Classical Field Theory

Prof. Suresh Govindarajan
Department of Physics
Indian Institute of Technology, Madras

Lecture - 3
Obvious invariances

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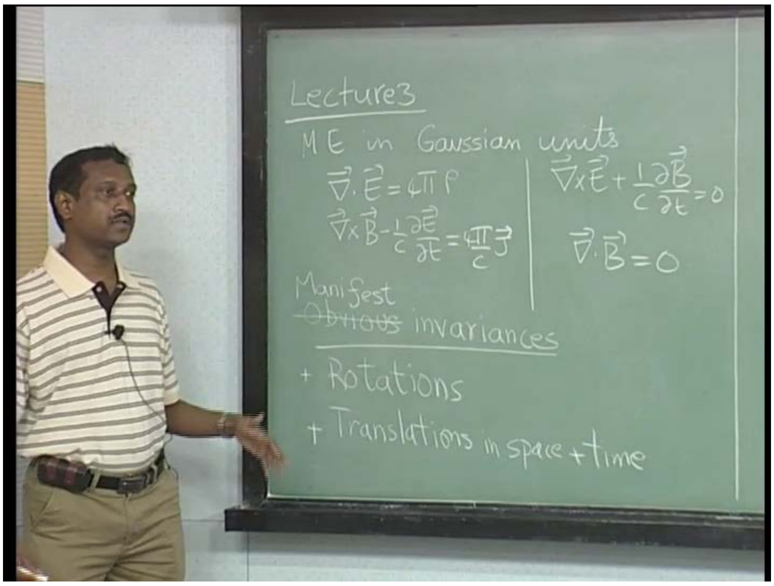

So, last lecture one of the things we accomplished was to rewrite Maxwell's equation, in Gaussian units, and I am just rewriting the same hopefully there are no sign added. So, the idea here is to ask what are the invariances of Maxwell's equations? So, obvious invariances are as follows, we can see that if you look at this particular equation, the left hand side of this equation is scalar, so is the right hand side. Now, this is the vec, so again each one of the three terms in this equation is a vector equation.

Now, you can see that since vectors are equal to vectors etcetera, you can you one says that the invariance under rotation is manifest. So, it is the obvious I mean (( )), so we should say the correct word is to say manifest, where you look at it and by inspection, you see it. So, it is invariant under rotations, it is also invariant under translations, and the easy way to say that there is no explicit dependence, either on time or special coordinates.

So, it only remains that we need to check, whether it is invariant under Galilean boost to check, if it is so the Galilean group was done by taking all these various these two things plus gallium boost., but it turns out that this is not invariant under, Galilean boost I will what I will do is rewrite these equations in a manner, where it will be variant under something different.

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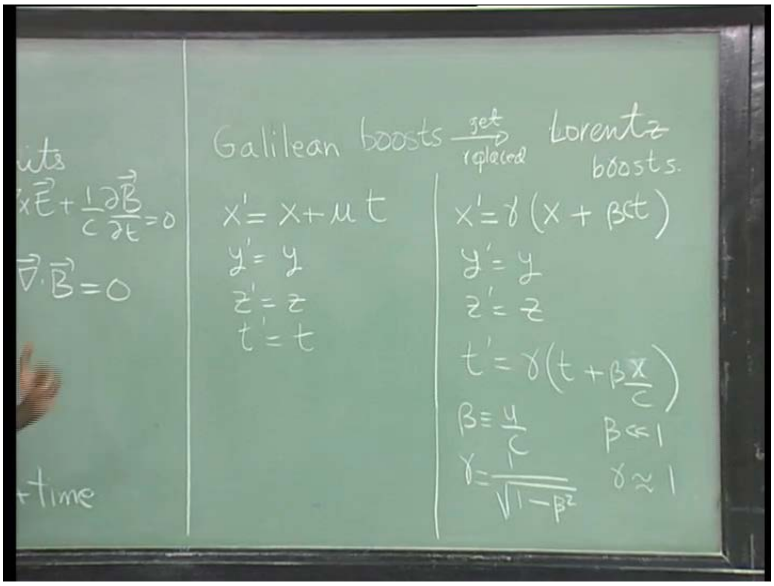

And we will see that Galilean boosts, gets replaced by what we will call Lorentz boosts, so let me write out what what was the Galilean boosts. So, let us say we had a boost in x a direction, and we would write something like x prime equals to x plus some velocity, so let us write u t, then y prime would be untouched z prime, would be untouched and t prime will be equals to t, this is what we meant by Galilean boost.

But, a Lorentz boost is much more complicated and I will write the form here, but we will derive it in a completely different manner. So, what it does is it makes x and t in a trigger manner. So, let us just write that out. So, x prime one writes as gamma times x plus beta t this two remains the same, but you get a change out here.

We introduced a couple of things gamma and beta this is standard in courses in special relativity beta is nothing, but u by c, and gamma is a square root 1 by square root of 1 minus beta square and see this is speed of light. So, now you can we can do something if you think of c as a parameter of the world we live in t is just a number, it is number, but for this purpose we could think c being formally being very very large or u being very very small. So, beta is very small dimensionally there is a problem here this should be c t, so beta is taken to be very small.

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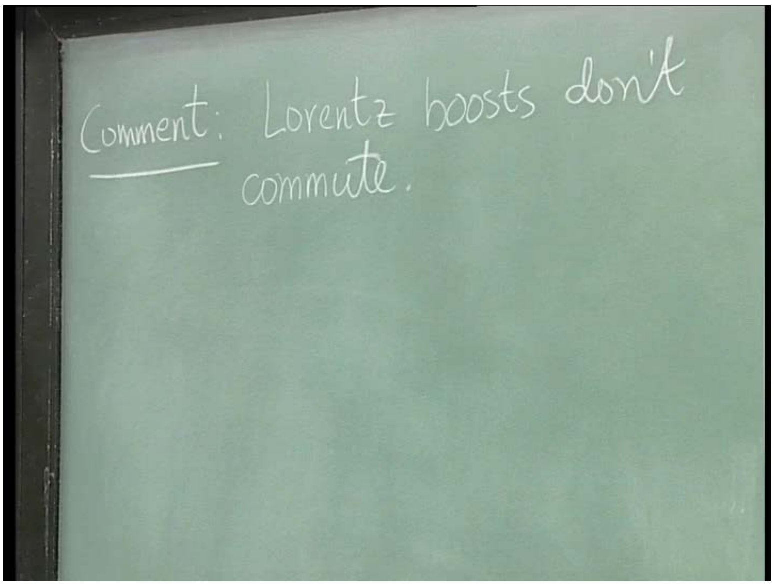

So, when beta is small beta is much much less than one gamma is approximately 1, let us correction of course, order beta square for gamma. But we can see that what we see is that when beta is much less than 1, and gamma this thing you see that you can actually recover this equation.

