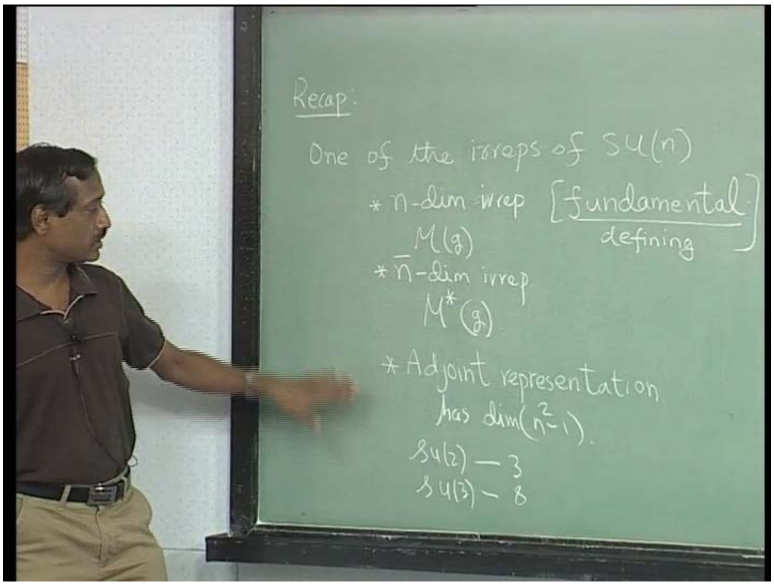
Classical Field Theory
Prof. Suresh Govindarajan
Department of Physics
Indian Institute of Technology, Madras

Lecture - 26

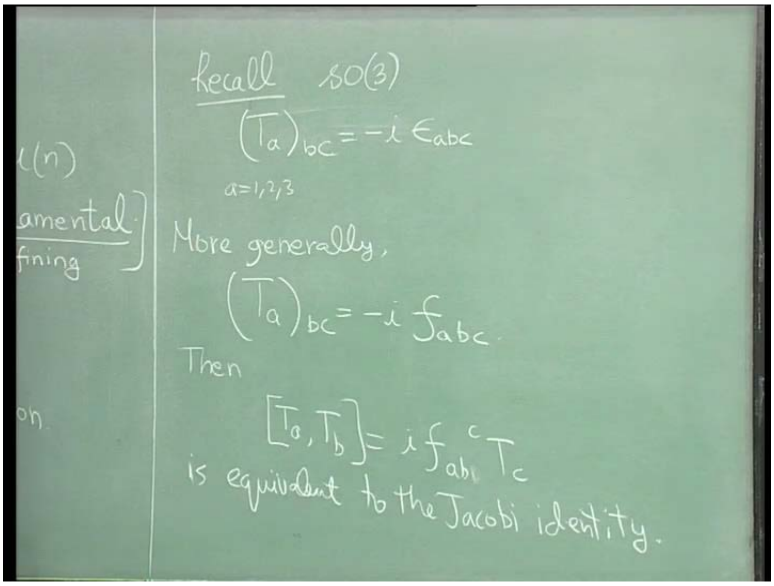
(Refer Slide Time: 00:19)



With examples coming from s u n and so, I will be going to continue that in this lecture, but let me just quickly recap something for you. So, one of the natural representations, which we saw was s u n, and that we saw, what we call the fundamental or the defining representation. So, that is how we actually, it is kind of, that was how I originally defined s u n in the early part of the course. I said take n by n matrices, which satisfy u u dagger equal to identity with the terminate equal to it. So, that was the definition. So, we saw that there is a n dimensional representation, irrep, which was called the fundamental or we can even call it the defining representation. But, fundamental is the more common usage. But, we also saw that the s u n for generic and other generic, these two has another n dimensional representation, which is not equivalent to this; the complex conjugate of that. So, we also got n star or n bar.

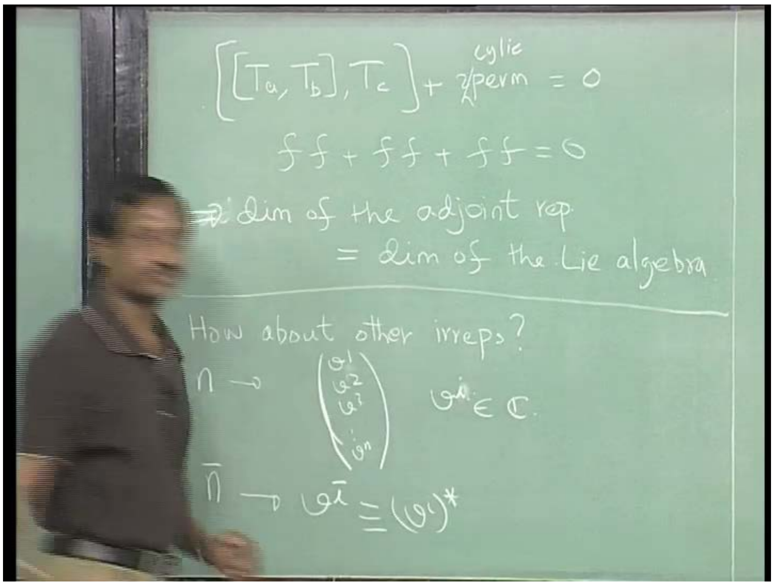
So, the idea was, suppose you have given a matrix m of g, which is an n by n matrix. This would be the realization of this; would come to m star of g. So, when you look at s u n, so, there are these two representations and the key point is that, there is a third representation, which is also very natural and this representation exist for any lie group. So, that is call the; so, the one which I am going to discuss now is called the adjoint representation. The reason I did not discussed it last time is because, it was part of your assignment and we have actually seen this in a way.

