Classical Field Theory
Prof. Suresh Govindarajan
Department of Physics
Indian Institute of Technology, Madras

Lecture – 31

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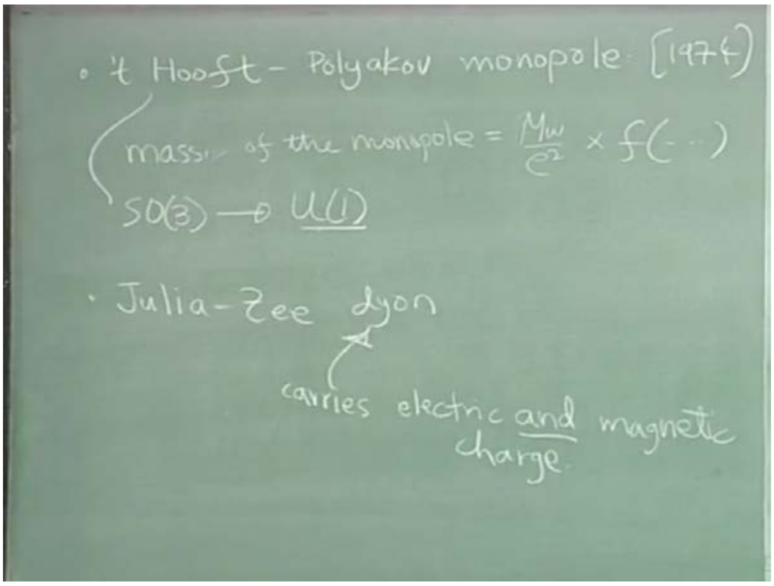

And this time around what we are going to see is that it can be embedded into in to an example where the singularity actually goes away. And historically the the, the first such solution was due to t Hooft and also Polyakov, those were the days of the cold war or whatever and so communication was not really I mean so this is from the eastern block and this is from the western block. So, lots of people follow this work, but it is truly these 2 people they constructed a solution. So, actually when you when I say they constructed a solution they actually t Hooft for instance doest quite right, he puts an anzats for it.

And he can estimate he can show that that has the correct charges etcetera, that it satisfies the boundary conditions. But he does not have an exact estimate, so for instance he cannot estimate the mass of the monopole. So, one-dimensional grounds he will get some some object, which looks like some something like this times some unknown function with some arguments. And what he did was to carry out some numerical integrations to have some estimate of these things. So, it was really an approximate solution where I mean he could guess a lot of features etcetera, just based on topology.

And but pretty much so this happened in 1974 and pretty much following at the heels of this. So, this so this is the solution where this, the starting point is S O 3 which is broken down to a u 1. So, the monopole here is magnetically charged under this u 1. In S O 3 there is no meaning there is no charge, but it is a non ebullient thing.

So, here when you mean a monopole or magnetically charged object is with respect to this u 1, pretty much historically following this Julia and Zee immediately generalize this solution to include something which has both electric and magnetic charge and it is called the Julia Zee dyon. So, this carries electric and magnetic charge. So, if you recall in our discussion on dirac quantization, we, we had only electrically charged objects and magnetically charged objects. You may wonder how to handle objects which carry both these charges, we will may be at some point discuss it, right now I just want to tell you that the word dyon is refers to something which carries both electric and magnetic charge again with respect to this u 1.

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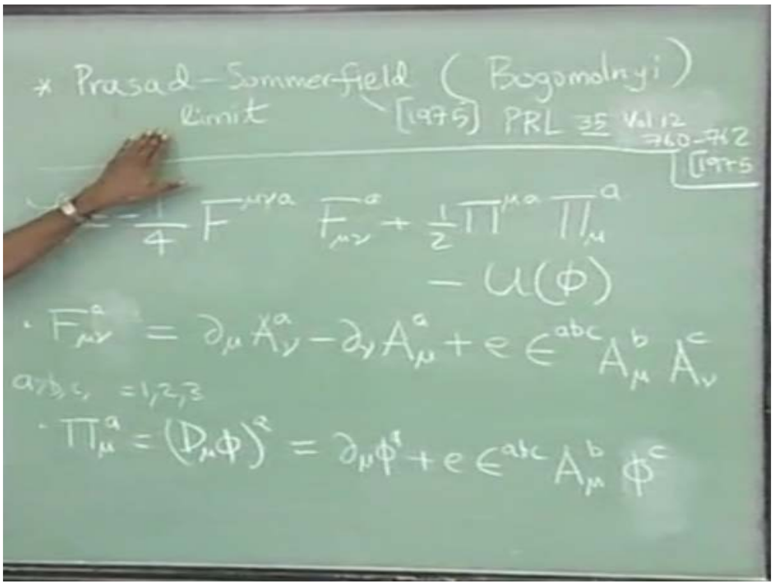

