Classical Field Theory

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Lecture – 32

The finite energy solutions involving just scalars in dimensions greater than one plus one; so, the thing is to we have already seen one example of vertices, where we got around their exterior we got finite energy solutions. And the thing is but I did not explain why that happened and the Hooft-Polyakov monopole also fits in that category. It is time independent, we are looking for finite energy or we will be looking for finite energy. So, what we will do now is to ask the same questions again and also in the process derive the boundary conditions; I only derived one condition yesterday which was to say that when r goes to infinity the value of the. So the phi square should take whatever is the minimum of the potential. So, let us go back to let us let us work with three plus one I will come back to two plus one to explain why even the vertex thing works.

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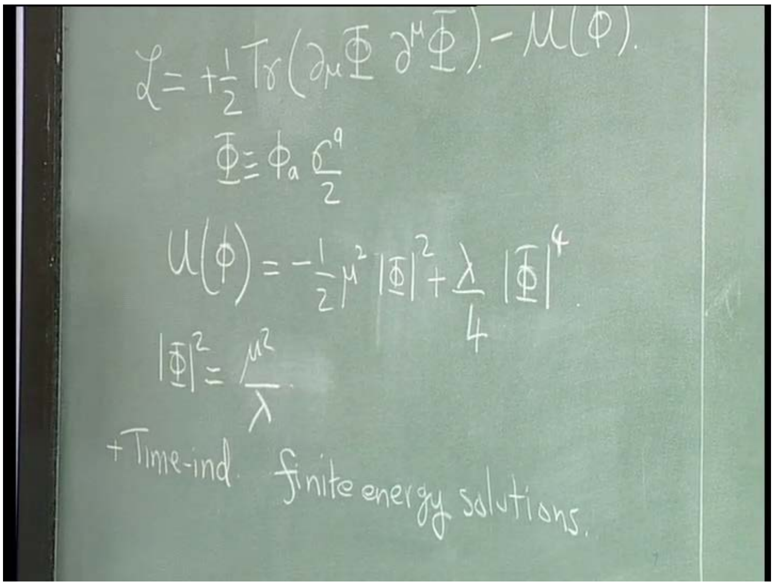

So, still we will be in 3 plus 1 dimensions and let us choose like we had the exact same setup we will have a triplet of scalars, so three real scalars which are in the adjoined of SU 2 and so the Lagrangian is just the usual thing but let me write it. I will keep going back and forth that we will write it as a column vector or the thing. So, let me write something here because some formulae become compact; I should may be put a transpose of this will do I think, yeah, this should do where phi is just the usual three guy's which you have phi a sigma a by 2.

These are real scalars, so I do not need to put star and stuff like that and we will have a U phi and U phi can be exactly what we choose earlier which was written in a slightly different form. So, let me write that and lets write it like this. So, this is the usual aquatic potential that we know and this thing has minimum event mod phi square is equal to a constant and that constant is the minimum of this potential. What was that minimum? It was some m mu by mu square by lambda. So, the idea is that we will look for we will do what Derrick said cannot be done; we will look for time independent finite energy solutions. These cannot exist but before we get to this.

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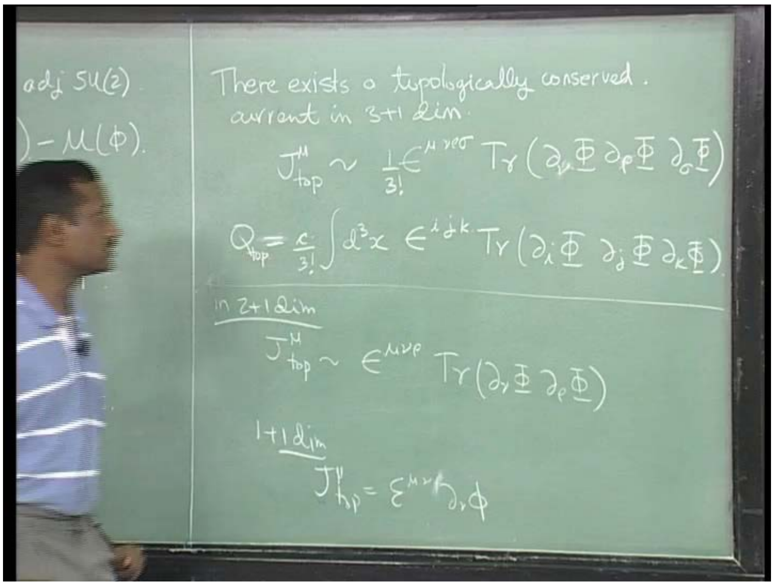

So, before we get to this we will see that there is a topological leads conserved current. So, in other words by this I mean something which is conserved without using equations of motion. So, I will not put an equal to right now, later we will put an equal to this thing; J mu topological which I will write as epsilon mu nu rho sigma trace of d mu phi mu d nu d rho phi d. So, this trace is put because these are this way when you write it this way these are matrix valued guys and because of the anti-symmetry between these things I could even write this thing as a commutator of these two guys with a half or whatever.