(Refer Slide Time: 02:30)



If you recall the lie group SO 3, sorry, the lie algebra SO 3, other than the lie group, so, for that, we had written out, we saw that, hopefully this minus i is correct; this is done from memory. So, T, so, here this is a a equal to 1 2 and 3. But, even the matrix indices b and c actually are also running over the same indices. So, in fact, more generally what we will see, if you take T a b c minus i f a b c, then this condition T a T b equal to i f a b c T c. So, this is what, this is the lie algebra is equivalent to the Jacobi identity.

(Refer Slide Time: 04:17)



So, you start with T a b c. So, once you go ahead and, so, what is the Jacobi identity? You write something like this. There are many ways of writing it. So, let me write it in one way; plus 2 other perms, cyclic perms or notations equal to 0. So, what you have to do here, you work out what that condition is. That was your assignment and you can see that, if I put this in, you will start getting, so, let us first see what a term, how this will look like. This will give you 1 f a b c. Then, there would be another commutator, which we have to evaluate. So, that gives you f a b c. So, this is like 3 terms which is f a of the form, f f plus f f plus f f plus equal to 0. I am hiding all in the indices. Thus, if you put this out here and expand it out, this will give you two terms and this also; this is one of the f f terms. So, modulo all the i's, it is really that same thing.

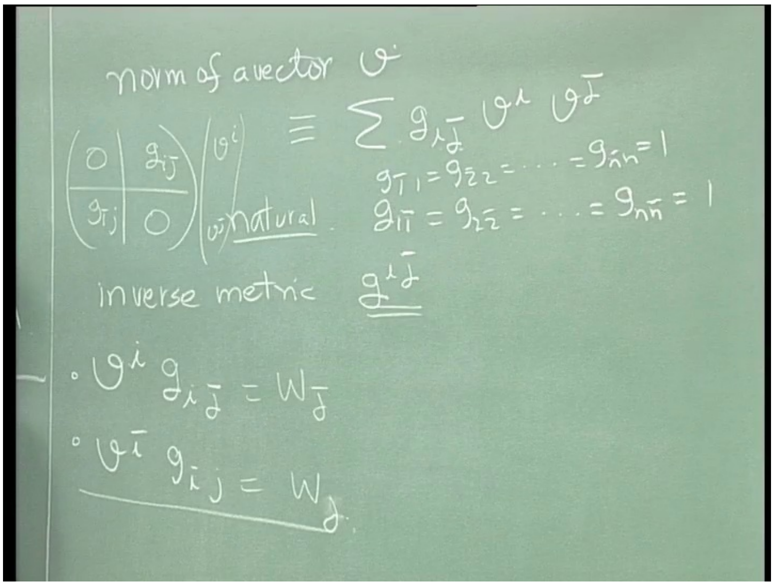
So, this is kind of natural. So, you can see that, I did not require any detail of the lie algebra. So, this is called the adjoint representation and the dimension of the representation is the same as the dimension of the group of the lie algebra. So, adjoint representation is equal to dimension of the lie algebra. So now, we are, so, what is the dimension of the lie algebra s u n? How many generators does it have? n square minus 1. So, this has dimension. So, if you put n equal to 2, we get n square, which is 4 minus 1, which is 3 dimensional. So, it says that, the s u 2 adjoint representation is 3 dimensional. But, we know the equivalence of lie algebras. It says that, SO 3 and s u 2 are the same lie algebra. Again, this was one thing which we checked in your assignment. So, you can see that it agrees with this identification. So, s u 2, it is 3 dimensional. For s u 3, it is 8 dimensional.

So, these numbers, just remember, these numbers will come back and so, of course, these are not the only representations. We would like to construct all representations of s u n and that is what I am going to do. But it is important to remember that these 3 exist. So now, the question is, how about other irreps? So, before I do that, I should, I need to introduce some notation. So, we will start with the fundamental representation. So, the idea, we will use something called the Tenser method, which is pretty much what we did. For s o n, we discussed tenser methods; the tenser construction. But, we will do that for s u n and in a sense, I think s u n is simpler than s o n and you will see why.

So, what we will do is, we will start with the n dimensional representation and we will construct the rest. But, so I need to fix some notations. So, let us choose the n dimensional representation. So, it acts as, so, that will consist of some vectors, which I should, I put upper case; let us be very easy. So, this is some n dimension. So, you think of it. So, the lie algebra and the lie group act on this vector. In short, I will write, I will write something like v a, like this. No, I would not use a because, I am using a for adjoint index. So, let us we call it i. But now, the question is, what should I, in this notation, what should I use for n bar or n star. So, I need to do that. So, n star would be taking complex conjugate or whatever. So, n star or n bar, I will indicate just by putting a bar on the ith guy.

You can take this to be equal to star of this, if you want, if you like. So, when we talked about vectors in r n, the normal vectors, n dimensional vectors in r n, we defined an, we saw the unitary, sorry, the orthogonal group was the thing which preserved the norm. We can define the unitary group also similarly, as preserving a norm, but, the norm of the vector, we wanted to be a real number. If these are, so, and this v i belong to complex numbers now. If they are real, then the v and v star would be the same.

