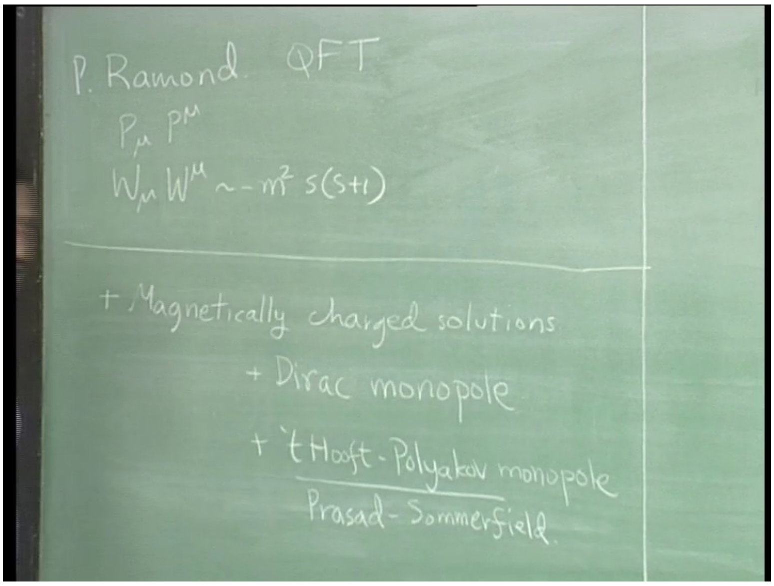
**Classical Field Theory**
**Prof. Suresh Govindarajan**
**Department of Physics**
**Indian Institute of Technology, Madras**

**Lecture - 30**

Last lecture, I did not actually get all factors correct and it was kind of deliberate. I want you to go back and work things out. I mean, because the structure is well known and you know the structure constant, for instance, for the group, for the lie algebra SO 4. So, going back, you can work it out and you have done some of it in your assignments as well. So, I just wanted to emphasis only the structure. Somebody asked me, is there any reference? Actually, any book on quantum field theory will have whatever I discussed.

(Refer Slide Time: 00:45)

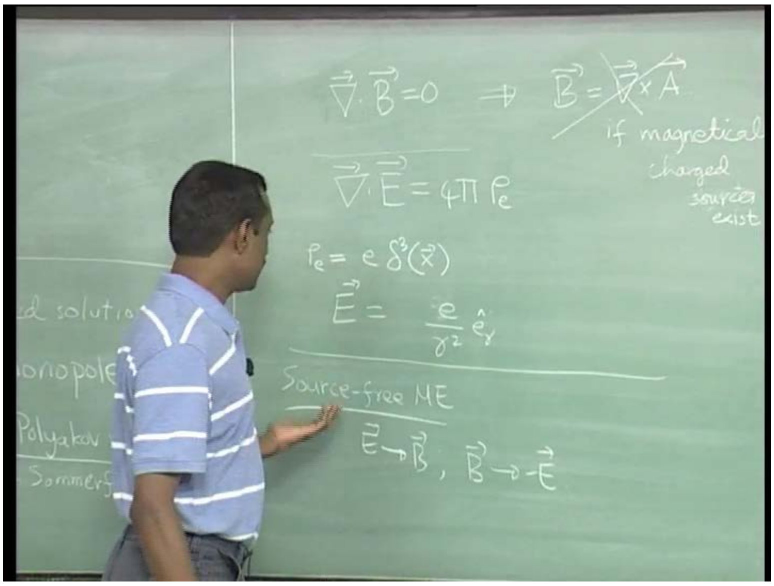



But, just, so, I give you a concise reference. This a single reference, if you wish, P Raymond's book on quantum field theory. It is, I think the fastest introduction to starting calculations in quantum field theory. It does not waste a lot of time on things and in particularly, he has everything, sort of, very concisely in the first 20 pages, all the stuff that you need before you can get started doing quantum field theory. Among other things he does, representations of the Lorentz group as well as the Poincare group out there.

Just one thing, I think, yesterday I said that this should go like m square times s into s plus 1 and I think there is a sign or minus sign in our convention. But, that is again something which you can, so, these are Eigen values of this Casimir should be there. One more caution is that, whenever you look at any book on quantum field theory, first thing you have to do is to probably look at the appendix, where they define their conventions. In particular, is a matrix convention plus minus minus minus or minus plus plus plus and all these will, so P mu, P mu would be minus mu square in the other convention. So, these are all details. But, I think the idea here is that, that the Poincare group has two casimirs, which are these two.

So now, what I am going to do is, discuss, go back to discussing classical solutions. So, what we will discuss in the next two lectures is magnetically charged solutions and I have already given you two assignments on this. One was on the Dirac monopole and this lecture you could say is related to and the next one is, I will say on the ‘t Hooft Polyakov monopole. But, I do not do it in their version. It is done in a simplified version, which is due to Prasad and Somerfield. This should take us about a couple of lectures.