So, let us even do that. Let us put a three factorial here to take care of certain commutories and so we can define j zero. So, there will be a Q which I will define with some constant which will fix in a moment trans integral d cubed over all space and put mu equal to zero it becomes just epsilon i j k; by the way is it obvious that this current is conserved. It is because when you do d mu of this thing you will have three terms but each one of them you will have d mu d nu of phi which this is anti-symmetric and that is symmetric and so it vanishes, the same argument as before and so for each one of these terms that is indeed true. So, while we are at it we can also write in 2 plus 1 dimensions I can write a similar J mu topological which would now just be similar to this but only three such guys.

So, one might think that this could be any phi will do but now for instance suppose I had just two scalar fields, what will happen to this kind of a charge? If you look at the derivatives, there are only two scalar fields, it follows effectively that whatever you do; like I said there is an anti-symmetry in the problem, you will find that there is not enough space. So, you need at least three scalar fields; you can have more than that that will be okay. So, that is why we needed three real scalars. A similar argument will tell you here that you need at least two. If you write them as phi 1, phi 2, you will see that you can write something and anti-symmetrize. If you anti-symmetrize two identical object it is zero; there is not enough space. So, you can see that even though I am writing a trace out here I mean and with three I could have done it with only two. Again you can see that this is conserved for the same reason that this was conserved.

And you can go down to 1 plus 1 dimension and you can just make do with one scalar one real scalar and homework exercise to write the ones in greater than three; it is not too hard. So coming back to this problem which tells you that there is this kind of charge which might be possible and so we are going to choose it to be what we have considered here. So, now the point here is that this is a total derivative again. You can see that because I can pull out this d i; any one of these guys I can pull them out and when it acts on it. When the derivative acts on the other two terms, it would be it again for the same reason that this vanished that would vanish, so it is really a total derivative. So, this particular charge can be written as a surface integral. So, Q topological; now this is just as a remainder of the other things we will continue with this.

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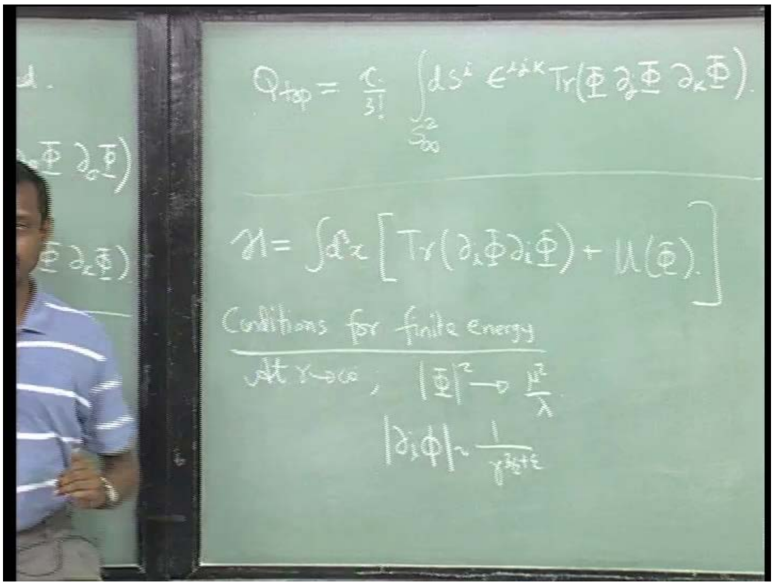

So Q topological will be some constant divided by 3 factorial times; should it be 3 factorial or 2 factorial, it does not matter, that we can observe in the constant. So, I could write this as a integral over the sphere at to infinity using Gauss's divergence theorem because we are just in number three dimensions times d s i. So, this is just because it is a total derivative. So, it is going to be this way. So, now coming back to this set up let us ask can we write time independent finite energy solutions which carry this charge. So, let us look at the Hamiltonian density for this thing. Now the point is there would be a time derivative parts, because it is time independent I am just saving time, I will not write the thing. So, I am just going to write out the special parts, it would be d i trace of d i of phi. So, now let us write out the conditions for finite energy. So, this is the boundary conditions if you wish at of course r tending to infinity; first thing you require is this U of phi go to its lowest value and remember there is a constant here which you need to add, so that the boring infinity is removed.

So, let us assume we have done that. So, we need mod phi square; this is something we discussed yesterday should go to mu square upon lambda; that takes care of this term. So, this term vanishes quite fast let us say. What about this? This tells you that you are in three dimension; so this better at least go like 1 by r power 3 by 2 plus some epsilon. So, if it goes like exactly 3 by 2 it will become the radial integral by r square d r divided by r cube, so it will be 1 by r. So, it will go away from the log and then you are fine. So, these are the two conditions that you require for this thing but now let us as what will be the conditions. So, now that we have written this topological guy as something at infinity, so first thing you can see is that this sort of fixes this guy to go to some constant value but now we have plugged this thing in here and ask how will Q topological go.