(Refer Slide Time: 09:49)



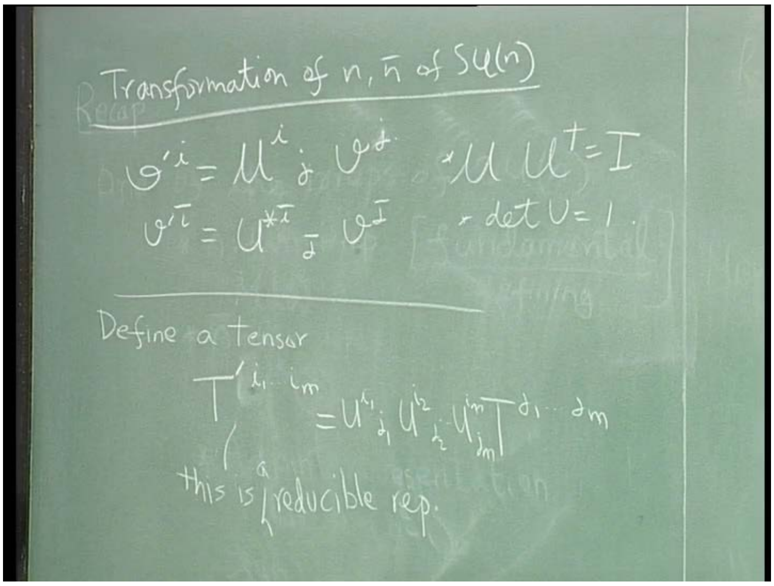
So, we take them to be complex. So, the norm of a vector v will be defined to be summation. So, here v j bar is actually defined to be v i complex conjugate. That is just inherited from exactly this simple observation. No transpose nothing. So now, you can see that and this is just the normal. So, I wrote arbitrary metric out here. But, really it is the natural one. The natural thing is to say that g 1 1 bar equal to g 2 2 bar equal to 2 1. All the other terms are 0. So now, let us just do, what you would have done to raise lower indices first. Let us understand how to raise lower indices and that is usually done by using this metric. I could write this as delta i j bar, but just to show you that, I mean to remind you that the indices are not the same, that is why I am using a g. So, and I want you to remember the index structure. One of them is a name index and the other is a n star kind of index. So, obviously, we can define the inverse.

It will also have 1 upper and 1 lower. By the way, also g 1 bar 1 equal to g 2 bar 2 equal to whatever, equal to g n bar n. I mean, I can, so, so if you think of the, so, if you think of this object is having 2 n, this things n and n bars put together, so, this metric is actually half diagonal. If you write this, because it takes i, it can take values which go from 1 to n and 1 to n bar. So, it is 2 n values. So, if you think this is matrices with two end values, then it is half diagonal. It is a symmetric. So, what I mean by that is, it is something like this. Acting on something, let this write. So, this has to some width v, the way I have written it is something like this. So, the inverse metric is this guy and which has the same proper. It is, just because it is an identity, we can see it is a same kind of thing. So now, the thing is that, what I want because, the half diagonal nature is important. I could use this and write a lower index.

So, question to you is, what would be an object, if I lowered this. You can see that I can only get something like this. So, like an upper n index is equivalent to a lower n bar index and vice versa, because there is no way you can get an i index down. These are pneumonics, but, useful and important. Similarly, you can see v i bar can only go with this. So, let me just call this for fun, in w j bar, these two. So, in particular, suppose, so, let us think of how did we go about constructing vector tensors. So, the idea was, we just load something which had a bunch of vector indices. So, here when we say vector, we will just define it to be the m index. Not the n bar. So, you write something which has, play many of those indices and each of those will transform in the, as if it is in the fundamental.

But if you remember, we were able to do traces in the s o n case. Now, we cannot do traces because, you can only trace a pair of indices, such that, one is an n and one is an n bar because, you need to use this metric. So, there will be no traces. So, that is the simplification.

(Refer Slide Time: 14:50)

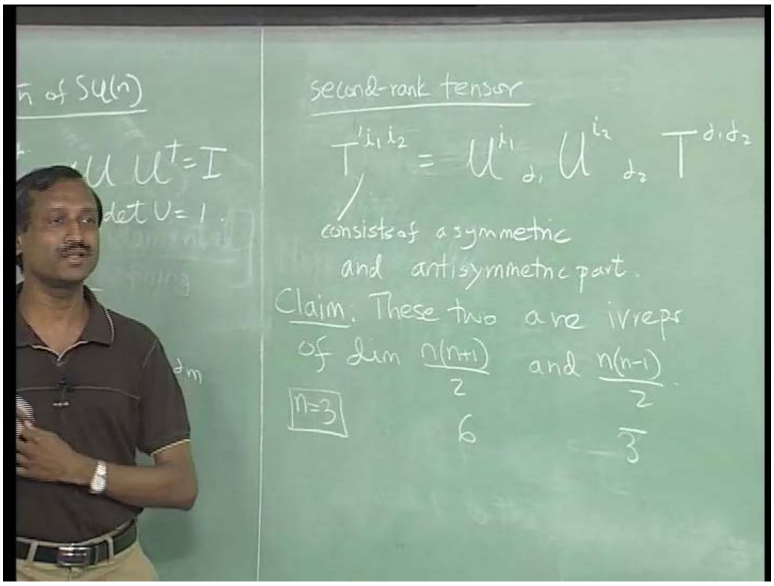


So, let us see how we would, so, let us write the transformations of the n and n bar of s u n. Just to fix notation, so, that would be just v prime i, should be mu i j v i and v prime i bar will be u star i bar j bar v, where, now I have just, instead of calling it m or something, I am just writing u and this thing of course, where u is such that u u dagger. So, this is going back to our earlier definition. But, only now we have, I have also written independently an i bar; how it transforms, just for completeness.