So, Lorentz boost turn out to be the invariance of this set of equations, and what we will do in the next 5, 10 minutes is to write rewrite these equations in a manner, where this symmetry will be manifest. So, by inspection we looked at these equations say they had to be invariant under rotations, we will do the same thing for these equations.

So meanwhile it is useful to look at how these things are different, the key point here is that x and t can mix we can change this, now so let us go go ahead and there is just one nice comment make, so let us come back to these things suppose we had a Galilean boost in x direction, and we followed it up with Galilean boost in the y direction, if you compose the two of them you get, another Galilean boost, you use the Galilean law of vector etcetera etcetera works here, but if you do the same thing. So, we have the Lorentz boost in the x direction, you can follow it up with a Lorentz boost in the y direction, you can compose them, there is no problem except the resultant, there is no longer a Lorentz boost in a some in a direction, like this and this has to do with the fact that Lorentz boost do not commute for example, in other words the order of the operations are important.

So, I mean I have an example, here of a operation which do not commute. So, if I have a rubes cube. So, I could I could rotate it by 180, here and if I followed it here with rotation by 180 on this things, since there is common edge, you will find out that it matters what I do first or what I do later, but suppose I did this followed by this this there is no common this things they commute, so this is example of operations which do not commute. So, we see this things even if you are solving puzzles, it matters the the ordering matters. So, it is kind of amazing that the generalization to this is not even obvious from looking at this, but like I mentioned towards the end of last lecture, it is important that we should be able to recover, this starting from this equations just kind of this limit.

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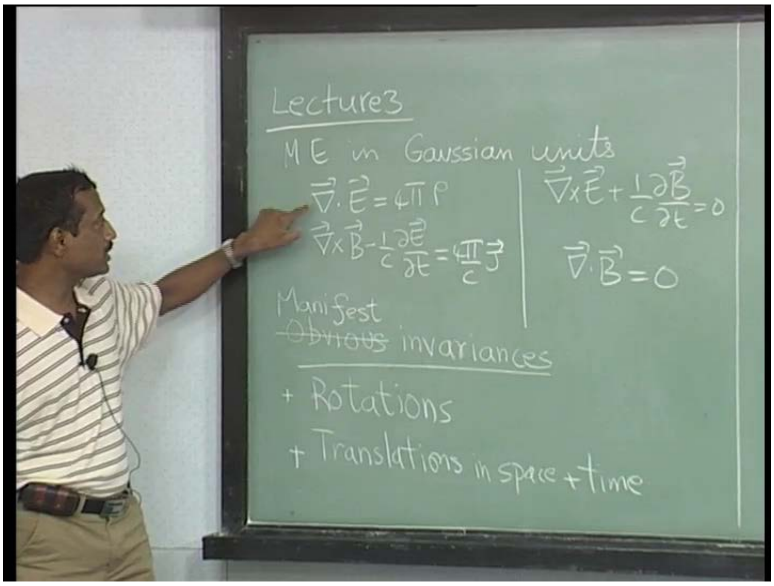

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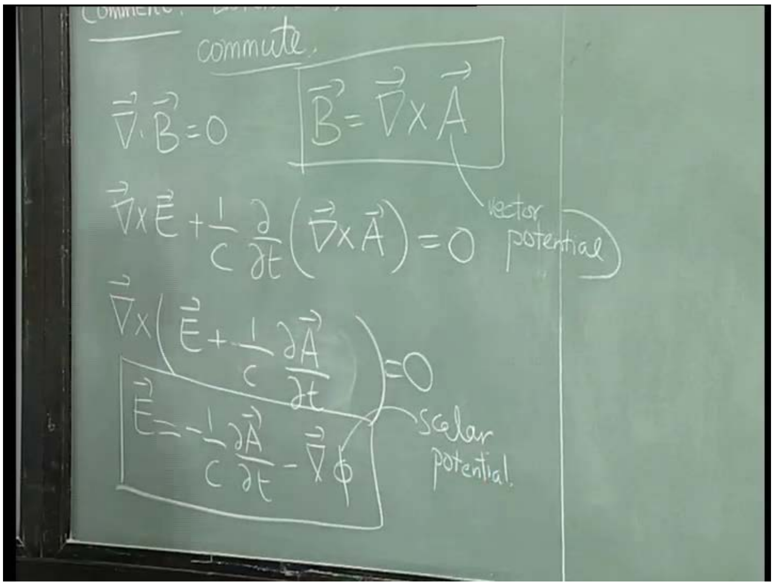

Because, this is the limit in which we have actually studied Newtons equation, we never ever looked at looked at situations, where beta is close to 1. so this is something which you have to sort of remember. Now, we just go ahead and revisit maxillae equations, and we will rewrite it and first thing to notice is I have written out two different, I have ordered it in what we might consider weird manner, because you might be thinking you should cross del E cross del E gays on same thing, but the reason is that these two have sources and these two do not have sources, so we can go ahead as for all these two equations, we can solve for them. So, we start with the obvious one, del dot v equal to 1, you can trivially satisfy this by which saying B is equal to del cross A.

Now, comes the interesting thing we can go ahead and try to do a similar thing out here, you have already solved for b in terms of a and this chest follows del dot B becomes trivial identity, and we need to do this, so let us look at del cross E plus 1 by C d by d t so, what you can now see is that we we can we can we can take we can put out a del cross and rewrite this whole thing as del cross E plus 1 by c into, So this is now looking like what we had did in astrostatics where this term was not there there’s no a while del cross e was equal to 0, we solved it by saying it was radiant of some potential, yes yes thank you, this is wrong thank you so this what you get.

So, we can solve for it now again you can write E plus d A by this term, which I will take to the left hand side and the convention is to choose it to be minus radiant of the potential, so the yes. So, the point is you are going to assume the functions are smooth so we can change orders of the derivatives I mean even here that is true right, we assume that the x and a y derivatives the commute,

So, now we can see that we have gone ahead, and solved for E and B in terms of two other objects, the vector potential and this is called scalar potential, now we can go back and plug these two expressions into these equations, what I am going to do now is to do the first equation, because if less tedious and as usual I will leave the next one as exercise for you.

