in thinking in terms of having a convergent factor. So, suppose we assumed that both u and q are nice these things then what we can see here is that this gives a minus sign. Previously we had to find the rotational cost from space time to space but here we are going to time through. No, the thing is I am going to put a minus sign here. So, that I get something convergent. So, then I just get this to be plus, but that minus sign will take care of; is this correct if I wick rotate d t goes to this thing. Which one? This will become plus that will not change this minus sign.

So, now what you can see here is that integral of d t L i integral d t goes to minus integral of. Oh okay okay, so I should going from here, I should be now tau is. Yeah yeah yeah, so that was fine right so the earlier thing was to go from space to it depends, so this is fine; this is okay. So, now tau is something like a Euclidean coordinate. So, it is closer to what we have written for an x. Yeah, good. So, what we get is a d tau of. So, now you can see that if you work out the equations of motion for this sort of a guy, this looks like a normal Lagrangian except there is one change, the potential is inverted. So, in this Euclidean problem we started out with this guy but what we see is that the potential is inverted and, but nevertheless we can I mean it is not so bad. I mean the point here is we are actually at the end of the day we are interested in this transition probability we would like to compute something like that. So, let us draw this potential out here.

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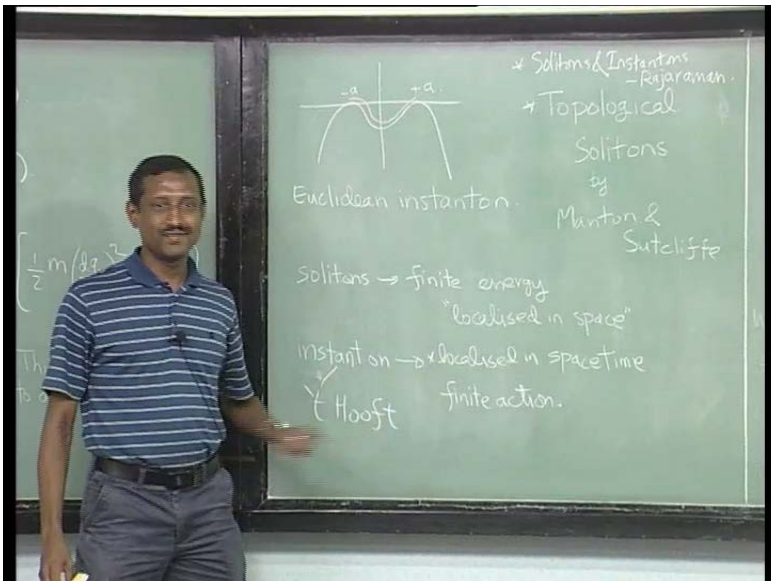

So, the potential would start looking something it is inverted. So, this is what you would get and if you are looking for a solution which is interpolating, did I get the signs correct, interpolating between these things you would look for a solution which does this. So, these kinds of a solution have a name. So, this called Euclidean instantons and I should explain to you the origin of the name. So, the origin of the name is first thing is if you remember we looked at; yeah, you have a question, because here if I think of this as a Lagrangian for some Euclidean this thing there is a minus sign out here, it is like a flame. Yeah in this in the Euclidean it is minus u of q if you think of this as d minus. So, the name is on is because it is somewhat similar to soliton except when we looked at solitons we looked at one feature was there at finite energy and we also notice that they were localized in space, but now if you think I will just take this guy and go back through that procedure which we did earlier which is to do this kind of Wick rotation. So, then something which was localized in space will become localized in time also.

So it is at an instant. So, it is not just localized in space but also in time. So, an instanton is localized in space time and if you want to think completely in space time what you would look; instead of looking for energy you look for action finite action which after you Wick rotate will become finite energy in some times. So, it is something which I have not proved but we will see a little later in probably in next lectures also that all the solitons that we wrote can always be converted in to instantons in a suitable setting. So, this the difference between an instanton spelling is wrong and this was coined due to by ‘t Hooft and I think Polyakov called it pseudo particle but that has not caught on for whatever reason but this is the name which is this thing. So, for instance this gives you the title of a very nice book on solitons by Rajaraman; it is called Solitons and Instantons. That book is slightly higher level than this course, but I would recommend now that you have done this course you can go and look at that book and you will find that you will understand lots in that, lots and lots of it.