So, let us define a tensor. Which one? This one. Yes, it should be. Anything else? So, you define a tensor of rank n or whatever. So, put a prime by its transformation property i 1 to i n is equal to, so, T j 1 T j n. So, the rule is that for; you focus on every vector index separately and just write a u for that. So, first i 1, write u i 1 j 1 u i 2 j 2, so on and so forth, up to the last fellow, which is i n j n. Let us call it m because, n is used up. So, this is exactly like we did for vectors of the orthogonal group. Except here, we use the orthogonal matrices. So, in that sense, it is exactly that. So again, even there, we remembered it was, but that it was reducible. So, the idea is to ask, what are the irreducible components of this? So you break it up into blocks, like we did in the orthogonal case.

So, what we will do is, I will work out explicitly for a second rank case and then, we will generalize. You might ask me, I could have defined things using i bar and I could have defined mixed tensors. The reason is that, I do not need to define mixed tensors because, what we will see is, this construction here, will get u even the n bar representation. So, in other words, I do not need to do a double duty. I can always find all the, any object which is mixed as some higher ranked tensor with some property, irreducible component or something.

(Refer Slide Time: 18:12)

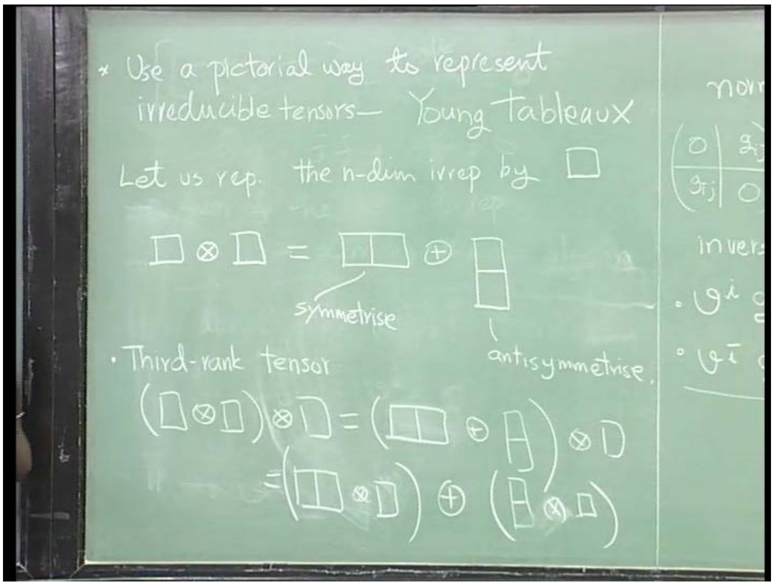


So, let us choose a second rank for example. So, I am very loose here, when I call second rank or nth rank. I just mean the number of vector indices that it carries. Quite often people, when they say second rank tensor, they mean really something like reducible component of that and etcetera. So, let us not get worried about that. So, look at this i 1 i 2 is equal to mu. So, what we can do here is, we can break this up into a symmetric and an anti symmetric part.

So, this consists of a symmetric and an anti symmetric part and these are irreducible. I will not prove these things, but, I will just claim. So, claim is, these two are irreducible representations of dimension n into n plus 1 by 2. This is the symmetric guy and the other one is, so, for fun, let us just go ahead put 3. What we get here is 3 into 4 by 2, which is 6 dimensional representation and out here, 3 into 3 by 2, which is 3 dimensional representation. So now, the question to ask is, so when we do, when n equal to 3, n equal to 2, I will leave it as a fun exercise for you, because you know lots of things about s u 2. So, for n equal to 3, this becomes 6 and this becomes a 3. Bt now, question is, is it a 3 or is it a 3 bar? So, that we do not know the answer.

So, I have to give you the answer. You find that this is actually a 3 star or a 3 bar. So now, the thing is, if you have, what happens if you have a third rank tensor? We have to decompose it into irreducible parts. So, this is where a pictorial approaches. Much easier at least for some things, I will show, I mean the general thing is hard in any case, but, there are some things which become easy. So, what I will do here is use pictures and those pictures generalize.

(Refer Slide Time: 21:12)