(Refer Slide Time: 03:35)




So, it is kind of concurrent. Due to, we know that, we do not see anything, which has electrically, which carries magnetic charge and that is really reflector in Maxwell’s equation, which we write out as del dot B equal to 0. This is what actually let us solve for. We can solve for B in terms of the vector field a. So, and just by analogy, we can, with electric field, let me put a subscript here e. So, the difference between these two is that, there are sources of electric charge, while there are no sources of magnetic charge.

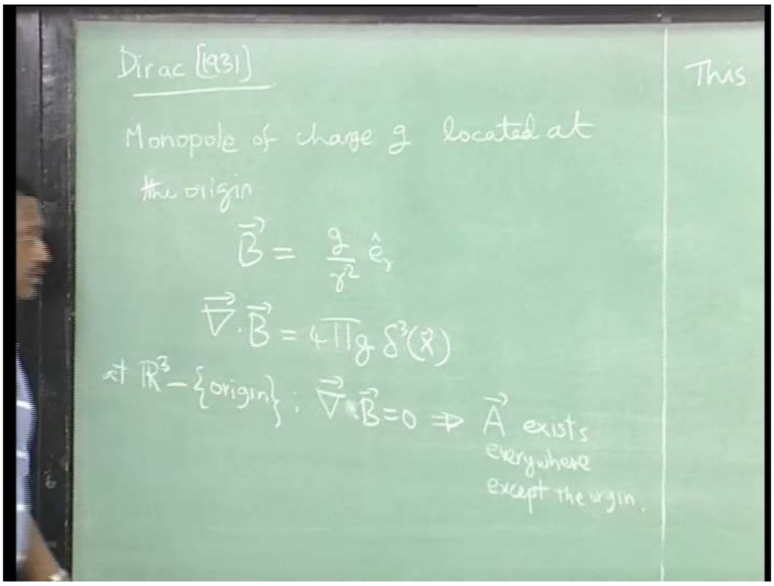
So, if somebody comes and asks you, if you, if there are magnetically charged objects, how would you modify Maxwell's equations? It seems obvious to write something like a 4 phi rho m out here and call that magnetic charge. But, if that is true, then this is not possible, if magnetically charged objects or sources exist.

That kind of and we saw that, when we did electromagnetism, it was very important for us that we could, we wrote that the basic field, there was the vector discombining with scalar potential to form a 4 vector and you got this vector field. It sort of tells you that, if you have things, which have electric charge as well, of course, you could write this thing. But, you can see that the Lagrangean description would be in trouble. We cannot write. The answer to that is that, that cannot be I think. A local description, where you have both electric and magnetic charges simultaneously present and if you think that, that is a figment of my imagination, that these things will never happen, at least theoretically speaking, there exist theories which are toy models, but, which have limits, where the low energy degrees of freedom of those things have, you know, are both electric and magnetic charged objects.

What we will see in the, for instance, in this is that, we will construct magnetically charged objects, but, typically they will be very massive. So, if you are looking for low energy degrees of freedom, it is ok. There are no monopoles. You can forget about it and work with this kind of setting and you are o. But, today we will be assuming something. We will proceed all a Dirac in 1931 or 1930-31, where he actually asked, what would be the effect of finding a single magnetic lie charged object? A point object which carries magnetic charge. So, his idea was, so, suppose let us just go back to this and if you have an electric charged point, electric charged, the source you would have written, the source for that as the charge times a delta function. This would be, so, this is like a point charge placed at the origin. What is the electric field due to this? Or should I put a q here or can we leave? So, this is what you would have written, right. 1 upon r square times a unit vector in this thing. We are working in Gaussian units, right. So, is there any 4 phi? No, nothing. 1 by 4 phi epsilon naught is 1. So, this is what you have.

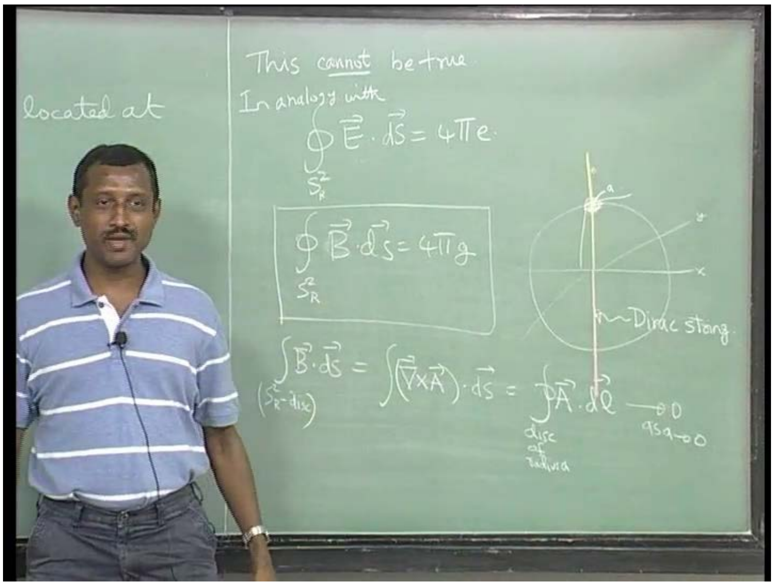
So, in exact analysis, so the thing is, one more point is that, the source free Maxwell's equations admit a nice symmetry, which is E goes to B and B goes to minus E. Presence of sources is what messes it up. So, if there were no sources, electric sources, this would also be 0. So, we are just talking about free Maxwell's equations. That is, d mu f mu nu equal to 0 and then, there is a symmetry. So, if you just see that, just using that analogy, you can see that I can always ask, I can write out what would be the magnetic field due to a point magnetic charge, just by using this symmetric. So, you could make this a symmetry of the theory even with sources, by just saying that, I exchange a electrical and magnetic sources, but, there are no magnetic sources. So, there is just go ahead and write del dot B equal 4 pi rho m and del dot E would be 0 in the new thing. That is ok. I am not writing the other two Maxwell's equations. They will also work out.