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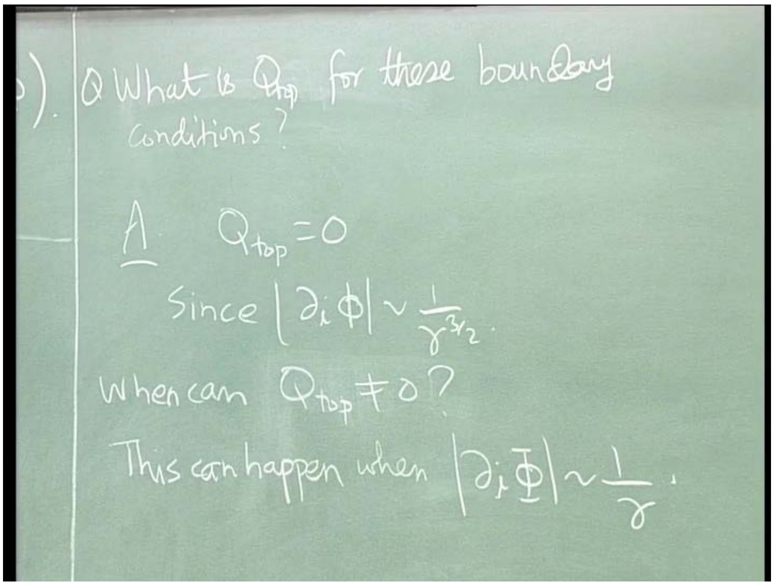

How does Q topological or what is Q topological for these boundary conditions? The answer is very easy; this is the question, the answer is Q topological equal to 0. So, let us say that this for large r goes as you think but now these go as 1 by r power 3 by 2, 1 by r power 3 by 2; that becomes one by r cubed but now this is a surface integral which goes like r square. So, it is just simple. So, basically what that tells you is that since it goes like 1 by r power 3 by 2. So, it falls of and strictly you remember this is where you have to put that thing and take it to infinity. So, what it tells you is that you might think of configurations which have topological charge but those cannot be realized in finite energy.

So, this is called I mean in some sense this have another way of understanding Derrick's theorem. So, there is no conflict. So, we saw this fancy thing; also we thought we could get something to carry that charge. So, but let us ask, so when can d i of phi go as or when can Q topological be not equal to zero. So, just we go back in just like we did finite energy we can ask what these things happened. So, you can see here that let us say that phi goes to this constant but the i of phi if it goes like 1 by r, then the r square can cancel and there is not even plus epsilon or anything; one if it has a term which is 1 by r you are in good shape. This can happen when there is a chance but quite clearly this cannot happen with just pure scalar fields.

So, there are two routes to actually getting around this thing. One route is to add gauge field which is what we going to do; the second route is to actually go ahead change your Lagrangian make it nonlinear then the energy, etc conditions would be different. Both routes actually work and so you can implicit in Derrick's theorem is the fact that you assume that your kinetic energy is this; the scaling arguments everything that we did really was based on this kind of Lagrangian. So, if you work with nonlinear models; for instance if you put something like some function of phi out here which is such that it will you know force things to work out nice but that is not that is not something we will discuss right now. So, the thing is so what happens when we add gauge fields which gets around this.

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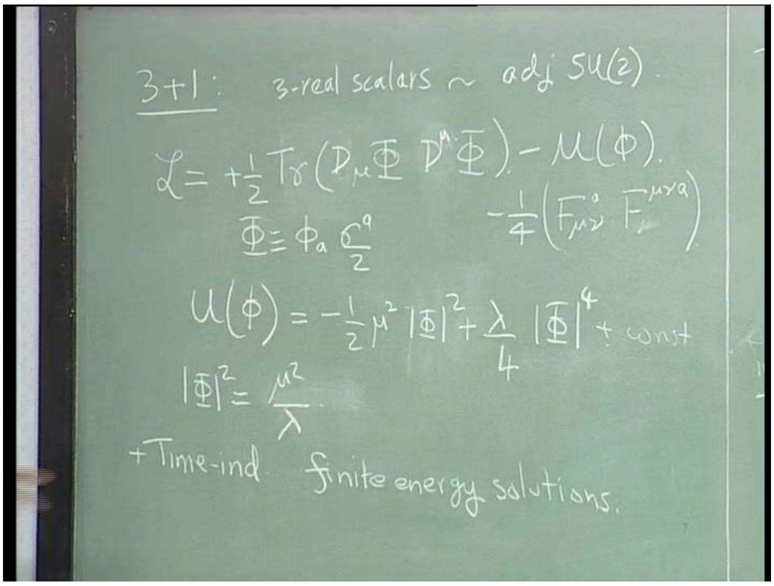

So let us do that. So, first step is to replace this by covariant derivatives and there will be one more which is minus one forth f mu nu a. So, this would be the Lagrangian density. So, this is what we started out with last lecture. So, this is exactly of that but the thing here is this. So, if you come back to the Hamiltonian density again the same thing happens.