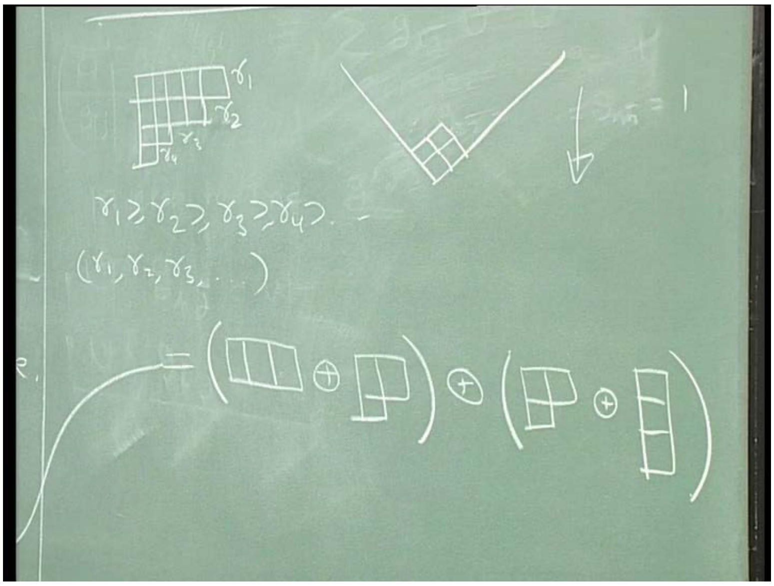


So, use a pictorial way. This pictorial way, as the name, it is called the young tableaux. So, the idea here is, let us call, and let us represent the n bar, the n dimensional representation by a box. So now, what we are doing here in the second, in this thing is taking two box representations and we are tensering them. So, the mathematical symbol is this. But, physically what is going on out here is basically this. What I am writing, this object actually represents this guy. Now, this is where I need to introduce extra notation. So, I will write something, so, the symmetric guy, I will write it in this fashion. I am putting two boxes together and the rule is, if there are two things like this, you symmetrise them, if they are horizontal attached. If they are vertical, you have to anti symmetrise. So, here you symmetrise and anti symmetrise.

So now, the thing is, now what I am going to do is, try to understand what is it we have to do for the third, for the rank three tensor? So, I will be, all these things will be a bunch of rules, which I will put in. Actually, there is a nice formal way of doing it. But, that will take me a few lectures. So, that is not relevant. The rules are easier to remember and I will not do the most complicated things. I will do the sort of things, which you will run across on a regular basis. So now, the thing is to naturally ask, what about the third rank tensor. So, you can start something like this. I am putting a bracket. The bracket could be put here, or it may be put here. It is not that so important. So, this we already worked out. So now, the thing is that, one thing you should realize is that, these symbols have, I mean, some nice meaning. So, here it is exactly like addition and multiplication. So, something like this can be written as, what is this called? The distributive law, right?

It holds even for this. It is easy to prove. Oh, that should be tensor. So far, I have done nothing. All I have done is use this. So now, the question here is, how do we compute, given an existing diagram, how do you compute this? So, what I will do now is to define something called a Ferrers diagram, which will become a young tableau once you put some indices into that. So, these guys without, these are all examples of boxes just put together, they are called Ferrers diagrams.

(Refer Slide Time: 26:00)



So, let us define. By the way, there are several inequivalent ways of drawing, not inequivalent, equivalent ways rather of drawing the Ferrers diagram. It depends on which part of the world you studied. If you studied in England, I guess you would call it Ferrers diagram and draw it in a particular way. If in France, it is another way. I do not know how it is in other. Definitely, I know that the English in French, so, there is some rotation of how they draw it, I will draw it in one particular way. I do not care which, where it comes from. So, the idea is that, you have a bunch of rows. So, let us, boxes arranged in some rows, like some boxes and the rule is the following.

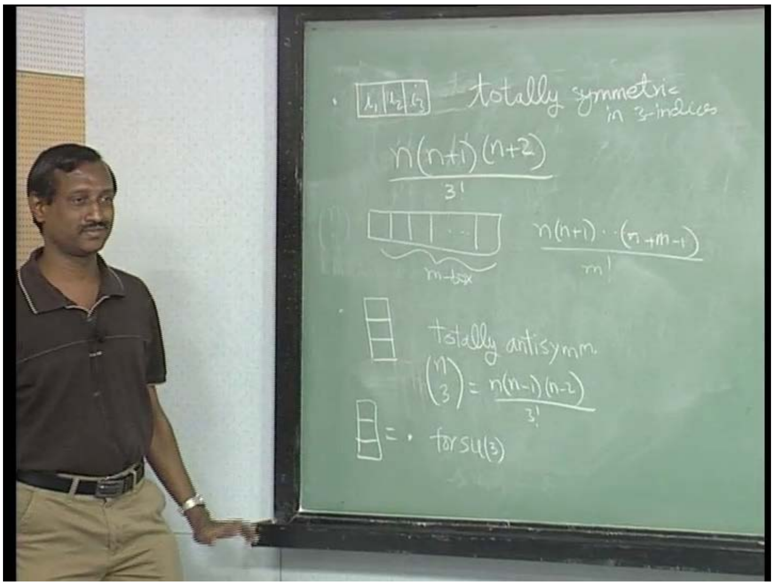
The number of columns or number of boxes in any upper row should be such, should always be greater than or equal to the one below so on and so forth. But, there is a very intuitive way of doing this is to turn this around and draw it something like this. This is like a children's game. So, gravity acts this way and you want stable configurations under small jiggling. So, suppose I mean, let us say I had something like this. I have put forth such guys. This is quite stable. But suppose, I do something like this. This is not stable because, if you jiggle it up, it will fall down here or if I put it here etcetera. So, you can see that, the rule which I am telling you out here is actually the best way to think of the rule is to understand it this way. So now, this involves taking this and rotating it by whatever number of degrees plus some 45, will that is some 45 or somewhere.

So, this is one way of understanding this. So, a Ferrers diagram can be written as, first row has r 1 boxes, second row has r 2 boxes, third row has r 3 boxes r 4. So, strictly such that r 1 is greater than or equal to r 2 is greater than or equal to r 3 greater than or equal to r 4. In fact, you can keep going to infinity till it hits 0 at some point. So, this is what is the Ferrers diagram. So, you can see that, this is a Ferrers diagram with r 1 equal to 2. So, this one we will write it as 2. So, what you do is, you represent this by r 1 r 2 r 3 so on and so, forth. So, as yet, this has nothing to do with these pictures. But, at least it looks like those. So, in this notation, we will be just 2. This would be 1 1. So now, you just, so, first step, first thing I want to tell you is construct all possible.