So, the thing this sort of thing got empathized largely due to work of Prasad, and some of his which is what we are going to discuss in this lecture as well as Bogomolnyi, we have discussed these names earlier. But this is the specific example where the actually were able to in a particular limit so which, which is called today is called Prasad Somerfield limit you will see what that is. In the Prasad Somerfield limit they were able to get an exact solution to the Euler Lagrange equation. And the anzats they used was basically the same anzats which they have used these 2 people have used but they were able to give an exact solution to the Euler Lagrange equation in a particular limit, that limit is actually very interesting. And later may be time permitting I will discuss the Bogomolnyi trick or may be the in the next lecture. So, so let us start with.

So, when was this paper, this was also in 1975, so I guess most of you were not born at that time. So, let us sort of work out the set up of this thing, the Lagrangian density is so the notation I am using is more or less what was in Prasad Somerfield paper, maybe I should give the reference since I have it here it is P L R physical review letters 35 number 12, volume 12. And the page numbers 760 to 763 pages of 60 60 162 in 1975. So, this paper is at least accessible from within our campus because we have a subscription to P L R but otherwise it is restricted. And so what I did was to work with more or less their notation except the one change I had to make was that their matrix convention is opposite. So, it is a minus plus plus plus while we use plus plus minus. So, I am taking care of those details, otherwise basically these are the sort of things.

So F mu nu a so it is an S O 3 or S u 2 lee algebra value of t in the adjoined, a is an adjoined index, and so S O 3 adjoined or S u 2 adjoined S 3. So, a b c d take values from 1 2 3, epsilon a b c is just the structure constant of the S u 2 le algebra or in 3 dimensions. So, this is the field strength co variant field strength and phi is just a co variant derivative which has and here just a b c we do not worry about upper lower, because it is just an Euclidean it is a Euclidean matrix raised and lowered with the chronic delta. So, really I do not even bother with that. So, I will just write it this way thank you. So, you may wonder what happened to the i’s etcetera, but remember the way we wrote things I have some minus there will be an i out here and the T is also have an I, so it goes away and this plus or minus really a sign conventions. So, I am just sticking to whatever they wrote. So, for these two are just definitions and so they they. So, let us look at this u of phi.

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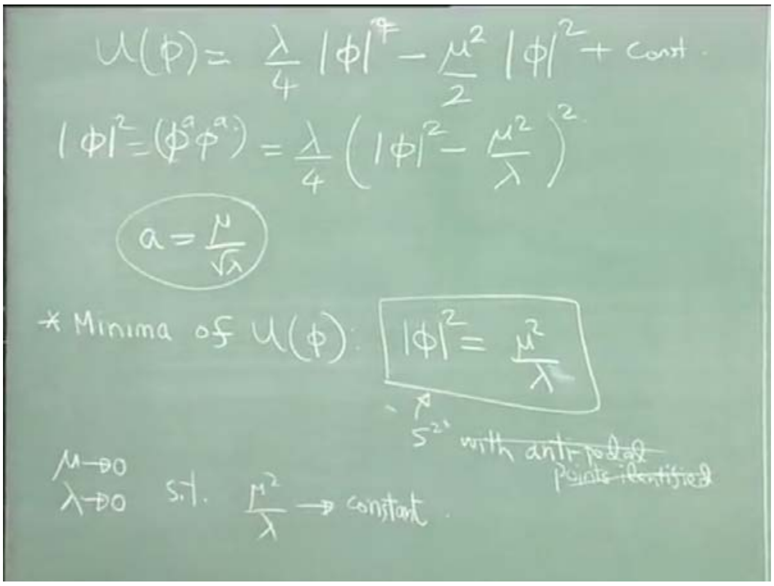

I just forgot to mention that u of phi is lambda by 4, this is the potential; this is phi power 4, so here by mod phi square I just need phi a phi a, it just summed over. And so this is a canonic these are normal potential which I can read it up to a shift plus some constant which I will you would have usually written it as a square, but a little bit of thought will show you that this is just nothing but mu square upon lambda time whole square. So, this so what we would have called a in the one potential which I have been using in this course would be mu by square root of lambda just to remind you. So, first thing to understand is what is the minimum or minima of u of phi? Of course, this is S O 3 invariant, so what is the minimum minima of phi would be min mod phi square equal to a square which is now thank you.

So, what is this an equation of so phi there are how many phi's there are 3 of them. So, this is this look like an equation of a 3 sphere, but there is also phi goes to minus phi which have to be identified with each other. So, it is really technically speaking its S 3 mod Z 2 with anti paddle points identified S 2, S 2 right, because all phi is going to minus phi is just a gauge transformation is not it no it is not it is not, because that has determinant minus 1. So, it is not an orthogonal transformation, so it is I will just its but if you look at the it is an orthogonal symmetry it is so in that sense there is so anyway we can take it to be approximately S 2.