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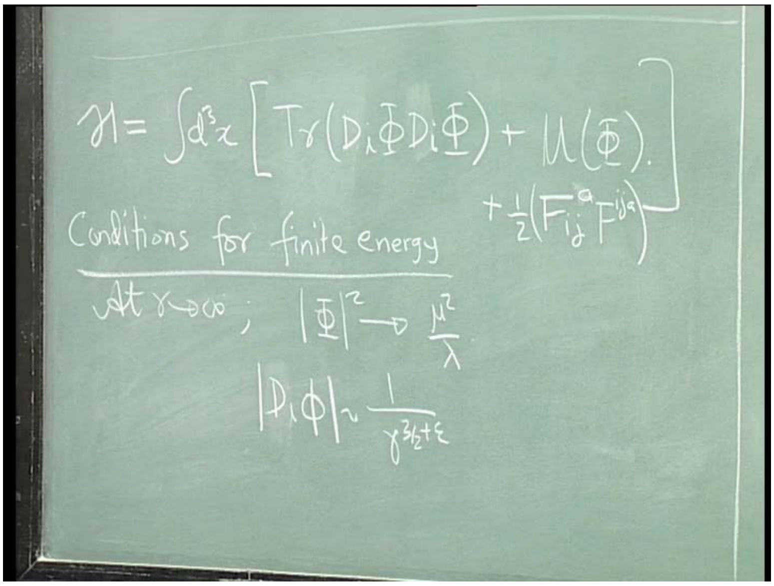

This becomes this plus we chose remember a zero was zero, everything's were independent, so there is no electric thing. So, again I am not writing this piece but there will be a piece coming from F ij half probably F ij square. Its non-abelian so I should actually be careful, I should write it like this. So, now you look at this thing your phi the requirement is exactly like this but the change here is now that this got replaced; the ordinary derivative got replaced by the covariant derivative but the topological charge still requires this. So, now you can see that what you need is something weaker; you can satisfy this provided the gauge field also conspires the gauge field configuration you also have that extra field has a condition.

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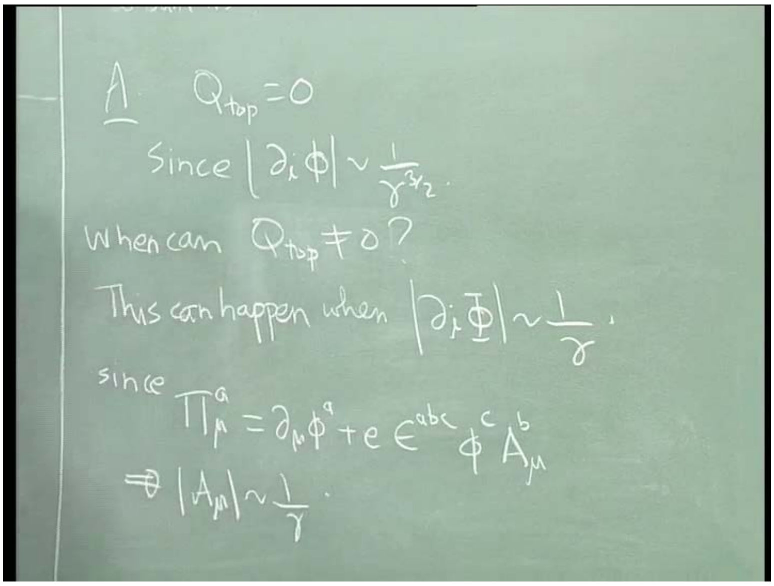

So, since d i phi or this was just we wrote phi mu a, what was that equal to; I just want to get some definitions correct. So, this is the same as pi mu; yeah, the most important guy I forgot. So, what this tells you is that you can arrange things such that A mu, so this goes like 1 by r, this should also go by 1 by r but some there should be this cancellation. So, in other words 1 by r part of d mu of phi A actually determines the 1 by r part of A mu. Now coming back to we still have to go back and check what about this. So, if A mu goes like 1 by r, f will go by 1 by r square, it will go by 1 by r power 4, you are in good shape. So, that does not still conflicts, so there is no conflict. This term goes to zero. So, there is no conflict. So, looking at this we need implies that in some sense A mu should go at least like 1 by r and you do not have freedom with the coefficients in front of that 1 by r; it better be such that this cancels this. So, I would recommend that you go back to the vertex example that we looked at and go through the same arguments; you will see that things work out even in precisely in the same fashion.