(Refer Slide Time: 09:08)




So, using that, this thing Dirac in 1931, he wrote, let us look for a point mono. So, he called the magnetic pole is what he called it those days, but, we will call it a monopole. Monopole of charge g located at the origin. So, just looking at what we have out here that would just be, B would be g by r square and del dot B would be; so, the thing is that, if you observe here at, so, space is just r 3. If I delete the origin, del dot B is 0 because, this was delta function. So, it is not so bad. I mean, we have a point charge. So, it is really at every point, except every point in space, except one point, things are normal. So, this kind of says that, so, this will lively suggest that, we can write A exists everywhere, except the origin. This is a nice guess, right. But, the important point here is this guess is wrong. This cannot be true.

(Refer Slide Time: 11:21)



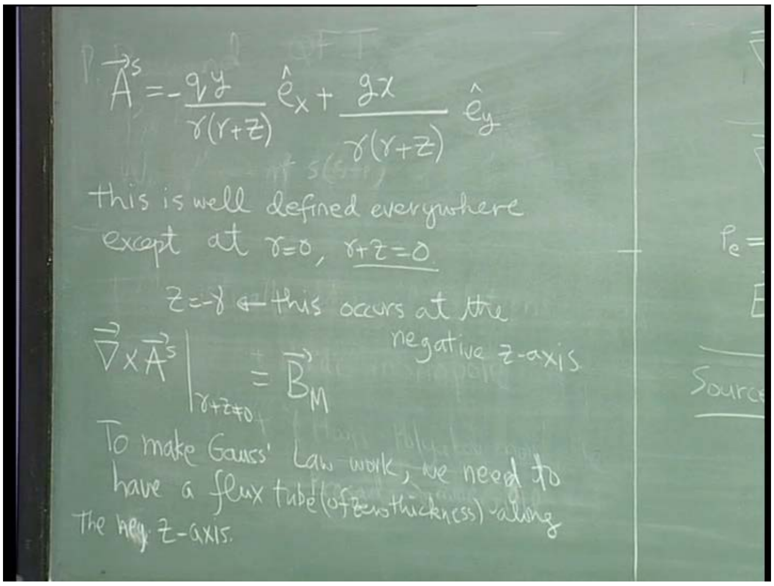

So, why is that? So, the analog of Gausses law again would be integral. Again, I will just write out first for, so, let us take a two sphere of any radius r centered at the origin, with some radius r e dot d s. What would this be? 4 pi times the charge enclosed. So, in analogy with this, we can write integral over any radius S 2. So, this sphere of radius r centered at the origin, I am just taking, I can choose any Gaussian surface that encloses that charge. Then, it is ok. So, I will get B dot d s should be 4 pi g. Now, the most important point here is, if you look here, this integral is done everywhere. I mean, you can see that, at no point, where this integral is being done is, I mean, it is not passing; the origin does not intersect any point of this point. It is a perfect; it is a sphere away from this thing. But, so, we have said that del dot B is 0. So now, the thing is, suppose an A exists, then I claim that it would imply that integral B dot B s equal to 0. So, how do we see that? Let us just do in the, let me draw some pictures here.

So, let us say that, I choose a sphere, x y and z axis and let us say that, I just, I create a hole of some small radius little a. So, what I have done here is, what I have is a sphere with a hole in it. So, really I have an open surface. I can evaluate this. I will evaluate the same integral. It is no longer a closed integral, but, it is an integral over S 2 of r minus a disc. It is a open integral B dot d s. This should be equal to; but, so, this is just the flux through this thing and this we can write this as a line integral over the boundary of this thing. So, this is equal to integral of a dot d l, a closed integral, where this is the integral over this guy. Some orientation will be there. We would not, I am not worrying. That, you can take care of that thing, integral of a dot d l. So, this is equal to the flux over this.

So, what is it I wanted to argue? I wanted to argue that, yes. So, this is just over l and now the idea is that, now I just take the radius of the disc. So, this is over a disc of radius a and in principle, if I take the disc to become smaller and smaller, in the limit of a going to 0, we will expect this thing to vanish. 0 as; this conflicts what we have just written out here. So, the only way this can go wrong is, if a become singular somewhere, such that, it will cancel even though, so, d l is something which is of size a going to this thing. If this goes blows up at some point as by 1 by a, then we can get some cancelation.