So, let us not worry about these issues. So, this is the set of (( )) and so and clearly it is a as long as mu is non zero and I mean this this quantity is finite we are. So, later we will see something particular limit which is to take, you take mu goes to 0, lambda goes to 0 such that the quantity mu square upon lambda and just to for convenience of their definition I think I should put in e somewhere so should goes to a constant. So, what is this constant do I mean you may think fine it says that lambda goes to 0 that means the potential is gone, but that is some memory of the potential and what is that? And that is remembered by saying that that that this quantity is kept constant.

So, it knows what you doing is in that let us put boundary conditions at infinity or whatever we will be looking for solution where this is not violated. So, this is the this is what is the Prasad Somerfield limit, and that is what makes life a lot easier, you can see that you all u of phi can be forgotten, but the only memory of that is the is this, and this is the biggest simplification in this problem so you are.

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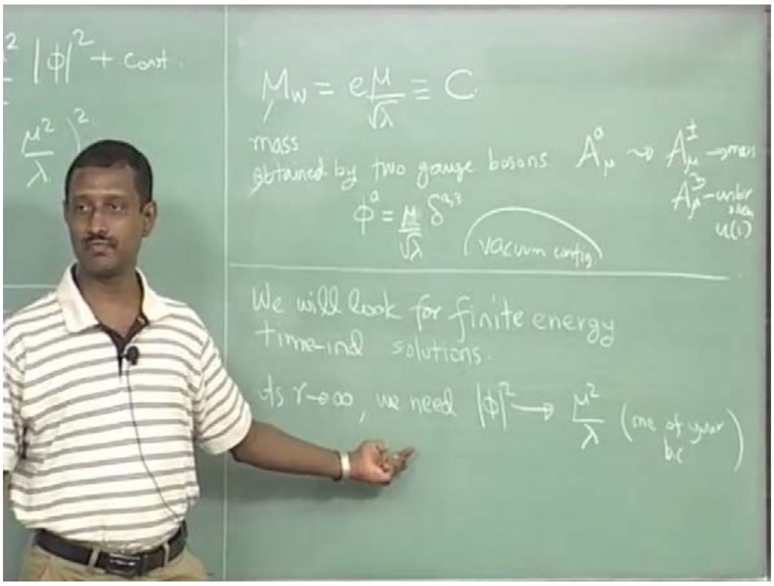

So, the thing is to find the Euler Lagrange equation of motion now let us do little bit more. So, let us just work out. So, so the thing is that so if you if you if you do a analysis this is exactly through higgs mechanism we will find that there are two, two of the higgs goes on become massive, two of the vector fields become the analog of w plus and w minus. So M w square M w will be e times a, but what is a it is mu by square root of lambda, and Prasad Somerfield define these constant to be C I think, just give me a second let me just sort this out yes. So, so this C is constant defined so it is the mass of the w goes on in.

So, this is the mass Obtained by the two gauge bosons. So, if you write the generators as A m u a and this way of saying it, if you if you chose this to be along z direction what you will see is that A mu plus minus become massive. And A m u 3 is the unbroken that is if you chose phi suppose we went ahead and chose phi 3 phi a to be a times delta. So, that is what you can see now mu by square root of lambda delta A 3 to just match whatever we we have discussed this particular example before I am just putting in numbers etcetera, not numbers putting a defining various things. So, suppose I did this I would get this thing. So, the idea is to look for finite. So, this would be the vacuum solution so this is the vacuum configuration. So, what we will be looking for finite energy.

So, we will look for finite energy maybe I should say time independent finite energy solutions, but time time independent solutions. And so the key point is that, so one of the things we will require is that as. So, at this point we will take the Prasad Somerfield limit we need not do this for this discussion, so as r tends to infinity. So, in spacia r is just x 1 square plus x 2 plus x 3 square as r at the this sphere at infinity you should find that phi mod phi a square should go to its well. So, we need should go to mu square upon lambda is that correct mu square upon lambda. So, this, what should happen, so this should be one of your boundary conditions. And there will be conditions on the gauge fields which I am writing out for now, but this is definitely one of the conditions that you will have to put it. So, the idea here is is to think what happened in terms of the in the vortex solutions when we did, because I want to think the way (( )) approaching this problem.