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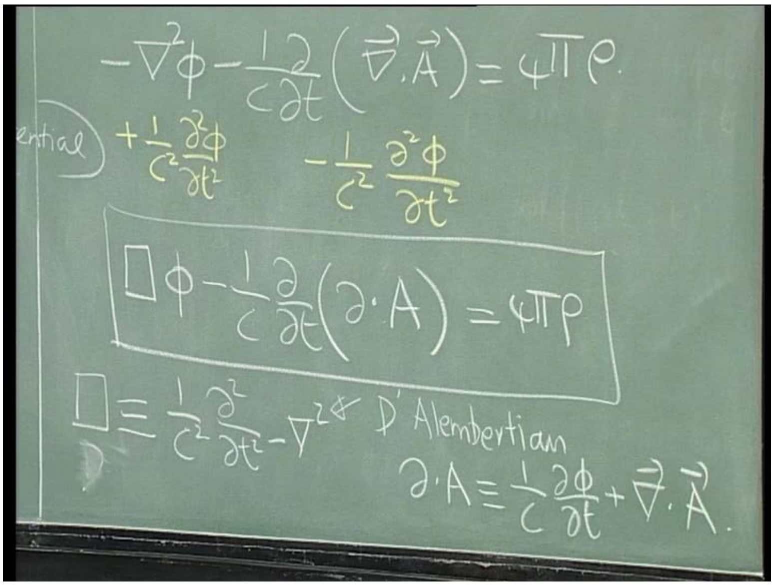

So, let us so what we have is del dot e is equal to four phi row. So, E is given by this. So, let us check let us do this, so we can write here del dot let me first write this radiant part yaah. So, what is del, of del is so we end up getting Gaussian square phi, and then again I will exchange the order of the 1 by C functions are smooth, what I will do now is to do is to add and subtract this two phices, and let me do that first. So, let me use different color, so you see what I am adding, so plus 1 by C really.

I am doing nothing I am adding 0, but I will combine these two and I will reorder things out here and I can rewrite this combination, I will write this as box phi I will write some symbols out here, and then I will explain what these symbols are, box is an operator which is called a d’ Alembertian, which is basically, so it is called the d’Alembertian and this combination D dot A is nothing, but 1 by C d phi by d t plus is this combination.

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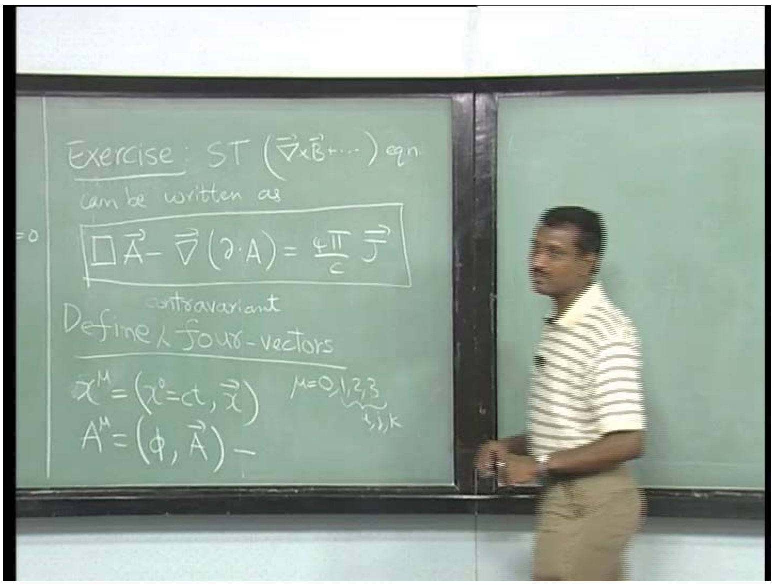

So, these are just definitions, so are there any questions. So, I will not disturb this bit, but I will erase this and like I promise, I will write exercise now the exercise is to show that, the del cross B equation, can be written as follows box a minus radiant. So, now there are need thing is that you can see that, there is a quite a bit of similarity between these two equations, all that is happening, all that is happening is that phi here gets replaced by A and of course,, row gets replaced J here, there are factors of C etcetera, so let us let us see how we can combine, what we will do now is to combine these equations. So, if we just do in rough counting this is one scalar equation, and there are three equations out here we will write this as one equation with four components by what we have done here and. So, we will define a bunch of objects, which we will call four vectors.

For the first four vectors is x mu, it is mix space and time coordinates and that is x 0, which you already defined in last lecture which was c times t the other 1 is just x. So, important to note I am writing this thing in upper index, we will see later on in this course, what the lower case would be, but at this point, all these case are upper indices, and with the upper 1 called contra variant vectors, and this mu this 1 runs with the index 0, 1, 2 and 3. So, 0 stands for the time coordinate.

And 1, 2 and 3 are special coordinates x y and z, it is better to write and we will follow the noted convention, that we will use symbols like i j k, the roman ones for special indices and the Greek ones mu, nu etcetera running over these, this will be the convention throughout this course virtually no exceptions now I define a four vector potential, lf minus 1 by 7, we will see you will see in a moment where there this is correct or wrong, we will see it it cannot be d by d t because it is a this is a vector. So, I will require from A operator d by d t is not a vector; that is a easy way to check, but you can go back and verify this

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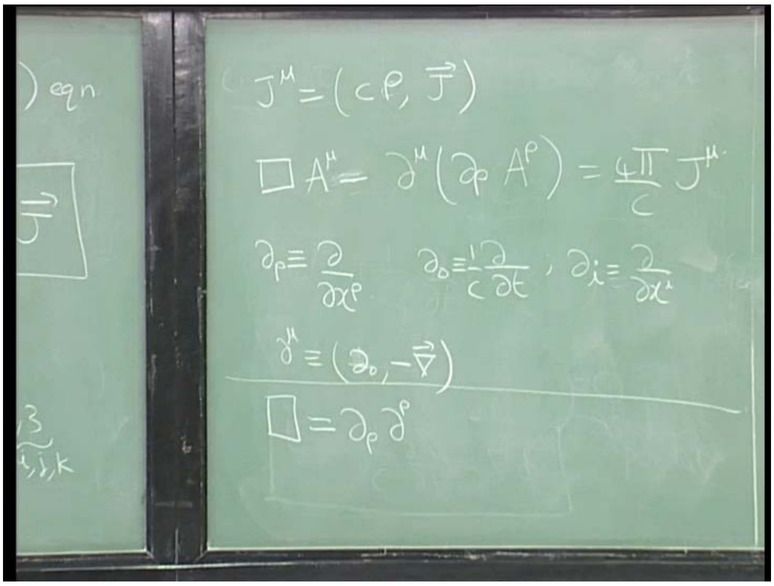

A upper mu is this phi, and we will also define a four current. So, this so this is a four vector potential, but on and off I will use the term vector potential, and you have to understand the context, whether I mean this vector potential or whether I mean this and there would not be an ambiguity, you will see and J mu c row.