So, what it implies is that, a will necessarily blow up at least at one point or some small region. It has to blow up. There was no, ok. So, even though we said that a exists everywhere, whenever we say mathematically when we say something it exists, what we mean is that, it is non singular nice smooth function. We can take derivatives and everything works. That is why I was going through all these, I mean, all these things assume these things. Implicitly, we never think of deeply, but, that is the point. So, that has to be violated. So, what we will see is that, the best case scenario is that, it will be violated at some point, one point on every sphere. Best case scenario would be this.

(Refer Slide Time: 17:37)



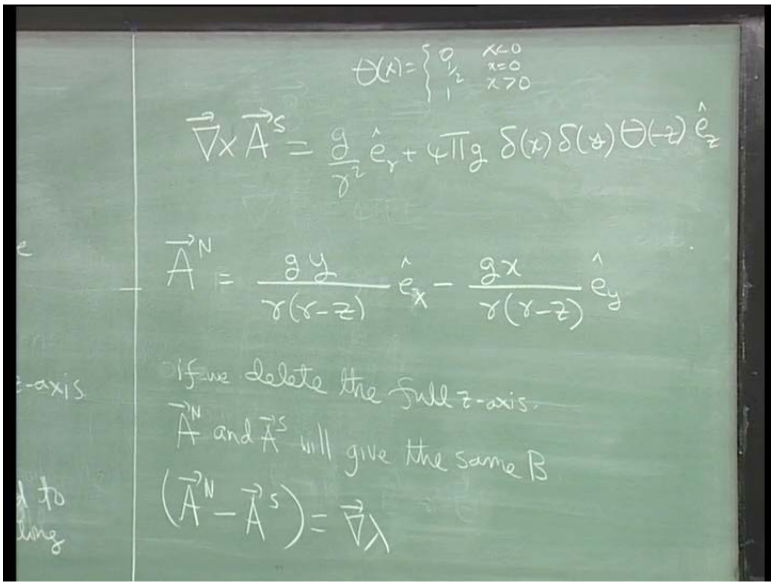

So, I gave in your assignment, I had given you two sub choices. First one I write as A vector south. So, let us see where does, so, first thing is, this is well defined everywhere, except when this thing blows up at r equal to 0 and r plus z equal to 0. These are the two points that blows up. r equal to 0 is the origin, but, we can live with that. We said we will delete that point. But, this is that which is more important. It blows up here. So, this is z equal to minus r. When does this happen? That is, this occurs at the negative x z axis, which is x equal to 0 y equal to 0. So, what it says is that, let me use a red chalk here or a pink. Is this pink? Yeah. So, what it says that, at every, the a, I have written is messed up at the south pole and then, you can check, so, what this says is that, so now, we can go ahead and if we agree to delete that z axis, we can just blindly go ahead and work out what is del cross a s for r plus z not equal to 0. At all points, where r plus z is not equal to 0 and you can check that this is equal to B. Let us call it m for magnetic monopole.

So, just following these arguments, you will see that it is necessarily, so, we will get roughly 0 from all the other guys. So, all the flux should really go through this to make the thing work. So, implies. So, to make Gausses law work, we need to have a flux tube of 0 size, if you wish, of 0 thickness along the negative z axis. This was Dirac's observation. Since, the tube of 0 thing is a string, so, this object has a name. It is called a string, a Dirac string.

Yes.

Along this single string, we started off with the B; through the B. But, we are looking at a. So, if you do it through B, so, how do I explain this. So, the thing is, good point. So, let us, so, let us assume that there, so, the idea here, yeah, that is a good point. So, the reason is that, so, the computation; there are two computations one is doing. Computation one is working with B and carrying out integral B dot d s. No problem with that. But now, if you. so, what one is doing here? If suppose I go ahead and assume there is an A, which works everywhere. Now, that has to give a 0 answer. I can even do something like try to regulate things. So, it will mess up. So, I could do one thing, which is to regulate this. If I regulate this, so, let me write out an equation and then, we will make sense of that.

(Refer Slide Time: 23:17)




That is a good point and this is exactly, so, what I am saying is that, if you actually or willing to go outside the space of just smooth functions and work with distributions, del cross A S will not satisfy. So, if you did a flux integration with this guy, so, let us say this was true, then this integral should give 0. So, what I am saying is, so, there are two contributions to del cross A S. One is it agrees with this everywhere, but, along the negative z axis, there is this contribution. So, if you go ahead and regulate things and do it correctly, what you will see is that, you will get something. So, let us have we regulate it. So, it is no longer, it does not, so, what will happen is, it will not agree with B in that region. But, it will be a perfectly smooth A. So, integral of that thing over the closed thing will be 0.

