Classical Field Theory
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Lecture - 25

... I think this is also a good point in time...I had given you an assignment a while ago; I think assignment seven, due 27th September it says. So, obviously it has been a while since you have looked at those things. So, I will spend a couple of lectures on how representation theory of Lie groups works. It is very useful and after which we will go on to understanding the symmetry breaking, etcetera, the setup for the Higgs mechanism, non-abelian Higgs mechanism that is a relevant for the standard model for the LHC.

So, it will become more like a particle physics course and because it will be a particle physics course we have to get every factor, everything right, so that we can match with the experiments and or at least try to understand what these various things look like. So what we will do now is two lectures I would spend on Lie algebras and then we get back to the kind of things we were looking at which is non-abelian gauge theories, because I realize that last time I was loosely using the word representation, etcetera and I will just make it more formal; I know I have been using it all through this lectures, but I will make it much more formal right now.

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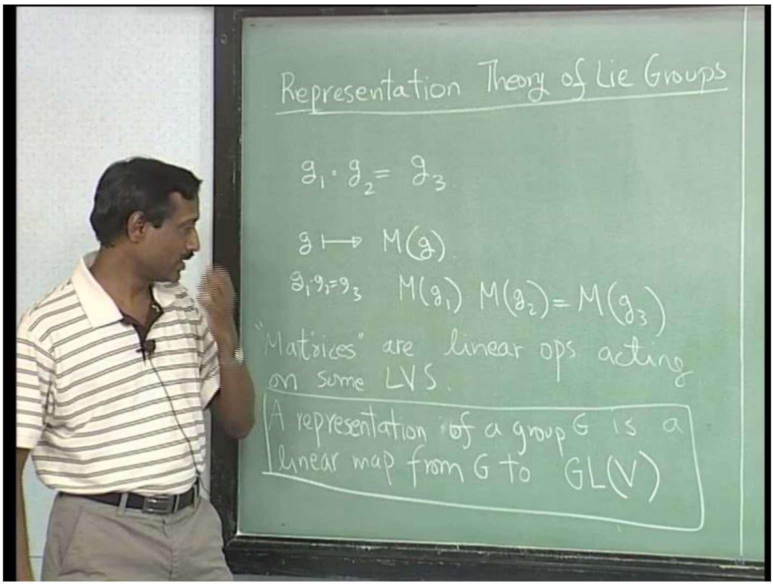

So already when we looked at, so what we are going to do is going to do representations. So, even though I am writing something very fancy we will never go beyond orthogonal and unitary groups because they have the necessary complexity that will capture more general things. So, when you have a group you have some group multiplication law; let us say which says that g 1 dot g 2 is equal to g 3 and originally I sort of introduce the idea of representation by saying we think of a matrix realization of this. So, if you give me any element g I give you a corresponding matrix and with the rule that if g 1 dot g 2 equal to g 3 then M of g 1 dot M of g 2 equal to M of g 3. And I said that this is what we mean by a representation of the group element but we can be a little but more abstract and the definition of a representation is basically so the idea is to realize that matrices are nothing but are linear operators acting on some vector space.

This is something which we know already from quantum mechanics, the Hamiltonian is realized as some kind of a linear operator acting on the Hilbert space which is an example of a linear vector space with extra stuff because it is infinite dimensional. But in most of our considerations we will be thinking of finite dimensional linear vector spaces. So, the general definition is to realize is that is a map. So, representation is a map of a group G is a linear map from the group G to GL of v. This is just fancy way of saying that these are linear out operators acting on some vector space v and the G condition tells you that it has to been invertible.

So, it has to have non-zero determinant and the advantage of writing it in this fashion is that if you give me a vector the way to get a matrix is to take a vector space choose a basis. Once you choose a basis you get a matrix element or you get a exact matrix realization but you want to write things in sometimes in a basis independent fashion. So, this is the basis independent way of writing it. This is how a math book could write it but really it is nothing but in physics literature all the time we always work with specific basis and so once you pick a basis for this vector space then this becomes exactly you will get a exact matrix realization for the group.