So, you see something very nice happening all these four vectors are made up of one element the first zeroth element, which is tether in the rotations, and the second element is the normal vector, under rotations that is true in every one of these, now you can see that I should be able to combine these equations in the following way…

So, when mu equal to 0, a 0 is phi. So, you end up getting this particular equation, and the rules are as always, we have a one upper row and a lower row, they have to be summed over, but now they are summed over 0, 1, 2 and 3.

So I need to still define for you what these operators are d row, it is a very natural operator it is just definition or equality if you write out, you can see d 0 is defined to be d by 1 by C d by d t, and d i is usual d by d x i, but the upper mu has a sign problem it is not a problem, but it has a sign difference and in notation, it will go like 1 by d C.

So, we will write this as 0 and minus del, so that is a sign issue which comes about. So, the upper case, which you see these two actually different, in this case it is only a minus sign now comes the rule well I will give you a working rule is whenever a index is summed over you end up getting object which is k not just under rotation, but also under Lorentz boost, we will we will formulize this whole thing in rest of the lecture.

But right now this phi is a scalar this operator box also has a can be written in a nice way, it shows it require this can be written as c row one upper and one lower, because the rule is that I cannot sum up too lower, or too upper, right now it is just a rule, but there will be lots of meet in that thing.

So, useful to remember the rules, so this is a scalar this is also a scalar, but this object is a vector. So, roughly it is again like before it is scalar under Lorentz boost as well; so this is a four vector, this is also a four vector, and this also a four vector. So, we have an equation which has four vectors, and I am just going to add this point again that four vectors transform nicely, under Lorentz boost and rotations of course, that is easy that is manifest already. So, we need not worry, so I just need to discuss how they transform under Lorentz boost.

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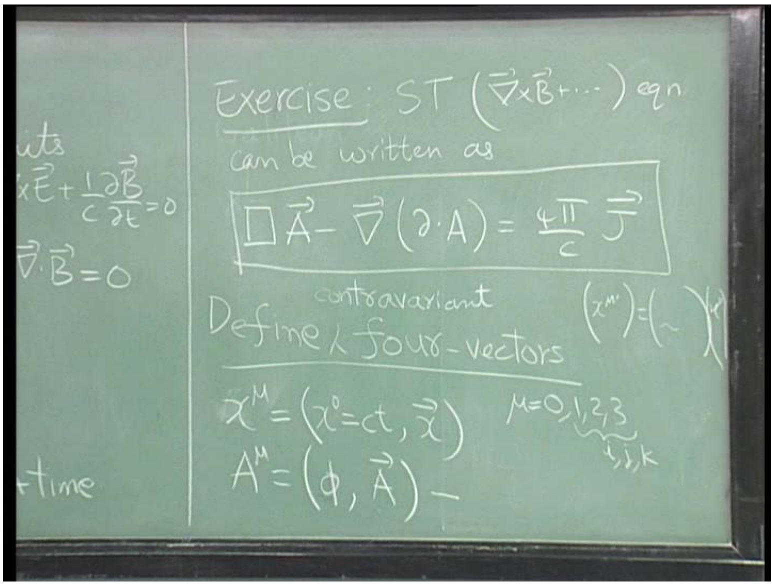

So, but that is already done for you I will explain to you how, x mu be I wrote out in this board where I just arranged something, and rewritten out how under a Lorentz boost x mu transformed, we saw that if a boost is in x direction, and the t components mixed, now the the statement is that all four vectors by definition transform exactly like x mu under Lorentz boost.

So, wherever you saw a zero if you want to work out that transformation, now this you put a zero, wherever wherever you saw the special x you replace it with A, that is the rule. So, I have given you how it transforms the important thing is that they are all linear transformations by that, I mean you can write x mu prime equal to some matrix times x mu, think of them as column vectors. So, i write organize it as x 0, x 1 x 2, x 3 there will be some matrix exercise for you write out that matrix it is a simple matrix. So, you can see it is just a linear transformation. So, this equation gets transformed under Lorentz boost to some linear combinations of itself, that is all it says, and I still have to tell you how a lower index transforms, but I claim that even that is obvious, if I say if you use the fact that I want this phi to be a Lorentz scalar, so if this transforms by some matrix by upper phi the lower phi should transform with the inverse.

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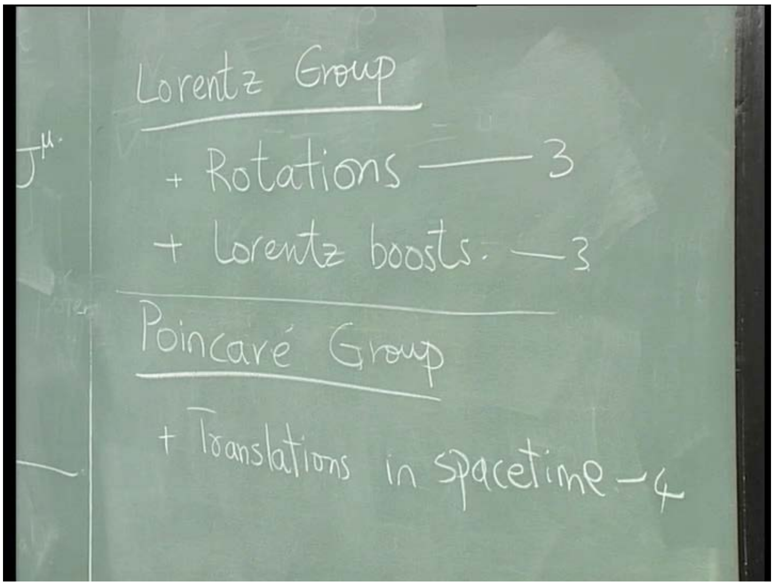

So, what we have achieved here is to show that matrix equations are invariant, under under Lorentz boost not Galilean boost. So, now we need to actually understand, what we mean by all these various things, but before that let me define for you the Lorentz group, this is the group given by rotations, and Lorentz boosts.

So, it is important this statement is a fact that it forms a group, it is it is important that a combine, a like I said two Lorentz boost, if I compose them together I do not get another Lorentz boost for what you get is a combination messy, combination of a rotation and a Lorentz boost another Lorentz boost, we will prove all these things, in a very nice manner, but at this point this is a and, so rotations had 3, 3 parameters like a Lorentz angles and a Lorentz boost, we have again 3 variant x y and z. So, six of them and you can also if you there is something called a Poincare group, you add translations in space time I no longer will say space and time, because now we have emerged them all we just use a single word, which is space time; so this has 4.