So, what one is saying here is, this will give a g flux. This contribution of this will be precisely opposite and give 0. So, the Dirac string, if you wish, carries exactly the amount of flux to cancel the other guy. But, the thing is that, what you can do is something much more. The thing is, you can look for other solutions. Let us look for; we will say something called north, which is more or less the same as this, except, it corresponds to changing a few signs here and there. So, this is just the assignment, which I have given you people, I am just looking at. So, this has the same property, except that it now blows up at r equal to z or z equal to r, which is a positive z axis. But, now you can see that on regions, if we delete the z axis, full z axis, then these two will give the same A N and A S will give the same B. So, del cross A N will give the same thing with

A plus out here. This theta function is just 0. The heavy side theta function 0 and 1. So, theta of x is equal to 0, for x less than 0, 1 for x greater than 0, probably half is what one writes for 0, right. That is a heavy side theta function.

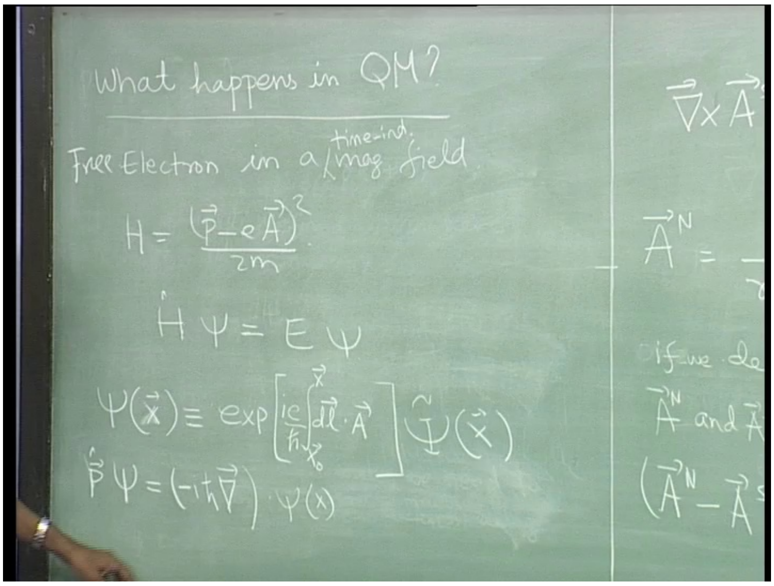
So, they will give the same B. So, what does that imply? That A N minus A S should give 0 field and should be gauge equivalent of some lambda because, B is 0. Then, you will just use the usual thing, curled curl grad is 0 or whatever that identity. So, you can actually, in this assignment, I believe I have asked you to actually compute this. So, what you can see is that, if you agree to use two patches, in some sense, one for which covers, so, it is kind of nice. So, let us focus on just a sphere. What you do is, geometrically, so, I will just take a projection. So, what we can do is, we can choose to cover the sphere by 2 things. One region is this, deleting the south pole and the other one is, so, if we agree to work with two, what you call two vector potentials, we actually can work nicely without ever, we can cover these things and we can get stuff which goes with this.

We can still work with the vector potential. So, but, the question is, so, you can ask can I write more general solutions. The answer is yes and you can show that by gauge transformation. So, another way of saying it, so, since these two are related by a gauge transformation, one way of saying it is, at the gauge effect of a gauge transformation is to take this Dirac string here and take it from here and rotate it and push it to the north. But, it need not do just that.

You could write more general gauge transformations and you could get, you know, a arbitrary line, string going away to infinity, which will intersect every sphere of a given radius once. It will still satisfy all these things, except, it will be a plane. If you give me some curve and ask me to write the thing, I guess, I can leave it as a homework assignment for you to work it out. Given a curve, which intersects every sphere once, what is the gauge transformation which will take say, this string to that. But, it is probably a better thing to do is to just take some arbitrary lambda, put it in and plot the string. That might be a more sort of a fun exercise. So, this is just a classical exercise and we see that, if you pick choose to work with one potential we do have, we have to live with a string.

So, Dirac did not stop there. This is nothing. It just looks like a homework assignment in Jackson or whatever, where you are regulating these things and working to show the, I mean, the hard part in this computation is just showing this. I think in my many years of teaching this course, so far only one student has taken the trouble to regulate it and show the delta function.

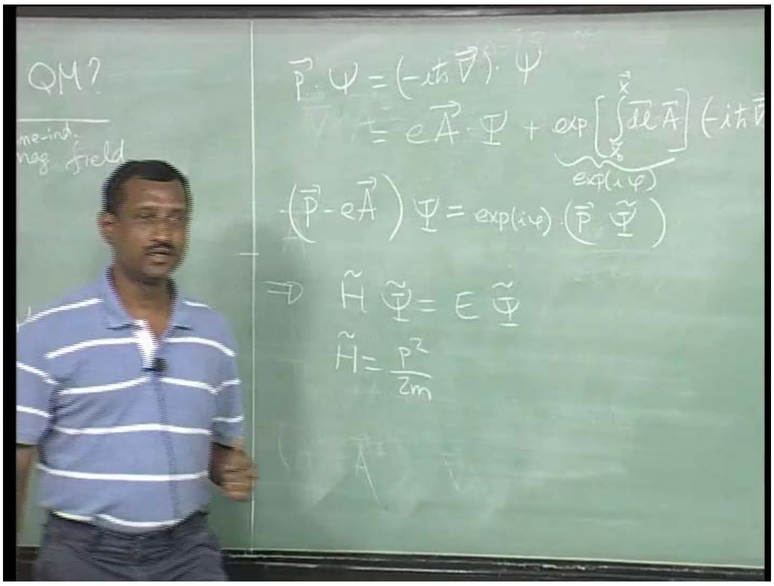
(Refer Slide Time: 30:26)