So, here I had just, I am given this guy. I the rule is, I am just, I am trying to playing this game. I am given this and I am asked, what are the things I can add to this? Add one more block, so, it becomes that children's game. So, now I can do this and then, I will explain what it means. So, here I can put this one here. That is one piece, one term. This can come here, but, it cannot come here because, then it will leave a gap here. That is the first one. The second one, I can put it here or I can put it here. So, that is it. There are no more. So, these rules are for a diagram, are important in telling you what are the possible these things. Obviously, if you can even do tensor of this with this, then the rules become more complicated.

But, since my goal is to, you know, sort of incrementally construct things, the more, only thing which I need to do is to see how to add this and it corresponds to this. Simple rule. So now, we have to understand what the rule is to understand what tensors these correspond to? Each one of these guys, which I will explain to you and you see here, that these two have the same structure. So, and the rules I will give will be incomplete. In the sense, I am not going to worry about details. I will just, so that, we can understand the structure. You can look it up in any book or whatever. So, let us first understand this. What is this?

(Refer Slide Time: 31:00)



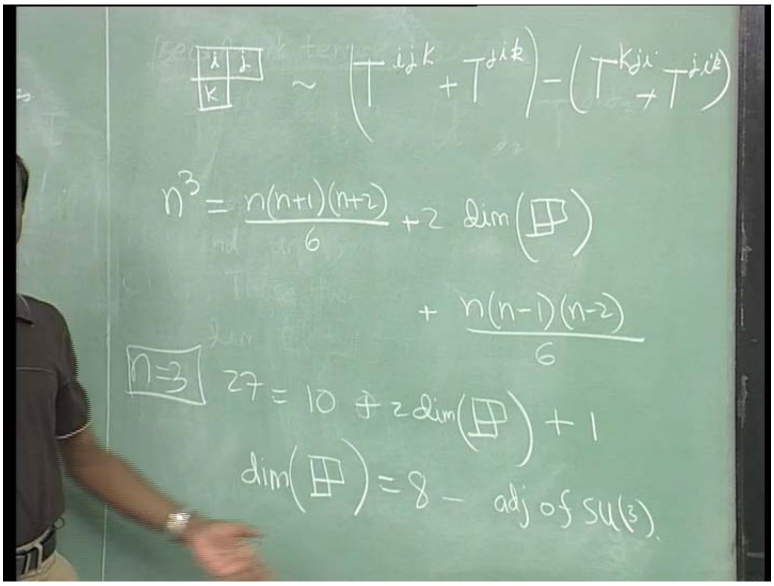
So, this is natural generalization of this. Here, what did we say, if you have two of them you symmetries. This is fully symmetric in 3 indices. So, we will make it a young tableau, which corresponds to saying that it acts on this index structure. You completely symmetrise it. Can someone tell me, what would be the dimension of something like this? This is an easy one. You can remember this. It is n. The next one is n plus 1. So, it is like the n choose 3, but goes the opposite way, where it keeps increasing. So, in fact, if you have something like this, m boxes, this will have dimension n into n plus 1 dot dot dot times n plus m minus 1 divided by m factorial. You can prove this. The formula is worth memorizing.

Another easy guy is this. What do you think this should be? Totally anti symmetric. So, what should be the dimension of this? n c 3, n choose 3. So, let us ask what happens if we put n equal to 3? It is 1 and what is the 1 dimensional representation? That means it is trivial. So, you see some nice rule now. That, first thing is, if you have s u 3, there is no way you can have 4 boxes in this thing because, you anti symmetries. There are only 3 possible indices. The maximum you could reach is 3. For that, this is, so, this is equal to dot. We use a dot for trivial representation for s u 3.

It cannot exceed. The number of columns, I mean, the size of a column can never exceed 3. In fact, 2, because of this simplification. There is no need to exceed 2, for n equal to 3. More generally, if you come here for s u n, you will find that you can never have more than n minus 1. What do you call? You can always reduce it to something, which has n minus 1. Is n minus 1 correct? Yeah, n minus 1 rows. Very good.

So, by the way, here also what you should do is, you should say this is i 1 i 2 i 3.I have not put those things in and then start you writing. So, when, here this became i 1 i 2 and i 1 i 2 came here. You work those things. You write them out. So, you should keep track of indices. But, I have hiding that. So, then you will see a difference between these two, when you convert Ferrers diagram to young tableau. So, young tableau is, this is an example of a young tableau. Now, I have to put entries into that.