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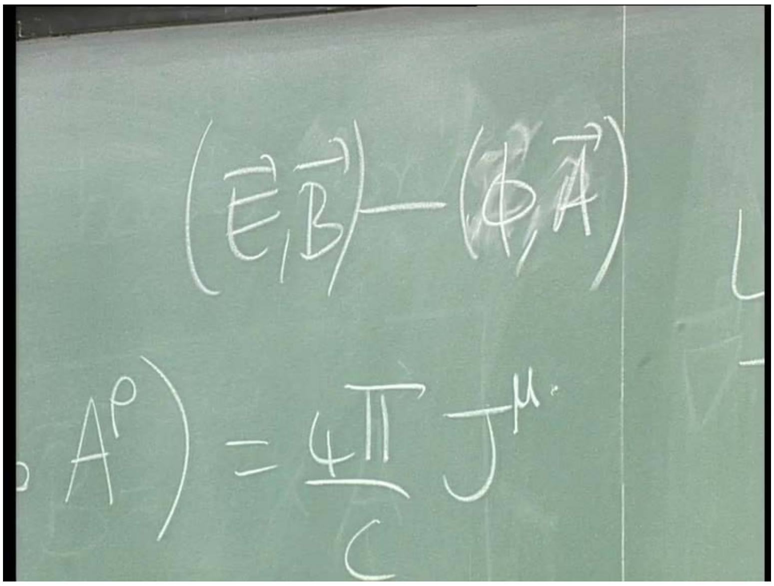

So, In some sense the Poincare group plays, the role of the Galilean group in non realistic and, so what we will do now is to understand, how to go ahead and define vectors, what do we mean by vector under some group or or whatever, it is so start with we go back to actually normal even, here I said under normal rotations, it is vector what do we mean by this what it is sense, what are all these things are they generalizations of that and how general can we get, so let us just one small bit, which I just forgot is that we made a change of variables.

We went from E and B, we went to phi and A by solving the equations, but this is not actually unique the phi and a are not unique, you can get, so I would ask you to say it consistently what is the freedom that you have and so, it is not the point here not unique, but it does not matter, which there are equal phi and A, rather equal phi and A, it gives same E and B as those are the physical.

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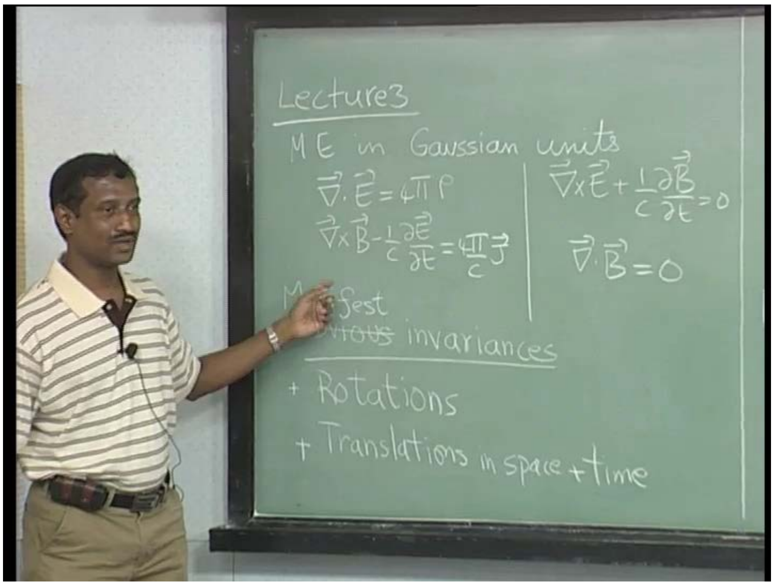

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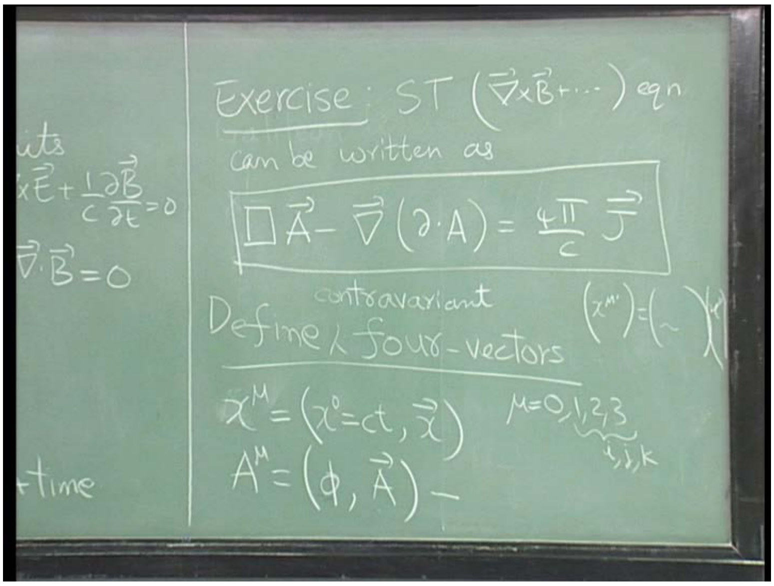

Another important point is all Lorentz equation is always second order, but these equations are not second order, they are first order equations, and so in some sense you can see that the right order checked in Maxwell equations are actually is a is four vectors because you end up with second derivatives, which you normally see in classical mechanics, and the most amazing thing is that there exists an action, which gives you the equation of all this again you will see..

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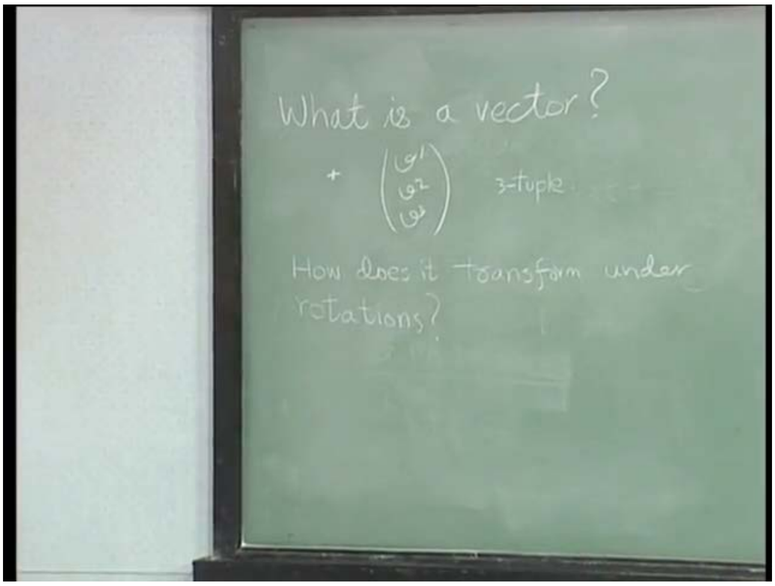

So, so we will go back and revisit what we mean by vector, so the point here is that we were very happy, when we see I mean when we said that things, were manifestly invariant when we wrote things in terms of vector the reason is the following is that first thing is let us first point is that something, which has three recited as a column vector, or it has it has which we call a 3-tuple, and it has certain properties under rotations.