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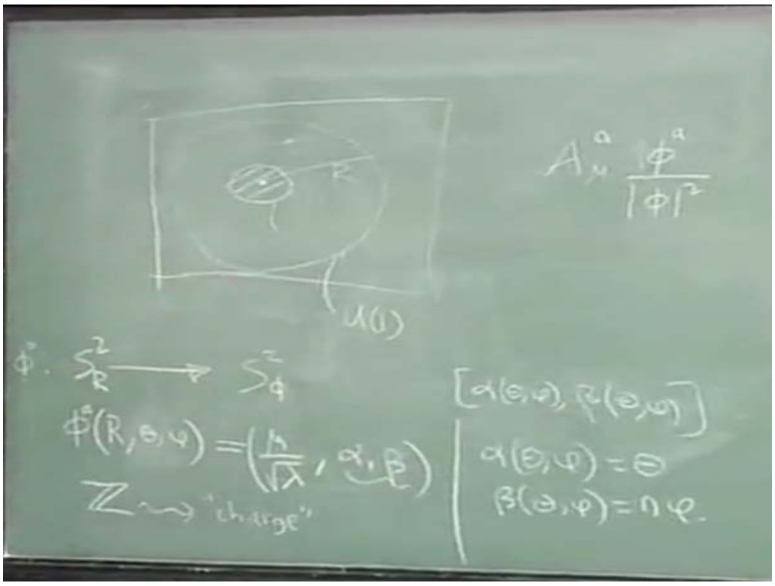

So, if you remember there what we found was that at infinity we just saw that mod phi had to take some value, but there was some region where finite region in which phi. So, bulk of the region phi was phi went to its vacuum expectation that is what is vacuum value, but there was some region, where it was it was deviating from that and that was your flux tube and in fact, you could we even worked out how fast it could die often we saw that in it was exponentially dying outside. So, that is really a phi a core out here. So, the same picture this is in, but this now is in is in to the mentions but you can think of the same thing in 3 dimensions, but you have a core which is a solid sphere kind of thing outside we thinks actually quickly as reached this value.

So, in that thinks it is actually morally speaking exactly like the key which had only so you know the support was finite I mean far away it might tannable it is hard to exponentially decaying. So, the energy was if you ask how much if you want 99.99 percent of the energy to be in this thing you will find that it will be there in a very small region whose size you can predict of course. So, the same thing we can do out here. So, what we would like to do is, so in this region u 1 is unbroken there is only a u 1. So, outside it looks like we have gauge we have only a u 1. And if you go ahead and chose in any region if you go ahead, and chose this we will say that A m u 3 is A m u 3 would be the u 1. But suppose you want to write it in a gauge invariant way you could say that I would write something like A m u a times phi a divided by mod phi. This this sort of, if you make any rotation global at least, you you can see that so this this will reduce to what I said when phi a is chosen to be along the 3 direction, phi 3 is non zero, but that is not necessary you may chose something else.

So, actually now you can ask what are the configurations which satisfy this thing? So, one thing we can see is that the set of so. So, you consider let R be the 2 sphere of some large radius so R is something outside all we really need is something very very large outside the core. And so what you have is this map this phi a's give you a map from S 2 of R 2 the S 2 in field space, because these are constrained to lie on a sphere. Suppose I permit my phi to rotate on this thing does not matter which one it need not always be that that kind of thing, you can ask what are all the maps so what you are interested in, you are interested in smooth maps which goes from S. So, what I have in mind is we just say phi a of R e power i phi how does; not R what am I writing, so it is R theta and phi. So, the so theta and phi are the point I just made it this thing, so you can see that this, so there will be this can also be written.

So, this should be equal to what is the value? It should be mu by root lambda times some other sphere. So, what should I call? Let us call them alpha and beta kind of thing. So, as as theta and phi vary, so you will get 2 functions, so these are the angles of these two this sphere. So, beta is like phi say and alpha is like a theta. So, what you will get is 2 functions alpha of theta and phi and beta of theta and phi. A and you can ask is there a way out to topologically classifying this maps. So, in other words you are not worried if I just go around from there and turn things around a bit. And we discussed this idea earlier when we took S 1 to S 1 in the case of the Torres, but even here you will find that it is the set of maps are classified by just some integer called Z. And this Z in principle should be related to so at some quantization should be related to monopole charge.

We will will make things more precise, but what I am saying is that there will be a topological class into which you can fix your t. And so that in some sense if you say that I am looking for a solution which has like vortex number 2 that implies that when you around went around once in that if that thing in field space it went around twice. Similarly, out here, so the simplest one way of writing some ending number thing would be to say that you know I will give you one example of this Z is that you chose alpha of theta and phi to be theta itself.