(Refer Slide Time: 35:32)

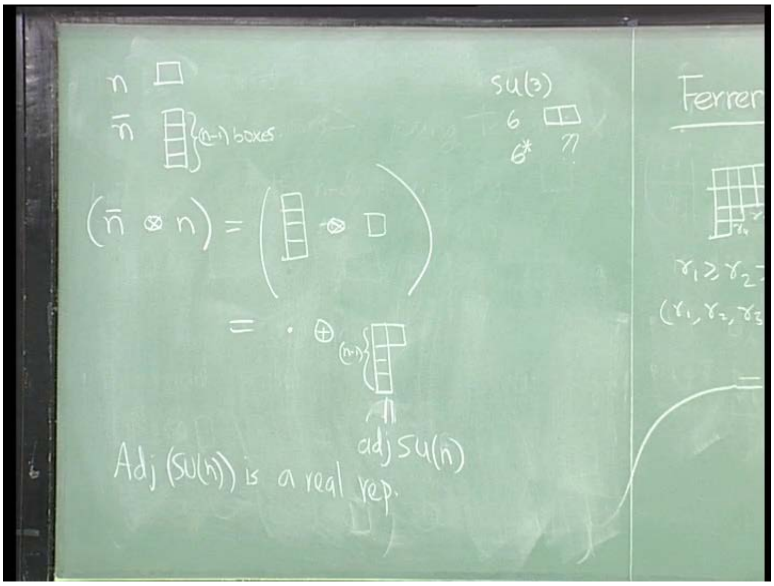


So now, the thing is, I need to understand what this means. Obviously, it is mixed. If you go this way, it is symmetric. If you go this way, it is anti symmetric. But now, we also need to have a rule as to how we write such a tensor. So, do you first anti symmetrise or do you first symmetrise? So, the rule we will do is, so, let us take something like this. Take i j and k. It says, so, what you should do is to first symmetrise with respect to this, i j k. So, suppose you are given a tensor, which has no symmetry properties. I am picking out this thing. So, what I have achieved here is taken care of this. Next step is to anti symmetrise i k. So, I just do that by; plus. So, this is what I mean by this tensor. I will leave it to you as an exercise to check that you get something different, if you anti symmetrise first and symmetries. You do not have to get the same. So, this is the convention. This is what I mean. So, now the thing is, let us go and do some dimension counting.

So, we know how these things break up. We know the dimensions of this. We know the dimensions of this. We got lucky. These two are the same. Whatever happens, because of their symmetry structures, these two have to be in the same dimension. So, we will do, just some simple jugglery and get the answer. So, first thing is, each of this is n n n. So, the left hand side is n cubed. So, n cubed equal to dimension of this fellow, the symmetric third rank, right? That is just n into n plus 1 into n plus 2 by 6 plus 2 times dimension of this fellow plus dimension of the anti symmetric guy, which we again know. You can simplify things and write for arbitrary n. I will give you a formula later. But, right now what I want to do is to just work this out for n equal to 3. So, as what happens for n equal to 3? We get 27 equal to, what was this 3 into 4 into 5? 10. So, this is a 10 dimensional representation plus 2 times dimension of this and this becomes plus 1, is it? So, you can see that this is, what is this representation? This is the adjoint.

So, you can see that, you know, slowly working through, you are able to get formulae for various representations and so on and so forth. But, the reason I pushed it to this end is because, I want to get the adjoint. I need to get the adjoint. But, of course, this is not the adjoint for, if you take n equal to 4, it is not the adjoint. You will get some other number. We will see in a short while, how to get the adjoint.

(Refer Slide Time: 39:32)

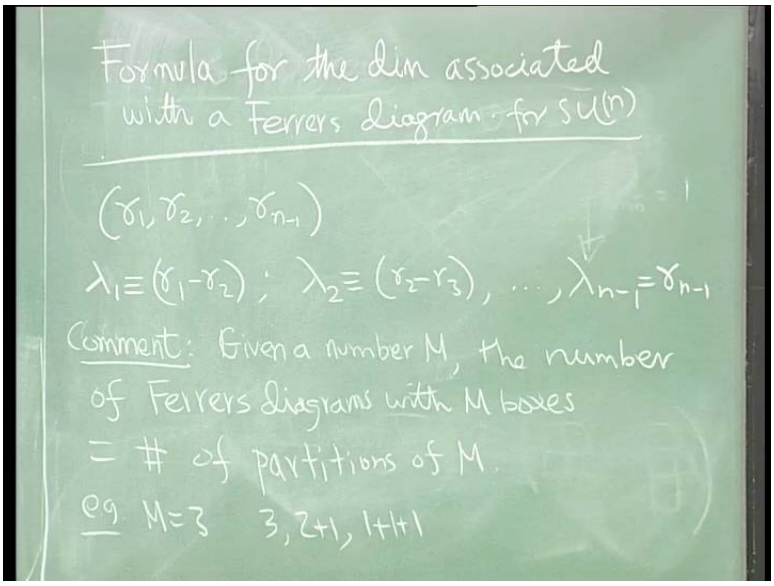


So, we will do another exercise. We will start with. So, we know some nice things. We know the following thing. n is represented by a box. n bar is represented by this, n minus 1 boxes . So now, what I want to do is to do this. Now, I want you to tell me, so, let me draw it in this pictures. So, this is n minus 1 boxes. I would not write it again. So, according to the rules for the Ferrer diagrams, what are the Ferrer diagrams we could write? We could put one out here. But, that is how many boxes? n boxes. So, that we can shrink and write it as one dot plus. What is the next one? What is the other possibility? Putting it here, there is no other place. There is exactly only one thing. What is the dimension of this? This gives you a dimension formula. What is dimension of left hand side? n square. What is dimension of this? 1. So, this is the adjoint. This is nothing but, the symbol for adjoint of s u n.

So now, I will pictorially, I mean, with this, I can actually prove one more nice thing. What happens if we take complex conjugate of the left hand side? n bar will become n, n will become n bar. But, that is the same. What about the trivial representation? It is equal to its complex conjugate. So, this side is real. This side is real. So, it tells you that, the adjoint and that complex conjugate of the adjoint are the same thing. Has to be. So, it comes for free, the pictorial way of doing things. Is this clear? So, we can see that adjoint of s u n is a real representation.