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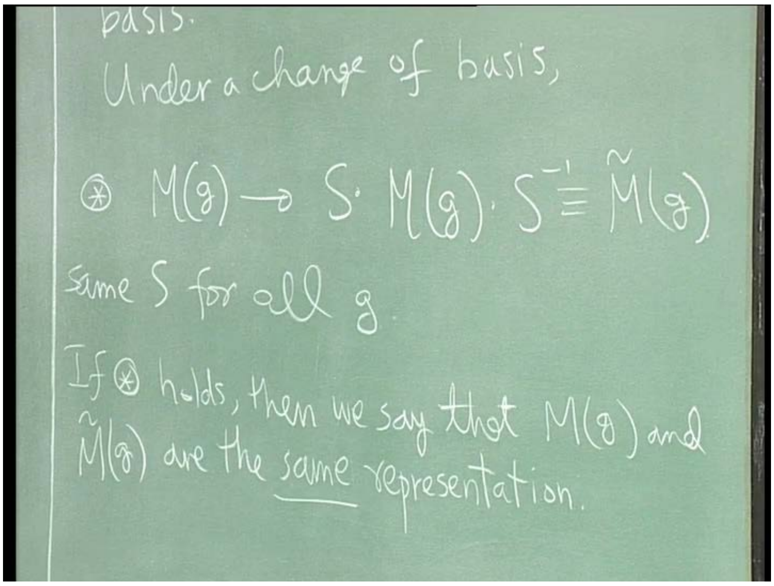

So, but clearly the thing is that things or definition do not depend on basis on a choice of basis. So, in physics way of saying it is that if I make a change of basis the representation should not change even those matrix would change. So, what happens to under a change of basis M of g would go to S M of g S inverse, same S for all, where S is some change of basis and it is of course it has an inverse determinant of this thing. So, if M of g 1 times M of g 2 is equal to m of g 3, then obviously that will also satisfy the same thing. But really if you go back to this definition, you do not want to distinguish between these two. So, you would say that two representations are the same if there exists an S such that they get related. So, we can just define this to be M tilde g. So, if star holds then we say that M of g and m tilde of g is the same representation; there are some other nice properties. So, suppose you are given M of g 1 times M you are given a particular M, let us just take complex conjugate of the whole thing M star not dagger you take m star.

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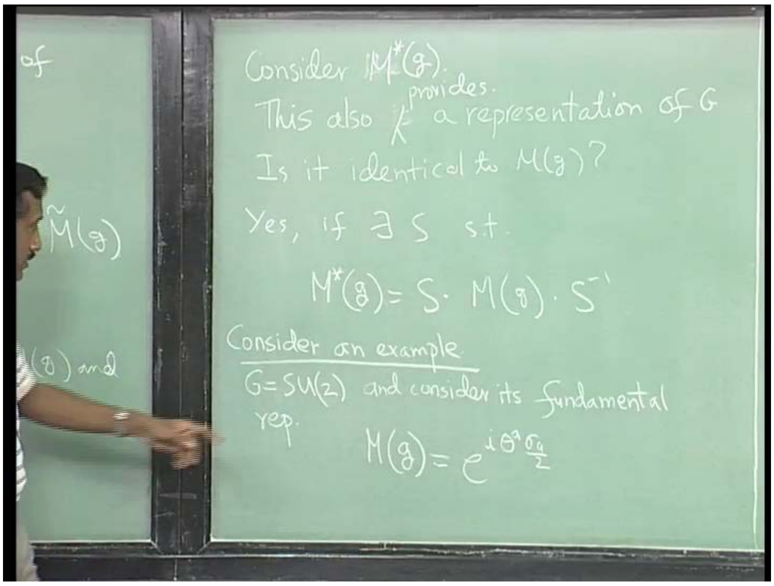

What I mean by that is every matrix that you had where you take it star complex conjugate. Now first thing is does it form a representation of the two? So, you just take star of this and nothing happens. We are not doing dagger or transpose; otherwise the order would change. But if you are just taking star, you can see that this also is a representation of G or provides but now the question is, is it the same representation? Can we answer this question; in general we do not know it could be it may not be. So, we have to ask is so the question to ask is, is it identical or same M of g in terms of representation?