I also a discovered a nice book called Topological Solitons by Manton and Sutcliffe. So, this is one book and there is of course the one I just mentioned Solitons and Instantons and for the idea of instantons, etc again something I mentioned all the time this for me has been very inspiring is Coleman Ericha lectures. I have very hardly used these books; I have looked at either original papers or looked at these things. So, now we have a working definition of what we mean by an instanton. We have to look for objects which are localized in both space and time and sort of give rise to a finite action as supposed to finite energy. So, lets us see how we would get a formula something which looks a bit like this. So, now I have to decide what I should erase. So, now what we have is nice path integral.

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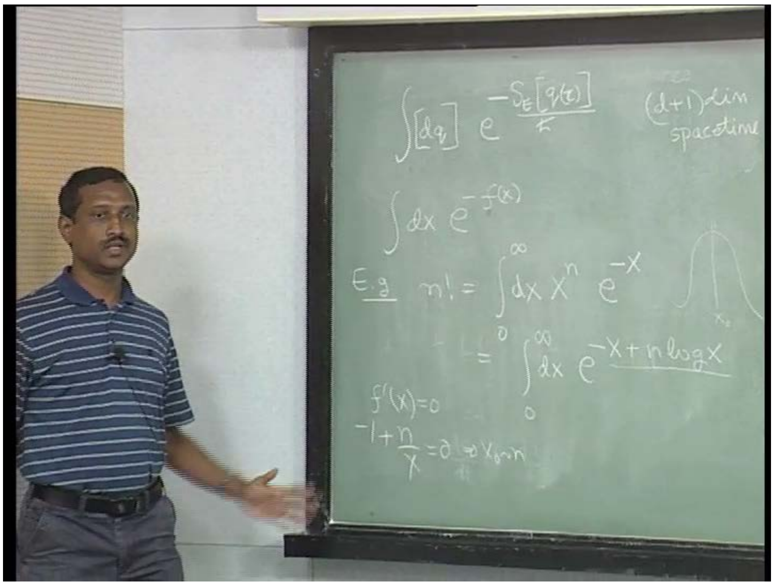

So, we need to carry out an integral of this kind d q e power minus S Euclidean q of tau; is there any question. What I am confused about is if the limits do not matter, then why the minus sign would matter because we would just look the limits and get a minus sign for the action. No no, I did not write the limits. So, I mean well it will matter, we will see. For something's it does not matter, but the order does matter I mean. So, let us see how we would do this sort of an integral and this is an infinite dimensional integral. So, what we will do is do something, look at simple example of, say, d x say e power, so divided by h bar of course, e power say some minus f of x by. We will look at something like this; a function which has something like this and we would like to evaluate this by the saddle point.

So, a nice fun example to do is to consider the gamma function or n factorial. So, we can derive. So, n factorial is equal to integral d x from in this case 0 to infinite x power n e power minus x is it; is it x power n or n plus 1 I never remember this; its right, I just got lucky. So, I can make it look like this, is this okay. I mean for my purposes n and n plus 1 will not matter. So, it is an example of this sort of an integral. So, we can just rewrite this as integral from 0 to infinity d of x goes like e power minus x plus you can just. So, this would be a case where it is f of x; f of x is this quantity x minus and log x. So, this is a nice integral; in the sense this is increasing function and this is a decreasing function.

So, if you plot this thing it has only one maximum. So, what you would do in evaluating this; so let us call this maximum x naught. So, you want to evaluate this and you want to evaluate it for some large n. So, the way you do solving formula is to say let us evaluate this guy at a saddle point and that is just taken by taking this thing and extremizing it. So, you just take f of x. So, you will solve for f prime of x x naught. So, we have to solve for this equation. So, if you just do that out here what do we get; we get minus 1 plus n by x should be equal to 0 imply that x naught is n and what you can do is to further expand it, I did not want to erase this. So, you just go head and expand it about this thing because it is truly x minimum.