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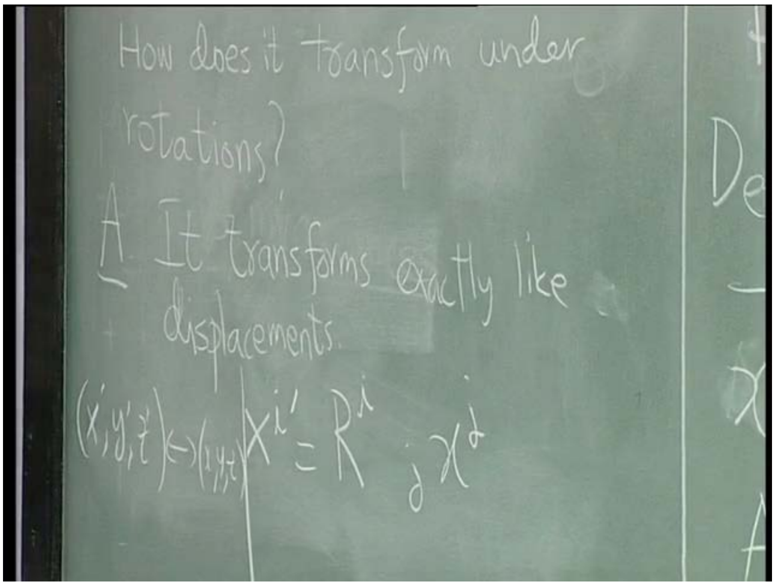

So, now ok and the answer to that is very easy it transforms, exactly like a like a coordinates, like displacements to be precise. So, by that what do we mean suppose we have a rotation, so let us say let us write this way R x prime is some matrix x j. So, it let us say some coordinates transform in this fashion. So, we are going from coordinates x prime y prime, z prime which x, y and z, which has x 1, x 2, x 3 as here.

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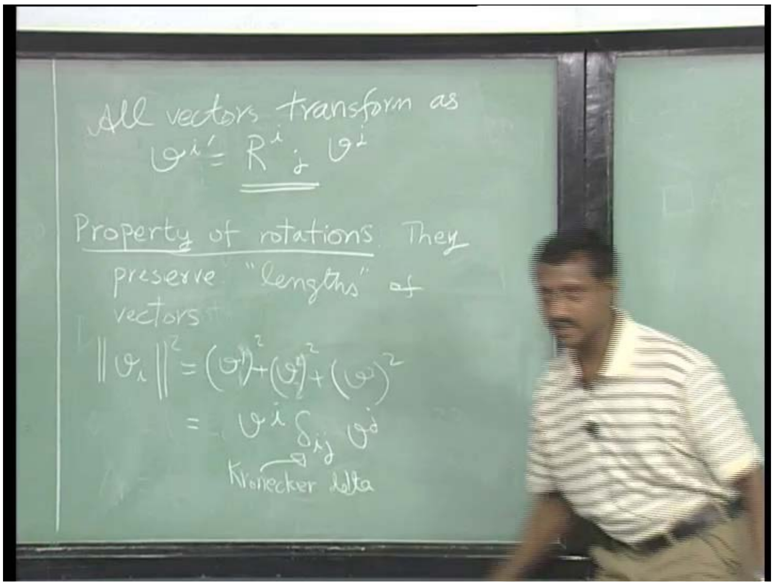

So, the statement is that any vector, we will call any object vector or better all vectors as exactly like this exactly some arbitrary vector. So, let us ask what are the properties of this matrix, what characterizes this thing we know that under rotation, the length of vector is preserved, so the property of a rotation it is very important actually lengths.

So, if you give me a vector, how do you define, it is length you normal definition of any vector, which has the sum of all it is components, but let us write it in index notation you would write something like this as where delta i j is the kronecker, delta and this i index is summed over, because it is one lower 1 upper repeated same story out, here this is called kronecker delta, however if you tried to write this if you call b as a column vector.

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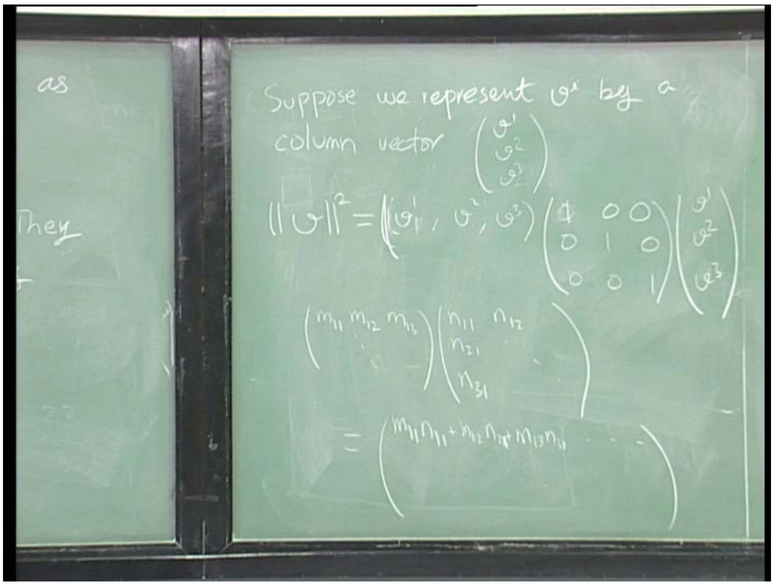

Then this term has to be written in the following way, v of i followed by a column vector, this is just a convention, which I will follow usually like I did there, then you will see that a we can rewrite this a prom prom this norm as a transform of this which is a row vector, and kronecker delta, which is the identity matrix, then you can see that if you carry out this thing you will; obviously, recover.

So, there is a little bit of work, we have to do to understand, how this happens and this has to do with how matrixes are multiplied, if two matrixes are multiplied. So, let us just take two matrixes m and n. So, let us say it is 3 by 3 matrixes m 1 1 and m 1 2 m 1 3 and. So, and. So, multiplying matrixes which is n 2 1 n 3 1 and. So, and. So, we can write more terms, but for me I just need to multiply out by first row and first column.

So, let us just concentrate on this you will start getting m 1 1 n 1 1 plus m 1 2 n 2 1 plus m 1 3 n 3 1, and rest of the comes not important let us focus on this, what you can see here is that the column index of the first matrix is done with the row matrix of this column.