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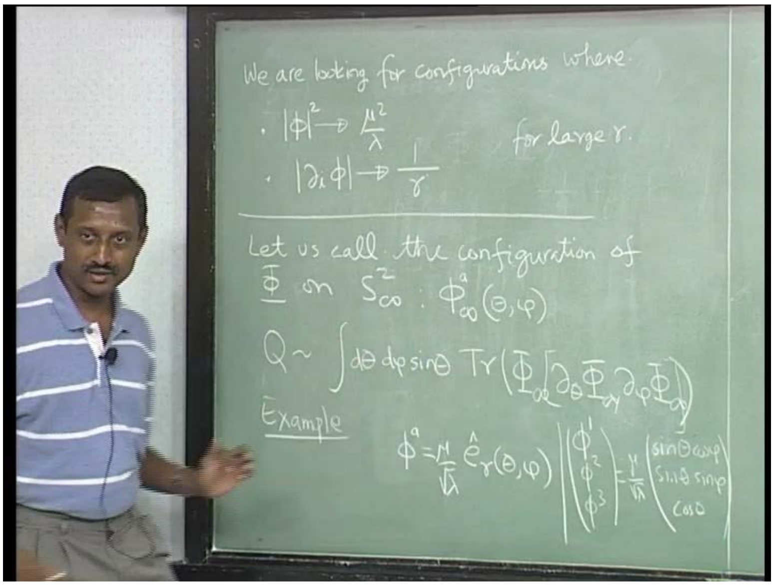

So, now that we understand that now the question is we have added gauge fields, we are looking for configurations where mod phi square tends to mu square upon lambda and d i of phi go to zero as 1 by r for large r, goes to zero as 1 by r. So, these are the things and so now the question is how do we go about, so let us make some definition. So, let us call the phi configuration the configuration of phi at on S 2 of infinity on the circular; at infinity there are only two variables, its theta and phi. So, we just call it something like phi infinity, I will keep going back and forth between the matrix and this thing depending on which is convenient for me. I will just call it phi a infinity of theta and phi and further so we want to ask what would be the contribution. So, let us look at this. So, what is happening here, we have taken a circle at infinity, so this i direction will be the r.

So, these will become the theta and phi derivatives. So then Q up to some factors we should define as integral over this thing which is d theta d phi sin theta which is the angle integrations that is a circle as we are at infinity times phi at infinity. So, trace of phi infinity times. Now the other ones are just d theta. This is it at infinity of course; this would be the topological charge. So, now what I will do is I will fix this by taking the simplest configuration where I take phi. So let us do an example which I know is minimal charge which will let me to fix the constant for you. So the example there is, so, yeah, thank you. So, I need to anti-symmetrize but that will just give me factors. So, I just put one of them, the other one will not contribute. So, I am just jumping some steps; is that correct.

Just let me get to the next step and then we will come back to this we will ask. So, it is like what I have in mind is you know if I have epsilon, no, no maybe I should anti-symmetrize; yeah, maybe I should write like this. If I am writing in matrix form so that it takes care of it. So, now what we will do is because of this condition it is like this is also. So, we know that this phi infinity of a is also set of points on a circle of radius square which is mu square upon lambda; that is what we have. So, we can just go ahead and choose for instance; let me choose phi of a to be square root or mu by root lambda and choose it to be along the radial direction. So, here is a case where I am doing the soldering.

This is also soldering because a here as we started out is the gauge index or the SU 2 adjoint index but on this side this e r has a normal space vectors these three components or if you feel uncomfortable with this I can just say that phi 1 phi 2 phi 3 is equal to mu by root lambda times; how does it go, sign theta cos phi sign theta sign phi and cos theta. You will see why it is better to write it in this fashion. So, I will keep working in this fashion. So, what we need to do here is to work out what is the d theta and the d phi derivatives of this but if you think in terms of unit vectors it is very easy whatever you write here as d theta of this d theta of this is e theta and d phi of e r is just one by sign, I think there will be sign thetas will get cancelled out, whatever I mean there will be, so this will just become e phi.

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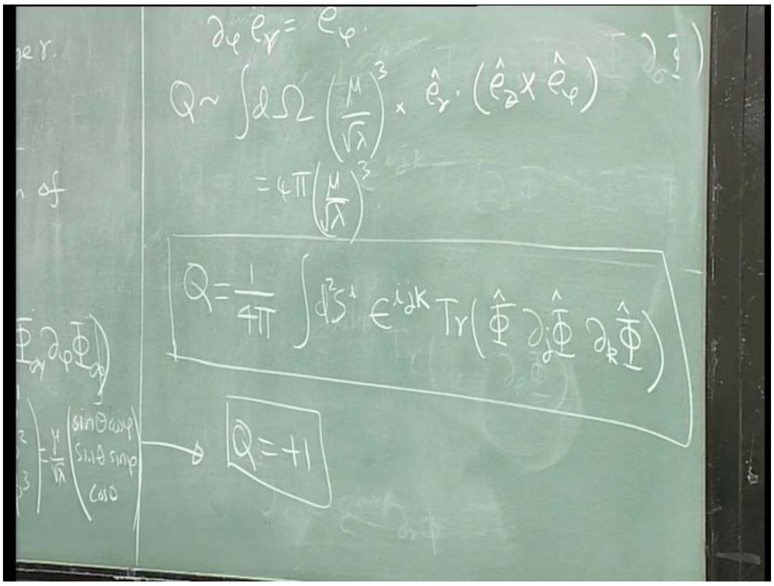

So, d theta of e r cap is just e theta cap and d phi of e r cap is along, there is no sign thetas, etc, right; I mean I think it is okay, yeah. So now we have to just understand how this thing works. So, this if you write out in terms of this thing you will see that this particular term that is so this is so Q will just turn out to be integral d omega where d omega is just thing times this is along up to each of these will give you these factors mu by mu by root lambda whole cubed times e r dot e theta cap dot cross e phi cap; that happens to be one. So, this will just give you four pi. So, we first think we want to do is get rid of this factor and we have to do this thing.