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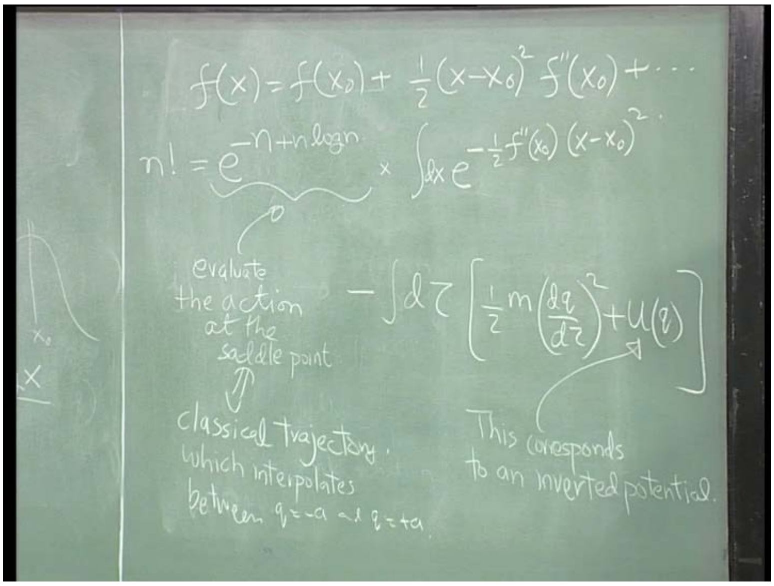

You will see that f of x will be f of x naught plus the first derivative vanishes and because it is a maximum it will have the correct sign; no one second I should be, yeah so minimum maximum. So I should just, yeah it will go the right way is what I am saying. So, what I want is. So, it has to whatever happen f of x is. So, actually in the case f of x and what we have in thing is a minus sign. So, it will work out correctly. So, what was I saying? So, this function is like this but what we will get up there because of the minus sign will be something which will be nice and upwards. So, you will get x half of x minus x naught whole square f double prime of x naught plus dot dot dot. So, first approximation is to say that you just substitute for this guy with f of x naught n factorial.

So, this is for large n we would like to evaluate this integral and you will see that the limit would not matters, but they were here this thing. So, we will see. So, what is it; so, let us just evaluate that. So, we just go head and put X naught that would be just e power, my x is just n times. Now the key is that in principle you should say yeah, I should do it from 0 to infinity, but this will get just give you a Gaussian. It is just a Gaussian integral which would be e power. So, now I will just write something and now I will change the limits; I will say that for large n this is fairly you can see that the width keeps reducing. So, in quantum mechanics you will see that this h bar will determine the width and you can see that in the limit when h bar becomes small you can see that you can make things as sharp as possible.

So, what one says is this saddle point is becoming better and better. So, in this case similar thing will be played by 1 upon n, you can convince yourself that is the thing. So, if you just work this out; you can already see that we got we got the leading term in Stirling’s formula, but you can also do this Gaussian integral and you will get this important prefactor. By the way what was Stirling’s contribution to Stirling’s formula; it was only the constant by the way. It was otherwise is general form which we know was already known. So, this is the kind of things you would do. So, we can do the same thing out here. So, what would be the leading thing would be the contribution to this would be just taking the solution to the equation.

So, the analog of x naught would be looking for a solution for the Euler Lagrange’s equation and you evaluate the action; the analog of this is just to evaluate the action at the saddle point. So, in other words but what is the saddle point in this case; it is just our normal trajectory intra this thing, this is nothing but the classical trajectory which interpolates between whatever in this example. So, in condensed matter balance one would say that this is the mean-field answer and these are Gaussian fluctuations. If you have seen it in condensed matter this what you would have done whenever you are doing; these things that though we will put in the fluctuations and get this that is what you mean, do this integral; it is not as simple as this.