Now, coming back to here let us look at delta i j, there the row is the first index this is a column index, this column index is going with the row index this is level. So, this works out correct, but out here you see this is the row index and this is the column index, so it does not work correctly, so you need to do the transform.

So, the transformation is exactly the operation, which takes care of that which converts a row index into a column index. So, this is some it takes some practice to get used to going from this form to this kind of form. So, but this is very important to realize is that when you have write something as a matrix you have to order in matters.

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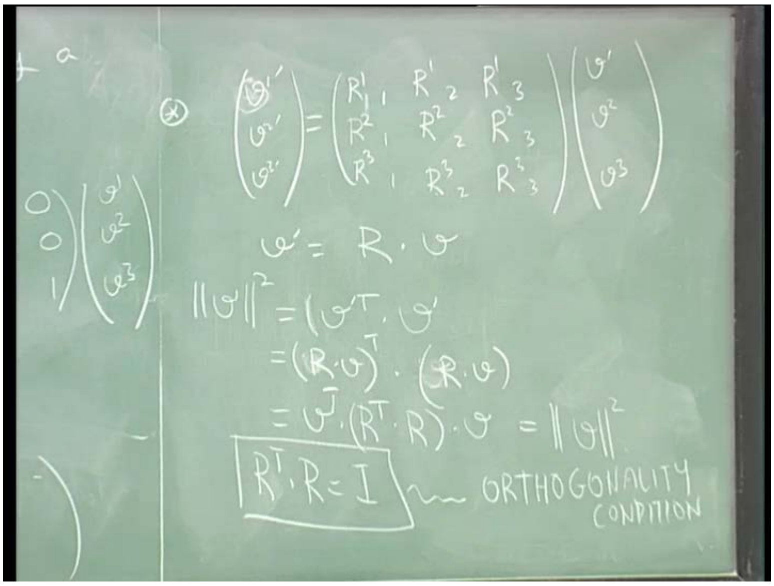

So, now we can go ahead and ask how this looks and this is nothing but, so, we could write that equation, so let us give this equation a star again star. So, star can again be written as simple matrix multiplication, so it has R 11, R 1 2, R 1 3 and look at the indent indentation, which is the row and which is the column or in short, we can write we can hide all these indices and write v prime is equal to the matrix r multiplying v it is a same thing now this now we can characterize rotations, we were told that rotations actually preserve norm, let us understand how that works, so we have to ask what is v prime square.

Since, this is a identity vector, so this just gives you v transpose. So, let us see what we get, so what we get is this is equal to v transpose dot v prime, but what is v prime it is r dot v. So, we end up getting under transposition the order changes. So, we end up getting this is e equal to v dot, and rotations preserves the length, so this should be equal to and this has to hold for all vectors not for a particular vector, for any vector for any vector this has to hold, and that can happen only when R transpose to R identity index y a a h

Now, we get sort of independent characterization of what we mean by a rotation matrix, it has to be some matrix, which preserves the length, but turns out to be this this statement, it is a sort of matrixes, which satisfy this condition; this condition is called the Orthogonality condition, does anybody know, why it is called orthogonality condition; it is a. Firstly, … vectors or orthogonal

And the final three, so if you choose a basis for these vectors, you choose actually a orthonormal basis, what you will find is that it will map, it is the defect of transformation linear transformations, which map orthonormal vectors, to orthonormal vectors; so that is why is called orthogonality condition.

So, this is the characterization of rotation matrices, there is little more structure in this which we will discuss, but before that there is one simple generalization here, we just said it is a sum of three components, but in principle I could have replaced with adding many more component, and let us do that generalization up to some d.

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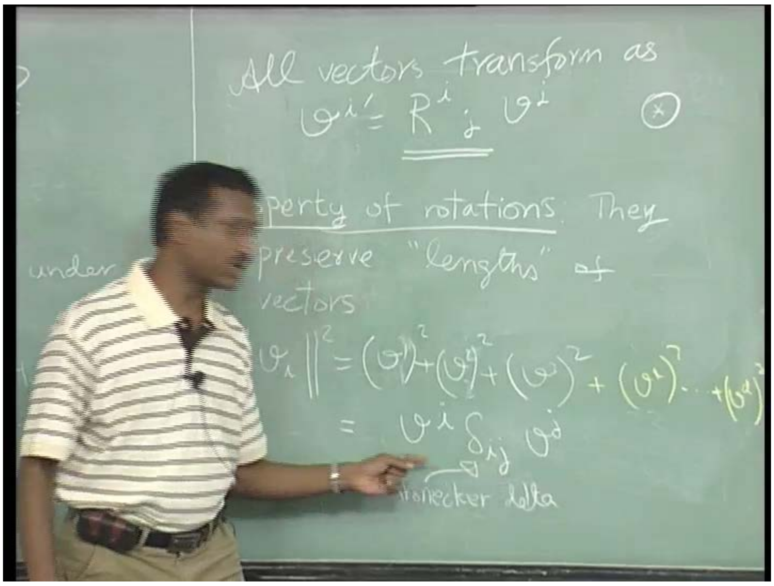

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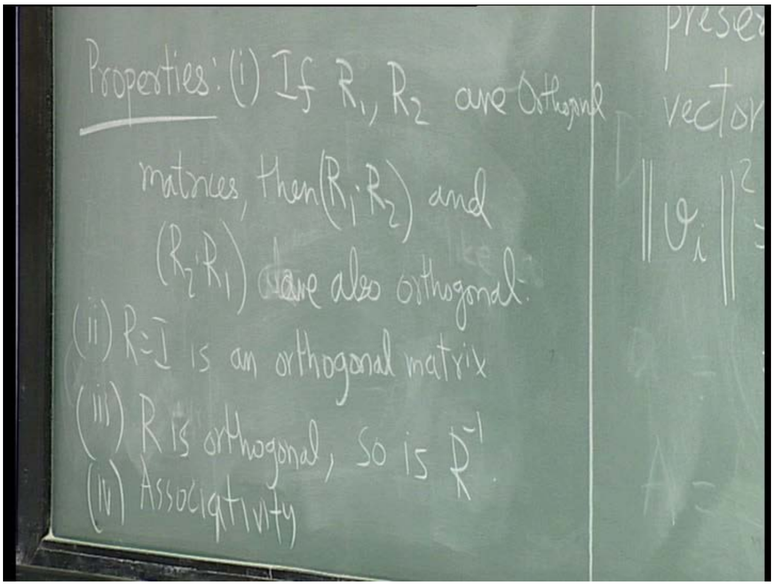

Now, what I will write as as a 3-tuple will become a d tuple, it is a vector of two components, but most of the things, which I said will go through which only thing is and this form nothing will change accept I and j is running from 2 to 3, they will go to 1 to d, and R will now become a d by d matrix, will be three by 3 matrix, but it preserves the same property it preserves, the length of a vector, so we can already see a very very nice generalization, and so we will do that right away, we will do for the rest of the lecture we will assume that this is a d by d matrix, which has just satisfy this condition.