The answer is yes, if there exists an S such that. So far we are very somewhat abstract but let us choose some simple examples. So, let us look at an example; so let us consider an example. So, let us consider the following example. Consider g to be the unitary group SU 2 and we look at it in the two dimensional representation and consider its fundamental representation. So, in that case we can write one possible choice for M of g would be e power i theta a sigma a by two. So, I am writing out the arbitrary group elements; we have seen this already. Now the question to ask is so let us look at m star.

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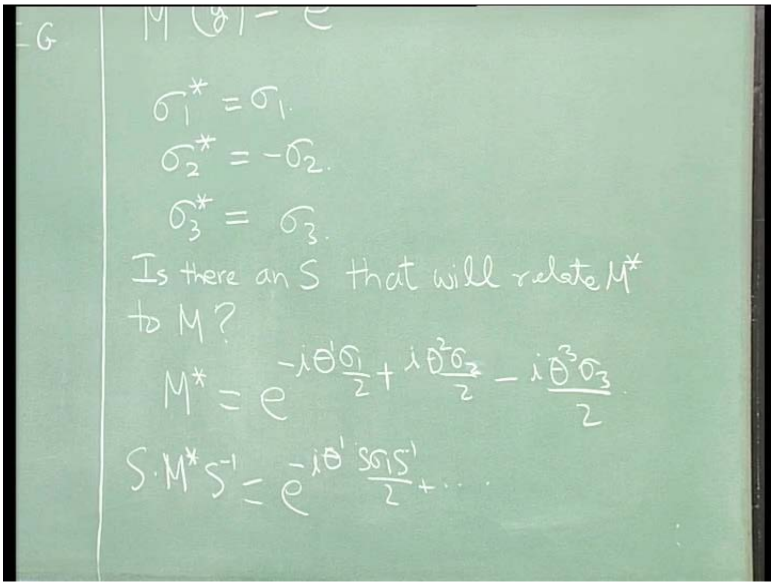

What happens to M star of g? So, i star is minus I, theta a is suppose to be real, sigma a star by 2. So, what is sigma 1 star equal to sigma 1; sigma 2 star is minus sigma 2. So, now the question is, is there an S that will relate M star to M? So, now coming back to this what we really want, so you can see that if you look at this thing. So let us look at it in some detail let us look at this guy. So, it become minus i theta 1 sigma 1 by 2 minus i theta 2 sigma 2 by 2, sorry plus, and minus i theta 3 sigma 3 by 2. So, what we need is some operator; if I find something which will change the sign of sigma 1 and sigma 3 but not change of sigma 2 I am done.

Why is that because you can see that S M star S inverse is equal to so on and so forth. You can use the definition of e power a and you can keep putting S S inverses in between. So, you can actually go ahead and see that it acts on this guy. So, now the question is do you any such, you know anything; any S which would do this for you; any guesses? Let me ask you a simpler question; something which acts on sigma two and takes it to sigma two, identity but it will do the same thing but something. Yeah, so S if you take to be sigma 2; take S to be sigma two, sigma two square is one, so I just leave it at that. So, S inverse is also sigma 2 itself.