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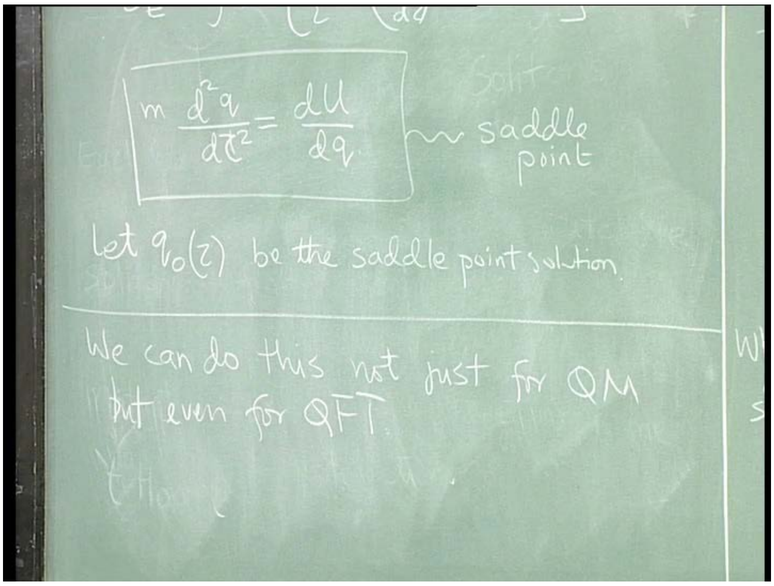

So, now you can see that what is the structure, what is S Euclidean. So, the equation of motion would be just d square q m d square q by d tau square should be equal to d u by d q; is this correct. So, just going back to you can see that if I go backwards and Wick rotate back or whatever you can see that the signs will work out actually with what you needed. So, this is the equation that you need. So, this is the saddle point and let q naught of tau be the saddle point solution. Actually we can do I mean this is the one dimensional system, we can do one integration very easily and so we could do that, right. So, in some sense you specify the energy or whatever and then you can do. So, you can go back and substitute for. So, equivalently yeah what am I saying? So you can see, yeah I would not say anything more.

So, this is what is going on. So, now it is very clear I mean next time I will make things more precise where I will take of a kink soliton and we will see how it works or whatever. So, the idea here is that we can do this not just for QM, but even for QFT, but what do I mean by that. So, here we did things in 0 plus 1 dimensional time; we could work with d plus 1 dimensional space time and all these arguments actually just go through. There is nothing different about the thing; it is just that there would be in addition to a time integral d tau integral there will be in all this formulae you start putting d d of x; thank god I used q so that it does not get confused with x. So, what we will do now is to I will just quickly jump back to 3 plus 1 dimensions to discuss what actually is our goal to understand what are called instantons in 4 dimensions or 3 plus 1 dimensions. So, we look at instantons.

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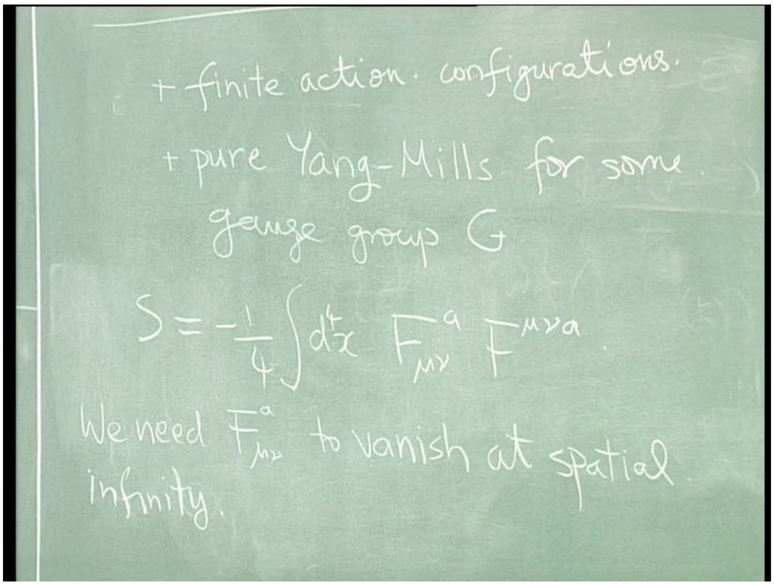

So, we will take a case in 3 plus 1 dimensions. So, now we will not do any Wick rotation or anything, we will stay with 3 plus 1 dimensions but we will remember what we did from this thing we look for finite action configurations. So, what would be that? So, let us just consider the case of pure Yang-Mills for some gauge field for some gauge group G; there is no matter nothing, we just construct pure Yang-Mills. So, in that senses it is somewhat different from what we looked in all the other cases. There were scalar fields going around but we also saw when we looked at the monopole we could think of the phi field as coming from something in a higher dimension coming from the gauges field.