So, you can see now Q should be now we can put equal to 1 by 4 pi and one way of doing this is to replace all these things by unit vectors. If you have just replaced this by the hat version of that, we would have been in good shaped. Then this will go away 1 by 4 pi integral d 2 S i epsilon i j k. Now I fixed the constant also and so this will be up to a sign its one. So, for this particular configuration I have adjusted it such that Q equal to plus 1. So, what this tells you is that if you want to get something which is topological charge one, it also give you some detail about what phi should look like.

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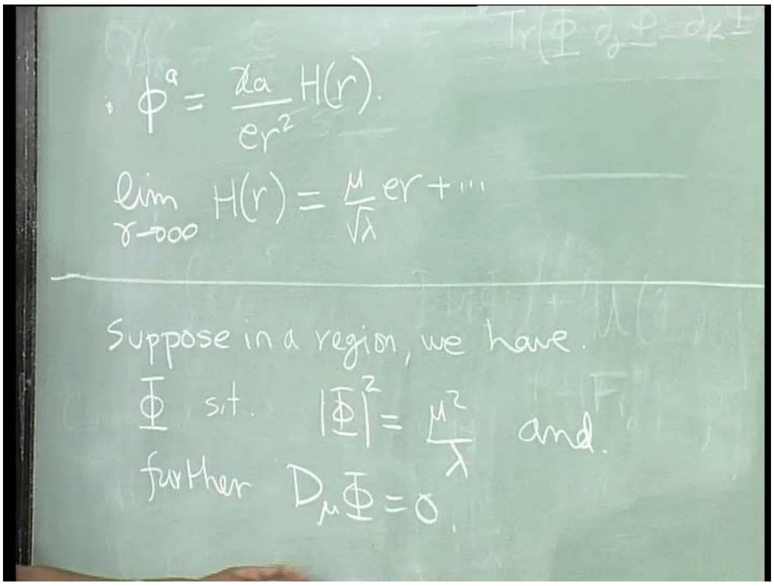

If you look at the ansatz of which was chosen by ‘t Hooft it was just this x a by e r square H over r so this is exactly, so at everywhere this actually goes to what am I trying to say. So, this is along the redial direction; it is exactly this because I can rewrite this if you wish as root M u by root lambda just to match this kind of convention x a upon r. So, it also tells you the behavior of h of r. So, we can again go back and say that limit r tends to infinity H of r should be such that this just becomes should go like e of r plus stuff. So, that this will match this; what? Oh, yeah yeah yeah sure. Yeah the best solution is to go ahead and define this should be its behavior for large r at plus subleading terms.

So, this gives us some intuition into the ansatz which reminds me by the way there was a question yesterday asking how did Prasad and Somerfield discovered the solution. Apparently reading somewhere I discovered there was a note which explained this. Apparently they were playing around with a numerical solution to Hooft equations and then they found their solution the hyperbolic solution was in a look like a pretty good fit and then they sort of worked backwards and found that actually if you take lambda equal to zero it is an exact answer. So, that just explains this thing. So, first thing is now we need one more thing which we need to understand is what is the sense in which we know that when you when you consider a situation where the symmetry is broken, for in this case it is broken from SU 2 to U 1. So, it is fine but the question is how do you identify the U 1? It was very simple when we just took the case of along the z direction or something three components was non-zero and did the thing it was fine.

But here you can see that even on the sphere at infinity the direction is changing. So, now the question is how do we identify. So, how do we identify the U 1 gauge field? So, actually the correct way to do that is to go back and to ask what are the allowed fluctuations in some sense and that is what we will do. What we will do is we will consider suppose in a region we have not just we have phi such that mod phi square equal to this quantity mu square plus lambda and further D i of I is also zero. No need, you write it like this, write the whole thing d mu of phi is equal to 0. We saw it falls of as 1 by r power 3 by 2. So, at infinity to so where we are trying to work out things is strictly I mean it is pretty close to zero or zero if you wish. So, the question this is where we expect to identify the U 1 gauge field.

Now the question is if you look at this equation this tells you that I can go ahead and solve for the gauge field in terms the phi field and this is also related to the statement which I made that you know these things have to get related. So, we will do it in this fashion and will work out we will see what is the most general solution to this. So, but d mu phi is nothing but pi mu of a should be zero and do you have an expression for phi mu of a, so this is what we get. So, we need to solve this particular guy going to zero and we are given phi. So, we can solve so in some sense we can. So, if you look here as long as this phi is non-zero we can sort of pull it out and divide by and stuff like that and solve for it. We will solve for it soon but key point here is that phi takes cannot vanish which is true on this; I mean the full vector phi cannot vanish that is what this is telling you, it has some fixed norm. So let us solve for this equation and see that we get some abelian fields. So, what we want to do is to solve phi mu of A equal to zero. So, what I will do this I can take this to the other side and put a minus sign.