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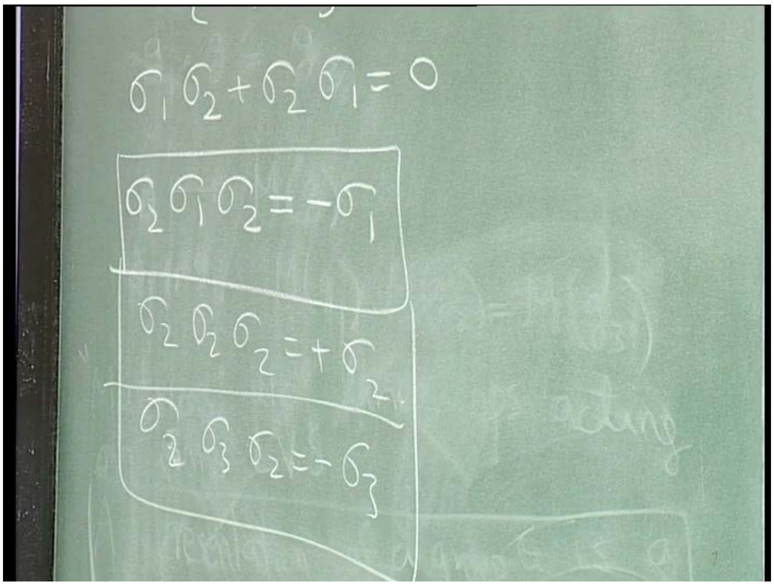

So, you guys should know this; sigma a sigma b is equal to 2 delta ab. So, in particular sigma 1 sigma 2 plus sigma 2 sigma 1 equal to 0 for instance. So, I can multiply from both sides; so sigma 2 sigma 1 sigma 2 is equal to minus sigma 1 because I would take it to that side. Similarly, you can see that sigma 2 sigma 2 sigma 2 equal to plus sigma 2 and sigma 3, oh sorry, sigma 2 sigma 3 sigma 2, so the same. So, the S should be just sigma 2. So, what can we say about this particular representation the fundamental representation which is two dimensional is the same as it is complex conjugate representation.

So, the statement is so one usually writes two for the representation; we say that 2 equal to 2 star. I mean now you may question which should I call two which should I call two star; that is up to you. You should call one of them two, the other will be called two star but here it does not matter; they are related to each other by a similarity transformation. You could not have guessed it just like that. So, now it looks very tempting; what about if you replace this by SU 2 by SU n; will it hold. So, I have given you a case where in fact M equal to M star. So, now the question is let us consider…

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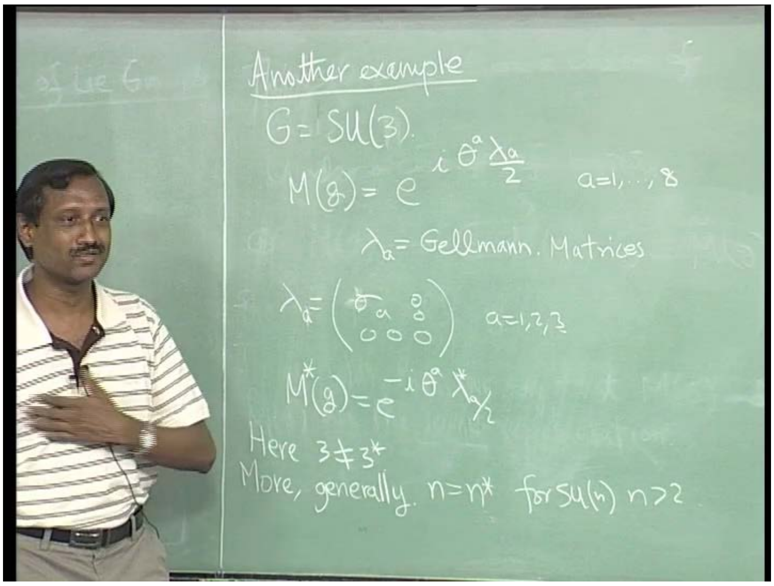

So another example are actually examples is a infinite family of examples. So, we consider G to be SU n and M of g. Now let me just be very so let us choose take g to be SU 3 and we can write this as some e power i theta a lambda a by 2 and in that assignment these lambda ’s so a runs from 1 to 8 in this case. And so this will be and l lambda a’s are the Gellmann matrices. One thing which you might have appreciated from looking at the assignment where these things were explicitly written out but it was supped up version of the polysigma matrices; especially if we looked at lambda one, it was lambda a first a was just halves rather sigma a with zero zero zero zero zero at least for the first three guys among that things. So, now you can ask what is M star of g would be? So, that is the first three guys if you look at them it is exactly like that and so it will give you some insight into what S could be, sorry M star, and you can go back and play with it and see if you can find one; the answer is no.