So now, the question which Dirac asked is, what happens and this is the most interesting part of the story. So, the question is, first thing is, let us consider an electron on any electrically charged particle in a magnetic field. The Hamiltonian, just a free electron. Nothing. It would be piece p minus e A whole square by 2 n. This is the Hamiltonian and restoring an equation and let us assume that we are talking about a time independent. Just for simplicity, a time independent magnetic field. So, this A also is time independent. We assume all these things. So, we want to solve H i equal to some E psi.

So, the neat observation is that, you can actually, in some sense, get rid of A and write it as an effective. We can, you can make a change if you wish. If you write psi equal to e power, let us write something and fix the sign e integral from, so, let me define things properly. psi at some location x, if I define it to be exponential of integral of some reference point x naught to point of interest d l, some line integral of A, put in e by H bar, i e by H bar and let us act on this with t, with a derivative and ask what you get. So, grade times. So, let us, so, p acting, the operator p acting on psi is nothing but, minus i h bar, del acting on psi of x . So now, del acting on this will just pull down. Let me just rewrite that.

(Refer Slide Time: 33:19)

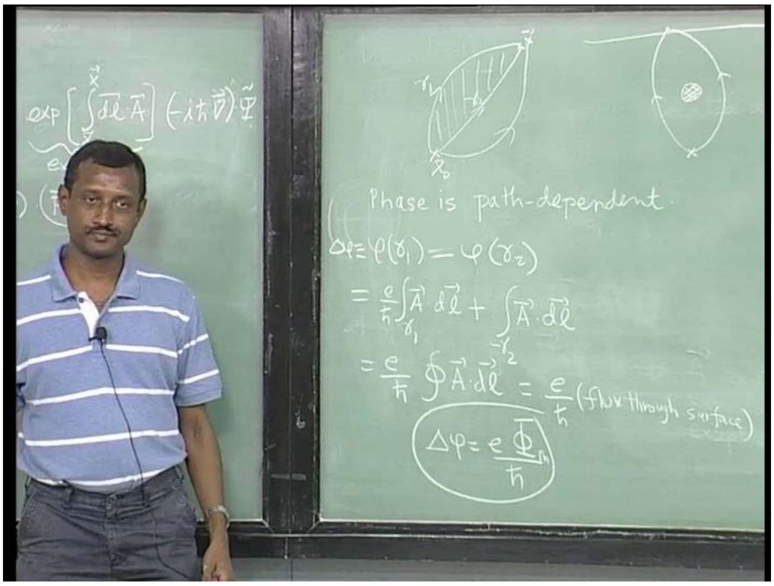


So, p operator is just minus i h bar, del operator acting on psi, some mysterious vector appeared, so, this acting on this is actually equal to, so, del, when it acts on this will have two parts, when it hits this thing and it when it hits this one. So, I will write them separately. So, the first one is when I hits this, what you get? You get A in a pull down. So, there will be an i h bar, which is minus i h bar into I, which will cancel. This h bars will cancel. I will get e A times psi and then, there will be a part which would be a plus, exponential of that line integral from some reference point x naught to x.

So, what this tells you is that, I can re write this now as P minus e A. I thought I have to fix the sign, but, by some fluke, I mean, there was a 50 percent chance that I would get the right sign; I have got the right sign. On psi is nothing but, pulling out this phase, let us call this as some phase phi e power i times some psi. I will just define it to be this, is equal to exponential of this i phase times B acting on psi tilde. A little bit of calculation will show that. So, what is happening here is, you can see when p goes through, passes through this exponential factor, it gets rid of this A and it is only as if it is a p. So, you can write out the Schrodinger equation.

So, this implies that h psi, so, I can write some h psi tilde equal to e psi tilde, where h tilde is just as if it were P square upon 2 m. So, the important point here is that, the presence of vector potential just corresponds to changing the phases of a, of the wave function. Now, the question is this. We keep saying that the phase of a wave function is not observable, but, actually difference of phases is observable. So, let us sort of do the following thing. So, you can see that this thing depends, has some reference point.

(Refer Slide Time: 36:24)




So, what did we do? We have to, let us say, we have x naught as this reference point and x is the point of interest. We can take any different path. We can go like this. I could have chosen this path, this path, each of these paths differ by something, differ by the phase. So, the answer is actually path depended. So, the phase is path dependent. Let us understand how it is dependent on the path. So, let us choose 2 paths, gamma 1 and gamma 2 and ask, what is the difference of the paths?