But I can do this beta of theta and phi I can chose it to be n times phi. I am giving an example of a map this is a nice smooth map nothing wrong with it. So, you can see what it is happening n times and there is an orientation. So, you can I mean I get you this phi, but to this I can add any smooth functions which will jiggle it around they will all be in the same class. So, suppose you chose you suppose we went ahead and chose one of these things, the point here is that there exists a solution of the Euler Lagrange equation even though we may not be able to solve it. There is no way we can by any jiggling take take solution with n going to n minus 1 or n minus 2. So, the vacuum solution would correspond to n equal to 0. So, clearly this now you can see why I am saying that I cannot if I choose something like this everywhere there is no way I would get solution which has which has n naught equal to 0.

So, clearly this would not be correct. So, I need to permit myself for phi to go in different direction, and so what I have in mind is that is why I am just sort of ah looking at this sort of a thing. So, locally if anywhere I can always make a rotation if you give me in any small region I can always make a rotation. So, this, this artery is this phi space think of them as co ordinate. So, at a given region phi will be in some pointing generically in some direction, you can always rotate that and make it Z action, but you cannot do it everywhere there will be an obstruction, somewhere you land into trouble that is what this is telling you. But make you intuition work this is useful, so what you do is suppose in some region it look like this we know that this has to be the u 1.

So, this kind of tells you that this should be a naturally the u 1 which you should look at, it should be mod phi, I think unit vector in this space. It is confusing, because there is space time and there this is and the space of phi that is also an r 3. So, what we will be looking at today is the one solution. We will do this I, I want to postpone it, because we will we will do the, we will do exactly like what we did in the vortex case. So, you just hold it for now oh I erased the one thing which I needed next, but does not matter.

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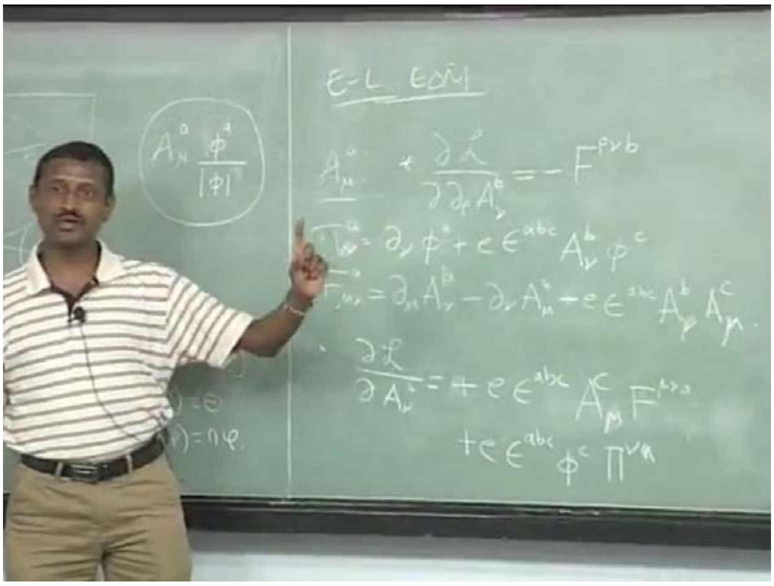

So, what we need to do is to work out the Euler Lagrange equations of motion. And I will show you some tricks in which we will tell you that there are some terms which you need not worry too much about, you would have got that has to work. And this is got to do with symmetry, so for instance suppose I wanted, so let us say we wanted to get the equation of the motions for A mu a. So, what you need to do is to do delta L by delta let me first write out F mu nu a equal to. So, now you can see that there are because of this pieces there will there will be this will be contributing even to when you do delta L by delta mu there is no contribution I mean there were earlier no contribution coming from the kinetic energy for the, the thing, but now there will be. But what I want to show you is that that term you could have written it without this thing there is way of fixing it without even doing the calculation.

So, so let us look at this so we want to do, let us say that we wanted to do, let us do it this way A mu of b. So, I I am just playing around so that I get something which, which I like. So, this is what let us start with this and so there was a minus 1 4 thing by its again as before that one will not change it is the same as whatever we got here, and it will be minus F, it is minus that is what it is. But we also need to work out in A u b, now comes the thing that we have exactly one more piece out here. And so again you can see that there will be 4 terms either from one coming from here, one coming from here and the other F F square, so the 4 will cancel with the 4. So, I need to only work out one term you know that the rest will work out.

So, what you will get is e so I will just look at one of these things so the, and from here will get. So, let us pick I will I will chose the part where mu and nu are exchanged. So that I will get a minus sign so that I get A nu b. So, I would write this whole thing like this the same thing. So, I can see that I get minus from this and there will be a minus from the original action itself, so they get a plus and e epsilon a b c A mu c.