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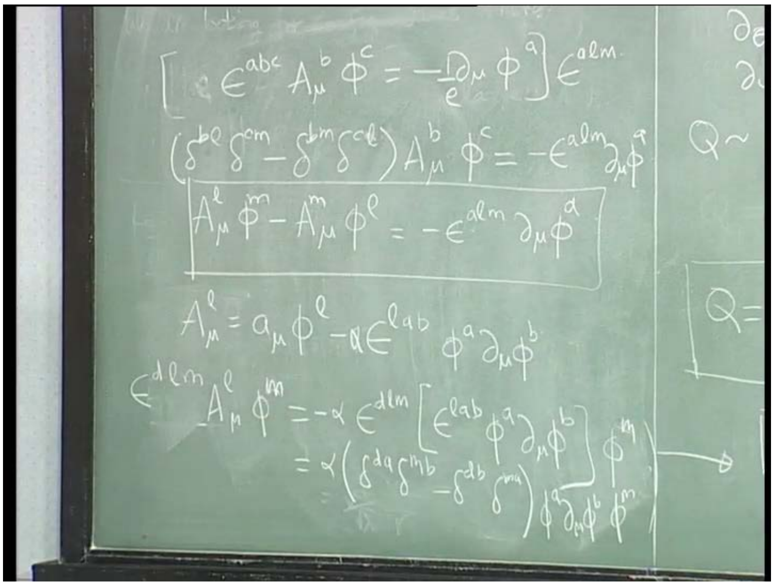

So we have e epsilon abc A mu b phi c equal to minus d mu of phi a in that region. So, the region we have in mind is a region close to infinity where we know that SU 2 is broken to U 1, everything is fine but we still have painful factors of phi floating around. So, we need to get around that. So, first step to do that would be to get rid of this epsilon from here to that other side. So, let us do that by multiplying this by epsilon, say, a l m. So, we have to use that and put this here 1 by e. So, this is just using the identity involving to epsilon with one contraction, this is just delta b l delta c m minus delta b m delta c m into A mu b phi c should be equal to minus epsilon a l m d mu phi a; nothing deep, I just have gotten rid of an epsilon but now you can see that there is some simplifications happening out here. This just tells you this is A mu l phi c minus, I did something wrong, right. There something should be m; I did the opposite replacement; are we okay now.

So, first thing you can see here is that because what you are seeing here if you think in terms of matrices this is just a commutator of A with phi. So, obviously there is one piece which will give zero to this which is if I take A mu l to be let me write something else let me write little. Suppose A mu l is some little a times phi l, it will give zero that will cancel out. Then I should write another piece which will give this guy, so this should be something like minus l with some constant which I will fix soon epsilon l a b phi a with some constant which I need to fix. I think it is a minus sign we will fix this constant. So, we just plugged this in here. Now for this whole exercise you can see that this term just drops out. So, it really comes down to doing this thing again but instead of doing that can we just go back to the top expression which we just started out with and see that it works; that is all we need to do, is this clear. I just go and plug this in here and then I do not need.

So, what I need to do now is to work out what epsilon I have to be careful with indices. So, let us put d and then I have A mu here as l, that is good, and is there any m’s here epsilon d l m A mu l phi c should be equal to; let me call this constant alpha so that I do not call every constant I write today as c. So, this will be minus alpha times epsilon d l m A mu l which is this term times phi c. So, I just need to again use this and nauseum this particular thing. So, alpha times so this I can observe the minus sign and bring l here. So, it will be d with a delta d with a and delta m with b minus delta d with b delta m with a times phi a d mu phi b phi m. So, there will be one term which is so let me just simplify that a little bit more. I need to write one more I was trying to jump a step maybe I should not.

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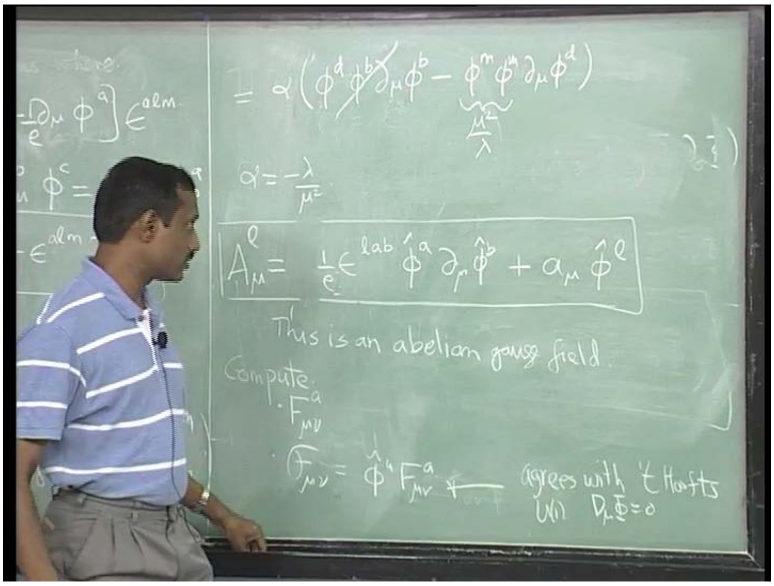