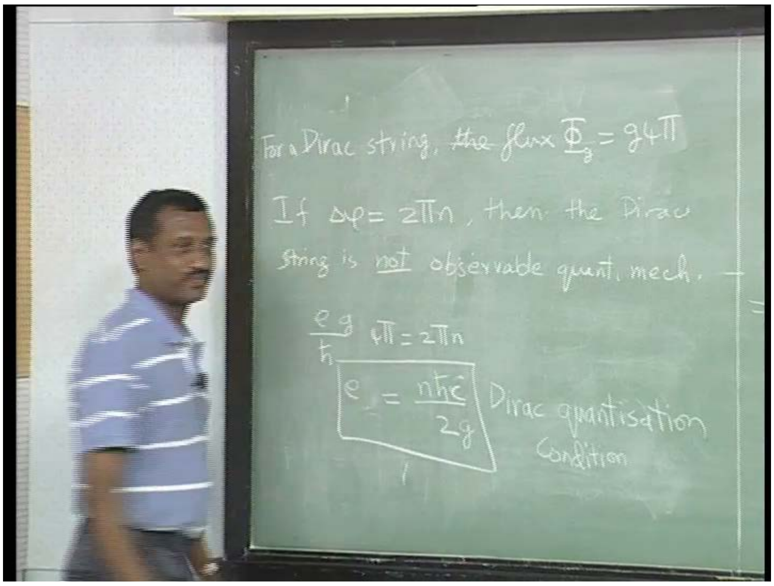
But so, what is the phase of, so, phi of gamma 1 minus, so, delta phi will be defined to be phi of gamma 1 minus phi of gamma 2. So, but, these are both line integrals. So, this is an integral over gamma 1 of A dot d l, with some factors. What are the factors? i e by, so, e by h bar minus integral over gamma 2 A dot d r. By choosing a nice property of the integrals, you can see that this has minus gamma 2, I can write this as plus of minus gamma 2, which means just change the orientation of this thing.

So, this combines to give you a closed loop, which is given by the sum of these two, A dot d l. But, just that same argument shows, going in the opposite direction, shows you that this is equal to the flux through this, the flux through this area. So, clearly this should be observable. So, let us sort of, so, and you can see that, if you were in a region where B is 0, then there is no flux. You did not observe the flux. So now, let us sort of be very, so, we can think of a slit experiment, where there are no slit, but, there is a source of magnetic flux. We can have a source or whatever. We can have a screen out here. So, there can be one path coming like this and other path coming like this. There will be interference and clearly the phase difference would be e by h bar times this B the flux. So, let us just write this delta phi is equal to e times the magnetic flux.

So, this is just standard non relativistic quantum mechanics. Nothing deep in this stuff. This thing has a name. It is called the (()) effect and this has been observed in the lab. People have seen interference changes due to these things today. I do not know the status in 1931, but, definitely today it is known, but, I guess. So, what, so now, the thing which, so, Dirac argued, see we have a string out here and this string is not a physical object. It is a given. Its location is a gauge artifact.

So, he asked, suppose there is a magnetic, suppose there exists one magnetic monopole, that will be a, it will have a Dirac string depending on what you choose, it is you are a. And so now the thing is that, so if you want the Dirac string to be unobservable, that would be unobservable only when delta phi would be integrals of multiples of 2 phi.

(Refer Slide Time: 40:53)



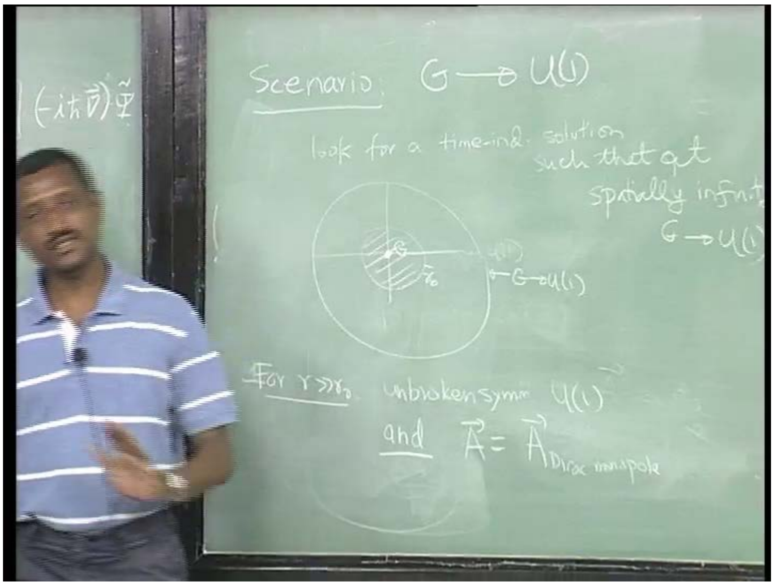

So, for a Dirac string, the flux is, the flux phi due to the monopole of string g would be g times d 4 pi because, we are talking about the flux, right, in the g 4 pi, g into 4 pi. So, what we want is, if delta phi is 2 pi n, then the Dirac string is not observable quantum mechanically. So, we can just, we note the flux. So, we note what is delta phi? It is e by h bar into phi, which would be g into 4 pi should be equal to 2 pi over n, 2 pi times some any integer. So, what that implies is that, e g equal to n h bar by 2. So, this sort of is an amazing result. I mean, the thing is that, Manikins' oil drop experiment told us that, the charge was quantized in units of charge. But we saw that in the standard model, there were things which were carrying one third charge, but, for some reason, we do not even seem to see one third charges as free things. But, you can say it as things are quantized in terms of one third e. Whatever is the number, it does not matter.