That is this part, but then there is also the part, which comes from co obedient derivative of this thing. So, let us let us call this dummy index nu, b a c A nu a phi c. So, when I take a derivative of this of pi this thing you can see that I will get I had it the way it should be you know and then. So, I just time adjusting so what I have in mind is pi nu a whole square is what I have half of that. So, when I take derivative of this there will be the 2 will cancel away and I will just get one piece which is this e I get a phi c, but this will be multiplied by phi mu phi nu with this, one second let me phi nu a now I am. That should n F nu anyway.

So, now you can see that so I have to take D rho of this and equated to this that is the Euler Lagrange equation of motion. But here I this is where I want to point out that this term here if nothing but it has to go to the other side and work out and has to be co obedient derivative of. You should convert this into a co obedient derivative of this, this guy as you can see is transforms nicely under local transformations, pi transforms nicely this also transforms, so everything is nice. So, right hand is a vector so this better be a vector.

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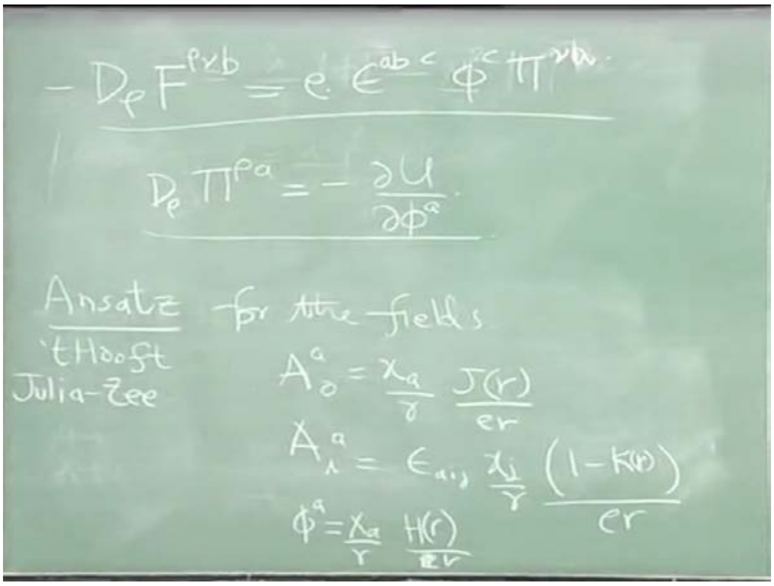

So, finally, have to assign the, so what we get is minus. So, I just write it this way D rho F rho nu b should be equal to. So, I am not worrying about anything out here I will just t H rho this term forget about this should be equal to this phi nu b. So, you can see that the this there is a it is not as complicated as it looks, because symmetry I mean this equation better be nice transform. So, your right hand side transforms properly left hand side will so the, that is what happens out here. Similarly, out here for what is it you would have written for you would have written for a free scalar you would have written something like D rho of phi rho a should be equal to minus d u by d phi a. Now, you can see that left hand, right hand side again transforms nicely, and so the only way that can happen is to just ahead and replace this by D rho. And it is not a mystery how to write D rho of this thing, because all these are in the same representation is exactly like this.

So, there is no need to worry any, but it is always a useful check you know what I do is when I do this computations I actually go back, and say you know as a check of my science I check it does it give that answer I know that in hand side. So, this is what you should get. So, so now the thing is we have the equations of motion, and one more thing we know is that in the Prasad Somerfield limit this term will vanish, the potential goes away just remember that, So, we need to make an anzat for the fields. So, let me write out the anzats, and this anzats is due to a t Hooft for the monopole case and Julia and Zee for the for the dyonic case.

So, in the assignment which I gave you people I was very kind i did I did not write the dyonic part, because it is a horrible mess to work through, but it is good for the soul. So, what is a what is a notation I used? So, so in the (( )) in here you can see that the phi electrostatic for pro potential not electrostatic the scalar potential also is non zero and so you will get an electric charge everything is time independent.

So, the electric thing will come from only this so if you put J equal to 0, you should get what we saw in the things, but the anzacs I want to write the most general thing and which was written out. So, A 0 so A i a is epsilon a i j X j of r into 1 minus K r K of r by e r. So, key point here is that this, this first term which I have written out here, actually has to do with the with the fact of this charge as 1 that is there taken care of. And the last bit is phi of a equal to X a by r. So, there are 3 unknown functions j K and H. So, you can see here that this phi is along the radial direction as large values.