So, it is not so wild to say that we look for only pure Yang-Mills, the action is just S will be minus one-fourth or whatever integral of d 4 x times trace in some representation; I do not know fundamental may be of or just write it straight as f mu nu a . So, this is the kind of action we are looking for finite this thing. So, what we should do is now is to put some behavior such that. So, the key thing is we need f to vanish at spatial infinity. So, we need. So, now the question is how do we do this, how can we achieve that? So, it should say that as you go to both spatial and time-like infinities your gauge field can go to 0.

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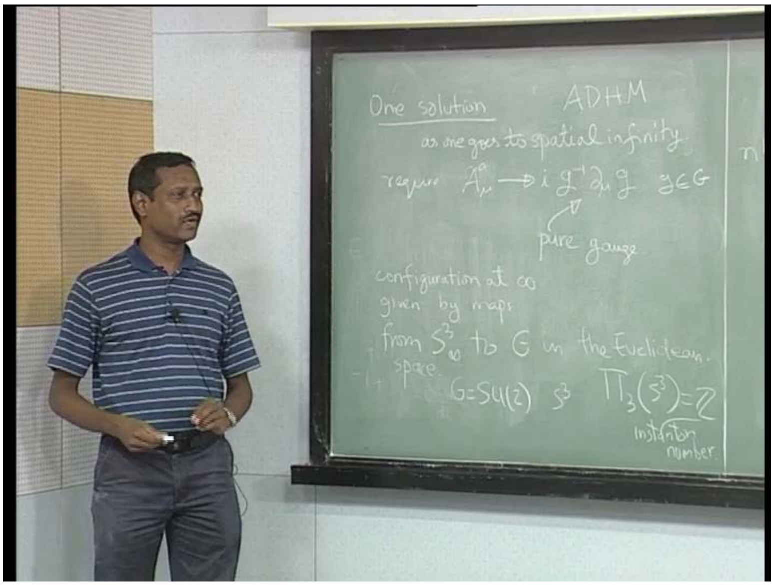

So, one solution say that as one goes to spatial infinity require A mu a to go to 0 but this not a gauge invariant statement. So, can we replace it with something nicer? If you start out with the gauge felid which is 0 if I make a gauge transformation what does it go to? It goes to pure gauge. So, instead of saying this we just say that it goes to something like g inverse d mu g, maybe there is some i or whatever; this is pure gauge. So, this is the kind of boundary condition that we have to impose. This is much weaker than this thing and just for purpose of this thing just consider I will keep going back and forth I started of this with Minkowski, but we will stick to Minkowski later. So, we will just go to Euclidean things where I can think of space, time as an S 3. So, what you will see is that you will have configurations at infinity given by maps from S 3 of infinity to g in the Euclidean version, not little g but the group element in Euclidean space.

So, let us take the case of G equal to SU 2 then the group manifold is S 3 also an S 3 and we know one thing; we know that pi 3 of S 3 is there is an integer worth of these maps. So, there will be some object which is quantized and there will be no smooth way of going from one or if suppose you find something which has this integer being one it cannot go to zero sector. So, what we will find is in the Euclidean theory that this is similar to what we had looked earlier that there is an integer worth like. So, this charge this will be like in the case of monopole pi 2 of S 2 when we had looked at that; there we interpreted that as a monopole charge, here this is just called the instanton numbers.

So, I finish with a question which is think can we write such configurations which have this instanton number and the first solutions for various instanton numbers were given by ‘t Hooft and then it has been generalized to by truly high power four mathematicians A D H and M and each of them is a big mathematician; they wrote one and half page paper I think in physics letter where they wrote out the most general solution and that solution. So, the ‘t Hooft solution is very much like you write out something, we will do some completing of squares kind of trick and you saturate some bound and you get some equation. You solve that equation, but this equation was amazing. It is not written in that language its written let us consider some n by m matrix which satisfies this equation and then the consistency conditions turn out to be the same equations we would have solved and out the blue they are able to write it. But today we understand we have a nice geometrical way of understanding it, but when they have wrote it out I mean it was just out of the blue. So, there is lots of fun stuff which will come. So, I will just stop here for today.