So, this should be alpha. So, this will just make phi d phi b, is that right, minus this will give delta d b that will make this d mu phi d, what should I do; I forgot something here, there is one more piece d mu of phi b minus this term phi m phi m d mu of phi d. So, I get these two terms; first point is that this is zero because phi b this is nothing but d mu of phi b phi b half but that is constant, so that is zero. So, this term drop out phi m phi m is mu square by lambda. So, we are done, so alpha should be minus lambda upon mu square. Now we can see that what we get is A mu l should be very nice lambda by mu square times but in fact you can see that I can simplify things. So, here you can see that this part is completely fixed but there is a fluctuation which is just one vector worth of fluctuations. So, in some sense you can see that morally speaking this is an abelian; in this limit this is an abelian. There is a 1by e.

In fact we can simplify this expression a bit we can get rid of this lambda mu squares if we work with phi hats and we can just redefine scale A again and the answer will be nicer with this 1 by e times this whole thing. But, what I wanted you to get through is that you can so this gives you a very nice handle on what the gauge field is. It also fits in with what I said yesterday, roughly speaking, if you did phi dot l will give you that fluctuation which is this guy plus but there is this piece which you cannot get rid of. So, you can go ahead and compute F for this F mu nu a for this and from this and then just define the U 1 gauge field F mu nu to be just. So, let me just do one thing; let me just get rid of, I want to get rid of these factors.

So, I will just make I will write 1 upon e here; there is no problem with that and this would mean just a new definition for a mu. So, maybe I will just call this a mu tilde and so this is what I have. So, you just define F mu nu as phi hat a F mu nu a where f mu nu is given in terms of this. So, this will almost agree with what at two fourth we had some extra pieces but those extra pieces were actually phi i a's. So, it is not written in that form in the assignment but you can show that those two forms are equivalent. So, this agrees with U 1 if you realize that d mu of phi is zero and if you just look through this structure this guy will.

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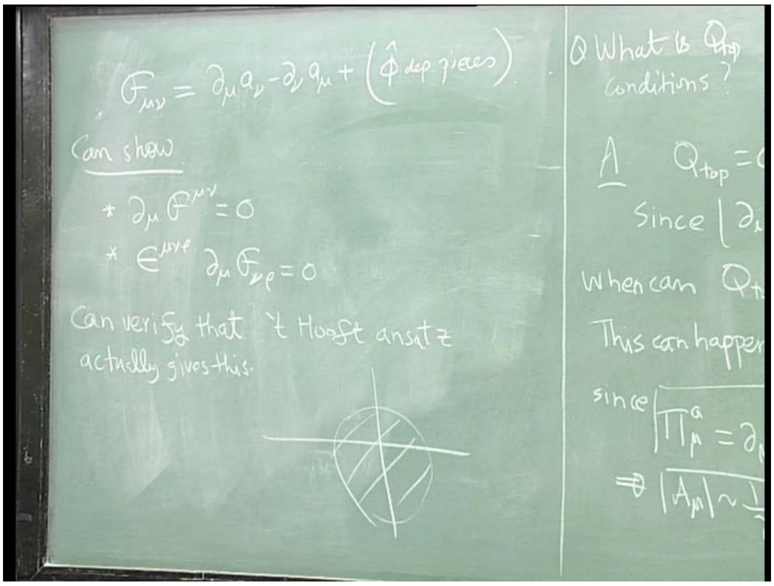

So, we will see that f mu new will have one piece which is plus the phi hat dependent piece and this f mu nu has certain nice properties you can show. First thing you can show is that d mu f mu nu is zero and you can also show that the Bianchi identity which is epsilon mu nu row d mu f mu row f mu row equal to 0 where we assume that d mu phi, etc is zero; you know all these things we have which is true. So, you can go back and so at this point we have not made any assumption about the ansatz or anything. You can verify that Hooft ansatz actually gives this. So, you will have to throw out I mean you should do something's with the k's and the h's; that I will leave it with you as an exercise. So, that is another way of checking this thing.

So, you can see that what is happening, so all this analysis was done in a region quite far away from where the full gauge group could be restored. We needed phi we needed this point that phi was a non-vanishing vector and so this gives you a concrete identification of what the gauge field is. I mean all these terms will be important; next time will see how the topological charge that we wrote out is indeed the flux, I mean, it is the magnetic charge of the monopole that more or less follows what comes from here. So, these pieces are important and will see how that works. So, I stop here.