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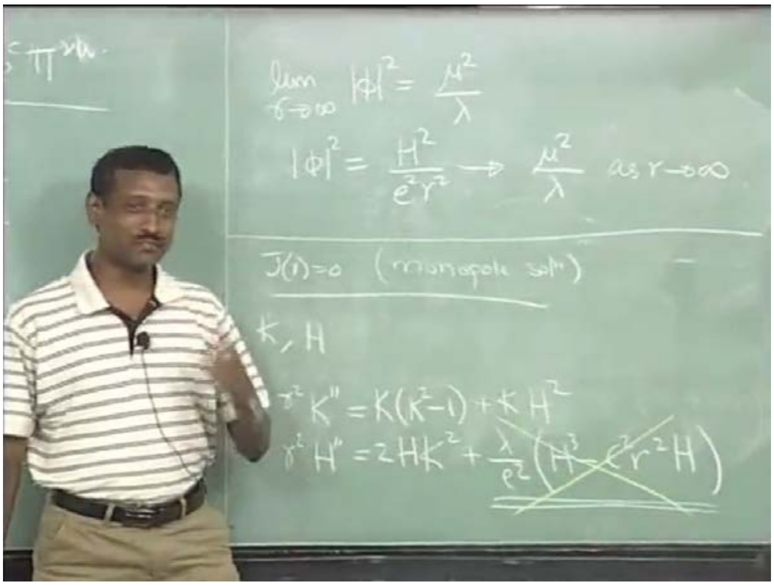

So, what we wanted was we wanted as limit r tends to infinity, we wanted mod phi square which was phi a phi a to become, what was it supposed to be it should be mu square by lambda, this is what we wanted. So, now we can look at this anzat and ask what it does for H, so what that tells you is that phi a phi a would be X a square which is r square, but there is a r power 4. So, it would be for the anzats mod phi square is equals to H square divided by e square r square, and this should tend to so H of r should tend to this should tend to mu square by lambda.

So, this is the boundary condition that one have. And now the point is is to go ahead and plug this anzats into these, these two equations and ask what does it lo like. So, what I will do now is for the rest of the discussion I will only I will chose J equal to 0 to get the monopole solution, because it has no electric charge it cannot I mean if A 0 is 0, everything is time independent. And there are simplifications which come for instance here D rho rho can be written as just the special part because D 0 A 0 is 0 and the I just becomes that. And so the equations become now we, we have said this equal to 0 so only unknowns are K and H, and the anzats says it this only radially on the radial variables, so it simplifies a bit. So, the equations become r square K double prime. Now, just for a moment let me write the full let us leave it no, no I would not write it sometimes a temptation to say more than is necessary.

Now the, this was given to you as your assignment to drive this thing and there are no shortcuts for this. So, what, what you have to do is to take this anzats; first step compute what is phi? Second step work these things out and make no a no mistakes to get this answers. And Prasad and Somerfield say that they with the bait of shimming it is a kind of understatement, because you need to get your everything right no sign mistakes nothing, then you can you will get this equations. So, for me I just copied from there, but it is something I would strongly recommend that you do. But now we can just look at this, this term is obviously coming from this term is coming from the potential.

So, you can see that if I if I take the limit, but lambda goes to 0 and the C is this is already written we already saw that C was related to our A. So really nothing is happening so lambda going to 0 this term drops out. So, there were questions asked by some people how did Prasad and Somerfield find out I mean that I do not know, because I have not open to them. But I think this was their observation that you know that the solution which they were able to write for this set of equation without this. So, this is so that is the simplification which comes out, this term goes away. So, then the equations actually have a nice solution. So, now we are ready to take the Prasad Somerfield limit. This would have been boring, but it was not boring, because in that minute they were able to write the solution and the solution is very nice.

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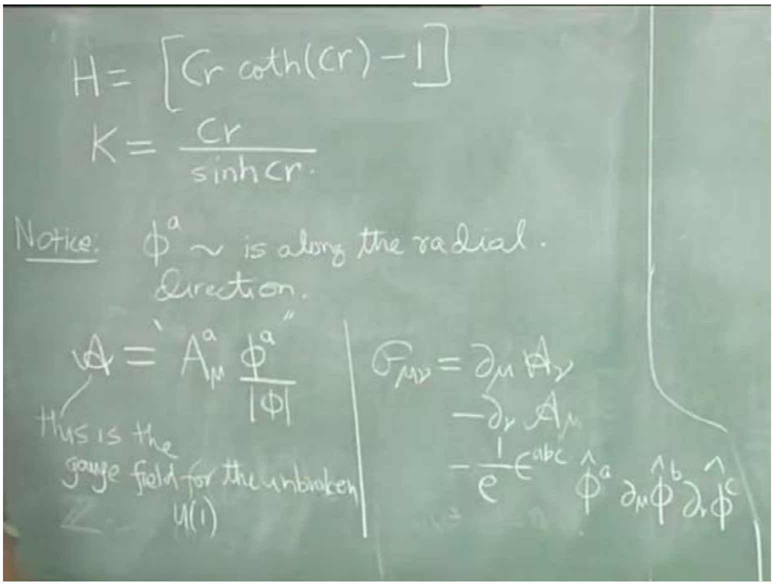

H, so it is a I think it is very easy to check that once you are given these two things that this equations are satisfied, but the actually the this high I mean if you think of the way of how Bogomolnyi obtained this solution the equation, there actually it is like the like what we did earlier in the vortex thing when lambda was equal to 1 or e square or whatever that was you remember that we could do this sum of squares, etcetera. And then we got this extra piece, you can even here you can do exactly the same thing and I will show this not today, because there is not enough time, but may be in the next lecture we will see how these actually these these two are solutions not to a second I mean they are solutions to the second order equation.