We also saw something which was 6 dimensional, right? Where was that? Somewhere we saw 6. I have forgotten which one. For s u 3, we saw 6, right? Anyway, I will leave it as an exercise. You can check. I mean, for s u 3, you will find that, yeah, it is just this, right? 3 into 4 by 2. So, it is, so, 6, for s u 3 for instance. 6 is this. This we have seen. But, 6 star is something else. I would not give you the answer. I will let you figure it out what that should be. It is not easy, but, not hard either. In general, so, you can use the, it really shows that there are pictorial ways of proving whether a representation is real or not. Of course, that assumes that this method is giving you irreducible representations etcetera etcetera. So, is this clear?

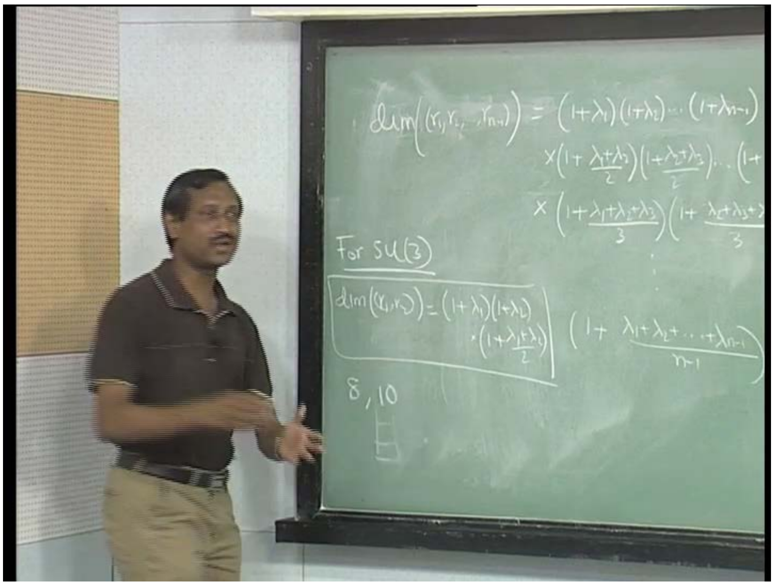
(Refer Slide Time: 43:32)



So now, the idea is, now we need a formula for the dimension associated. Of course, for which you have to specify which s u 1. I mean, you are not expected to derivative it or anything. So, I will just tell you. So, what you are do done is, you are given a Ferrers diagram, which has r 1 r 2 and since, we are at s u n, there will be , the maximum number of rows will be n minus 1. So, just define from this, define lambda 1. You can think of the next entry. I mean, has been 0.

Whether I forgot to mention there is a one to one correspondence between Ferrers diagrams and partitions of a number? So, just a comment. I have just forgot. Given a number n or m, the number of Ferrers diagrams with m boxes is equal to number of partitions of m. So, example will be, let us take m equal to 3. Partitions of 3 will be 3 2 plus 1 1 plus 1 plus 1. So, it is exactly three such things and with 3 boxes, we have already seen those things. So, in these things, the easiest way is to construct explicitly the map. I will leave it as an exercise to construct the map. It is a fun thing. It is very easy. You just have to stare at some few examples. You will get the map. So, this is just a random comment. So, coming back to this, I will give you a formula using lambda 1 to lambda n.

(Refer Slide Time: 46:15)



So, let us, so I am, instead of writing out the, drawing the picture, it is simpler. It is more economical to write this thing. I need to write the formula and this is one of the things, which I do not memorize. But it is not very hard to memorize by the way. Is this by 3 or 3 factorial? By 3. n minus 3 n minus 2 plus n minus 1 divided by 3. Keep continuing till you hit the last guy, which is 1. This is what? plus.

So, there are other ways of, there is another way of writing it in terms of what is called Hook length and stuff like that. But I am just writing out one such formula, just to tell you that there exists a formula given the Ferrers diagram, given the group s u n and you can you can write this answer. It looks very messy, but, for s u 3, it is not so messy. The Hook length way is better for general n. For s u 3, you can see there are only two things. So, for s u 3 dimension of r 1 r 2 is equal to, just; that is it. Very compact. I mean, the reason to write this is to, at least, so that, once in your life time, you will see something a proper formula and as you can see, I have to look it up and not make any mistakes. But, this kind of thing is very useful. So, this way, you can see that the tensor method actually gives you ways of constructing various representations and you can see the simple way, which I was doing, you are able to reproduce lot of numbers. Especially, numbers such as 8, for s u 3, 8 and 10 appear in particle physics all the time. I think we have seen both these numbers already.

So, what I will do is, in my next lecture, we will actually do a hard core particle physics application of this and it will give you a physical way of, you know, I wrote some arbitrary vector, but, what we will do is, we replace them with co arcs and we will not call them v 1 v 2 v 3. We will call them u d s or whatever it is and we will see that, how to organize what is called the Particle zoo. This was really the way; first, I think nontrivial application of group theory, in any area of physics. Even now, I think in condensed matter of physics, I have not seen too many applications of non abelian gauge groups. The maximum they seem to use is u 1. But in particle physics, it is the norm. And so, it depends on what your interests are; I have also seen beautiful applications of group theory in areas such as quantum optics and things like that. Where again, people use non abelian group symmetries etcetera, in a very, very beautiful manner. So, I am done for today. So, next lecture we will do applications.