So, what it says is that, if you find, if you observe even one particle with g, all the charges in nature should be such that e g is this thing. So, if you are given this g, so, e will become, so, we can flip this around and we can write it like this. You can see that all charges in nature. So, e could be, in this case electron, but I can replace it with some other particle, any other particle, all charges will be quantized in terms of h bar by 2 g.

So, this is the first ever sort of explanation of why charges quantized and it is called the Dirac quantization. We use the English spelling because, Dirac was English. This is an amazing amazing result. Very very simple and we can understand it at a very elementary level. So, actually I would like read out what Dirac said, "there is the mere existence of a of 1 pole of strength g would require all electric charges to be quantized in units of h cross c by 2 g and similarly, the existence of one electric charge would require all poles to be quantized." So, there should, if always put c equal to 1. So, I am just looking up from here and putting the factor of C out here. So, C, I have written capital C here, I mean that speed of light.

But coming back to this solution, we know that this looks little bit sick. In the sense, it is not a real, I mean, that is not a solution of Maxwell's equations, if you wish, because, if you wish, there is a definitely a singularity at, I mean, something which violates the normal Maxwell's equations. But we have already seen in this course that electromagnetism comes from some other thing. It is some s u 2 cross u 1 broken down to u 1. So, the underlying theory is something else at a fundamental level and the u 1 which we see is something else. So, both (( )) simultaneously actually ask this question.

(Refer Slide Time: 45:20)




So, suppose we have, so, the scenario is the following. We have some non abelian group g and it is broken down to u 1 electromagnetism say. So now, the thing is that, so, the issue we had was, so, let me just draw again. So, the issue we had was really at the origin, where there is a pole. But suppose we are asking for a classical, so, in this thing, you look for time independent solution, such that, at spatially infinity, this is the pattern. So far away. So, let me just draw a 2D picture, but it is a 3D picture. So, what we have is at the origin. We have a problem, but, at spatially infinity, let us say there is a scalar field, which gets a well. That is a Higgs mechanism. So, this non abelian local symmetry gets broken and we just have, there is only one mass less gauge field and that is an u1. But, suppose in this region, that is not true.

Like, if the vertex we saw, right, far away, the vertex took values, where the symmetry was broken. While inside, it was restored. But, the thing is, so, in this region g is broken down to u 1. But, in this region, you could have the full non abelian, the singularity gets replaced and what you get is the full non abelian thing coming back. So, you can construct such a solution. They look for a solution of this kind. So, when I say that something is charged under magnetism, it is charged under this, at broken u 1. So, what they did is constructed. So, the thing is, so let us call this some r naught. So, for r greater than r naught, the unbroken symmetry or mass less symmetry is u 1 and the configuration and A agrees with A for a direct monopole, outside that region. Then, all your all your things, whatever you did, you can see that this will give you a magnetic charge because, as long as I choose my sphere to be larger than this, Gausses law, you can use apply standard Gausses law and you will and anyway, the charge, so, you can think of this as some object lump, which is not point like, but, as some size r naught and carries magnetic charge. Because, how would you measure the charge of a finite size object? You would take some Gaussian surface which encloses that surface. So, I will do the analog of that and this will be magnetically charged, except this is non singular in this region.

There is nothing strange happening. It is just that the wave of the scalars etcetera, which do this, give you, end up giving you a nice smooth solution. So, that is what we will discuss in the next lecture. That is also your assignment. So, here you would see here that unlike a Dirac string, this is truly embedded properly into a nice field theory. You cannot say the solution is singular and so we can through it out. Nothing. It does not violate the equations of motion, which it did. The Dirac monopole did it. It violated at one point. But, it violated the equations of motion. Here this will not violate the non abelian equations of motion. The Euler Lagrange equations will be a solution. But, for Dirac's argument, I do not think there is any difference between this one, this monopole or that monopole. So, the quantization will hold. So, the example we will look at is to take the group to be SO 3 and then, we will find that there is a solution.

But except I think, the charge, the quantization of that will not be minimal. It will be two times that. It is related to the fact that we saw that s u 2 cannot be broken down to u 1. By giving in the fundamental, so, we have to take something in the adjoint, which is the same as saying, working with SO 3, at the level of lie algebras. So, this is the one we will look at. So, we look at the details of the solution. It is obviously much more harry. It is not so simple and also gives you an example of what we will call soldering of space and gauge indices. It is a non trivial example of that and 't Hooft's paper is I think eminently readable. Polyakov's paper, I think might be one page. So, it has everything. The ideas are there, but, much harder to read.