There actually solutions to a much simpler first order set of equations, and for first order equations is easy to see what the solution should be. So, so we are more or less we are, so now you can see now we can ask what is the u 1 gauge field. Now first thing to notice is that that phi a is along the radial direction. So, now we what we will do is we will define, so the analog of mu So, let us def let us use script notation, let us define script a to b that quantity which I wrote phi a divided by mod phi. So, this we will identify this is the gauge field, almost this is almost right, the sphere the character. So, if you ask what is the field strength?

So, I will put this in quotes because there could be derivative pieces etcetera So, the field strength turns out to be, where this hands your corresponds to taking unit vectors. So, you can see that if you are in a region where you have turned all of them to become along the 3 directions; obviously, this term will go off. So, then it may lo exactly like that. So, that trick here is not going to tell you exactly how this stuff comes, but the nice thing is if you go ahead and evaluate what this F mu nu is. So, the, the one we have is, is to work this out and see for this solution and ask what it looks like. So, what you do is of course, the time dependent part F 0 F 0 i is 0. So, the only term which would contribute are F i j which is like the magnetic field strength and then you can do epsilon i j k and the answer turns out to be very nice.

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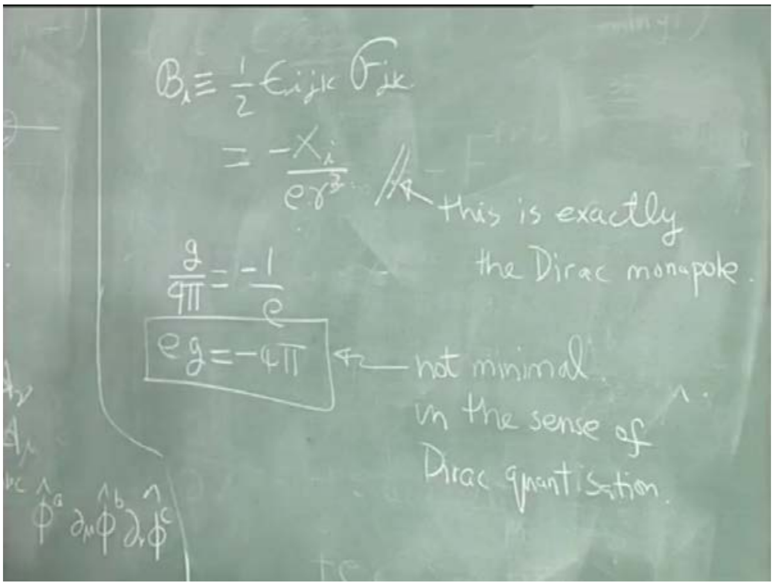

And very very simple form r cubed correct. So, this is exactly like this is exactly what you would have written we wrote for a dirac monopole. So, this is and this is being remember this being computed outside region where everything is, so what we are doing is doing this computation if this is the region all these definitions make only sense is the region where the u 1 S O 3 is broken to u 1 all the others become massive, so this is exactly the dirac monopole. Now, what is it charge? Its charge is 1. So, it, so how did we what was the thing we wrote? B so g by 4 phi should be equal to the sin h is minus one by e, so e g is equal to minus 4 phi. What was the dirac monopole condition e g was 2 phi. So, this is this is not minimal it has it has a a it could have been I would say it is minimal if it was 2 phi.

But there are again topological arguments which tell you why this is true should be the case we may be if it is if I can distillate into a form which I can explain to you I will explain it next time. So, we will stop here, but now we still have to ask, what is the energy of this solution etcetera? Are saying that for this sort of a calculation you cannot get less than 4 phi or. No you cannot for the S O 3 you cannot get, this is the minimal mini but it is not it is not minimal in the sense of dirac quantization. So, I will stop here will continue next, next lecture we will see how these things come. We will also see how these things and these are actually solutions not this, but rather this is a solution to a first order equation. And this will connect up with Bogomolny and this will really give you a example, why Bogomolny Prasad, why these were called Bogomolny Prasad Somerfield stuff?