So, we will now use the we will give definition a d by d matrix, R is called orthogonal if this is just what I have done is what is called abstract way, the what was what was a normal definition, you have just gone from that we have come to something more general, what are it is properties.

First property is that if R 1 and R 2 are orthogonal matrices, then R 1 dot R 2 as well as R 2 dot R 1 are also orthogonal, this is extending, whatever you know, we follow two rotations by another rotation, you another combination will give you another rotation, second bit is identity, R equal to identity is an orthogonal matrix, the third property is that is R orthogonal, so is

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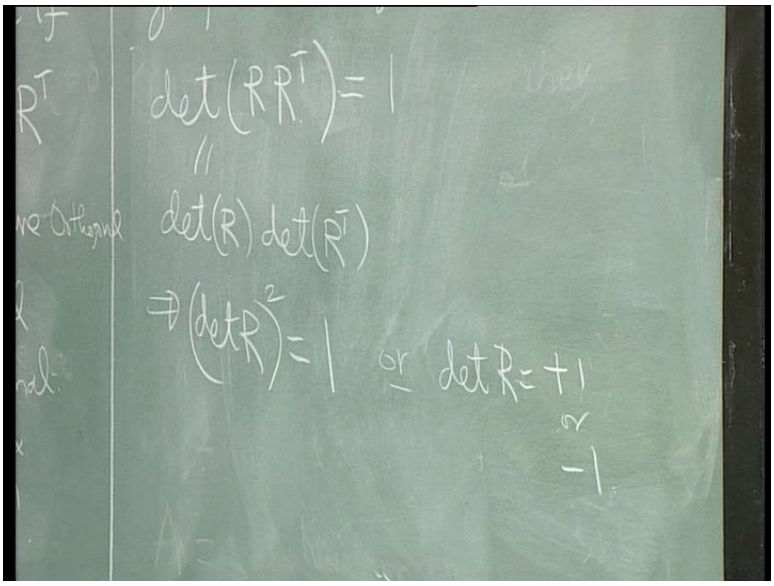

R inverse, that is not hard to see, if R R transpose is equal to one, it implies R inverses R transpose and for fine matrices ordering I mean I mean the inverses are the same, so this also implies r dot r transposes and of course, the last bet is a associativity, it means that if you have three matrices R 1, R 2, R 3 the ordering does not matter, and you will get another rotation matrix, so this defines for you actually a group in this properties, we will this four properties determine a group, this is called a orthogonal group, the symbol for that is capital O, and in bracket d important, that it is in capital letters, because the lower case means something else.

Let us look at this thing and let us take a determinant of this what is determinant of R R transpose, if you use this fact it is orthogonal it is equal to one, but what is the determinant of product of two matrices it is the product of the determinants.

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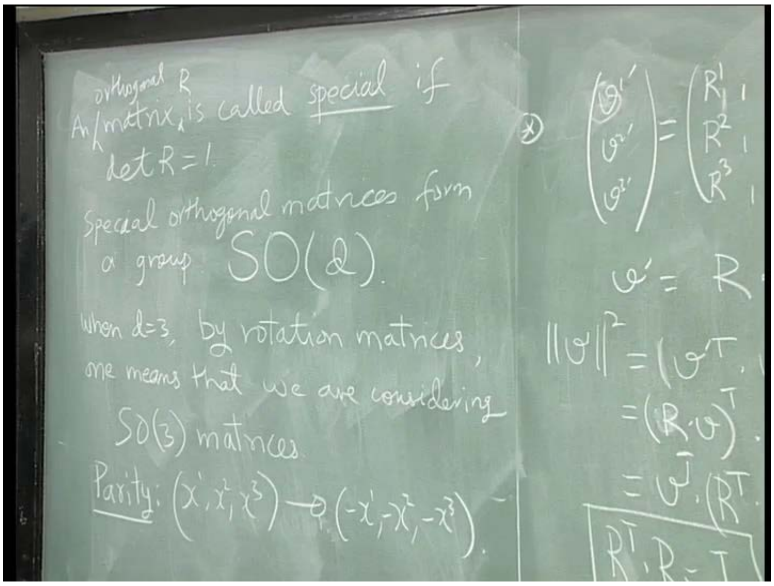

So, this but any matrix under this transpose and determinant, so this implies that determinant of R equal to square equal to 1 or determinant of R is equal to plus 1 or minus 1, there is one more definition, that is we will call a matrix, either an orthogonal matrix is called special; if it determinant orthogonal matrix R, the reason of looking at plus 1 not minus 1 is that, if you if you restrict yourself to this subset of a of orthogonal matrices, the determinant R equal to one under composition.

It closes if you restricted yourself to saying I look at the subset of orthogonal matrices, which have determinant minus 1, you take any two guys from that matrices, and compose them you will end up getting one way plus 1.

So, you get everything. So, this is so that is why you choose plus 1, and that gives you closure the rest of that it is same story. So, you get something called, so special orthogonal matrix from a group. S O d and, so usually in three dimensions, when d equal to 3 by rotations by rotation matrices one means deter 1 puts a one means, that we are considering SO 3 matrices not O 3 matrices.

But, but of course, you are interested in objects, how they transform under O 3. So, you need to look consider examples of matrices or which are orthogonal, but we determine minus 1, can you give me some examples, anybody from here some simple transformation, that we know parity goes to minus…

But this is truly odd number of dimensions, if it the even dimensions suppose you are in two dimensions, x 1 x 2 goes to minus x 1 minus x 2 that has determinant plus 1. So, that is not a determinant minus 1 matrix, so parity is not defined into dimension by issue to remember, this important point parity is not defined by just reversing the signs of all coordinates, this is a very common mistake among people, but in three dimensions you could do this, but simpler is just choose a mirror reflection, so one of them x 1 goes to minus x 1, all the other coordinates, here in matrix itself.

So, that would be an example, so I will stop here we will continue from here, we will understand, what what we will go back and revisit, what we mean by vectors, and we will add in the fact there are orthogonal transformations, which are not special orthogonal, it puts extra flavor on to what you mean by vectors therefore, vectors scale vectors and so on and on. So, that what we will do, we would also give the formal definition of a group, which will basically be these things, and we will move on to giving examples.