Here, yeah you need; this is the most important thing, it is the same S for all group elements here. It was independent of the value of theta; it did not dependent on theta because even out here if I just switch on only first three thetas, then it is the same as SU 2 and then you can map it but there is only a SU 2 subgroup of SU 3. So, there are whole bunch of elements which are not of that kind. So, here is an example but the complex conjugate representation is a representation; the groups here are not the same. In fact more generally n is not equal to n star for all SU n for n greater than 2. So, this is actually the more generic case not. But there exist representations of SU 3 for which that is true;

in other words there might be something else which is say six dimensional or eight actually eight dimensional, yeah eight dimensional for instance. Now that would be real; there exists an eight dimensional representation called the adjoint or whatever; that is real. So, if you find a representation which is like this you say that the representation is real and if it is like this you say that it is not, it is complex and not equal to m star; thank you, is this clear. Yes, where does the linear map come; what do you mean by an EMR term?

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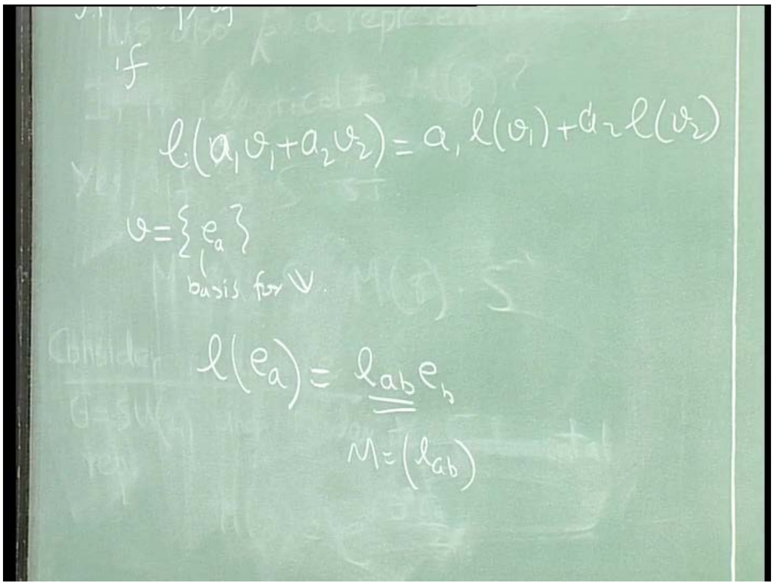

So, a linear map; so, a map is linear of a vector space to the vector space is linear if, a map l lets say, l of this is what one needs. If we are considering a group derivative; yeah, yeah wait. So, first let us understand linear; this is what me mean by a linear map. So, once you have something like this if you if you have a map which is linear, then it is a physis to look at the basic elements how it maps. So let us say that you choose a basis for v which is say e a's. This is a basis for the vector space, then this linear map can be written as a matrix because you just say that l of the basis vector should be l ab e b. This is some vector; any vector can be expanded in that basis vector.

So usually so this is the matrix or the matrix element as we would write in quantum mechanics. So, every linear map gives you this thing. So, here you go back to this group for every group element I give you a linear map. The map changes but that is the matrix so l ab or m ab I should have written. So, this would be the matrix element. So, the matrix m I would call as equal to this the set of these guys. Are there any, there is one more important idea which again you may have seen we have discussed it in specific examples.

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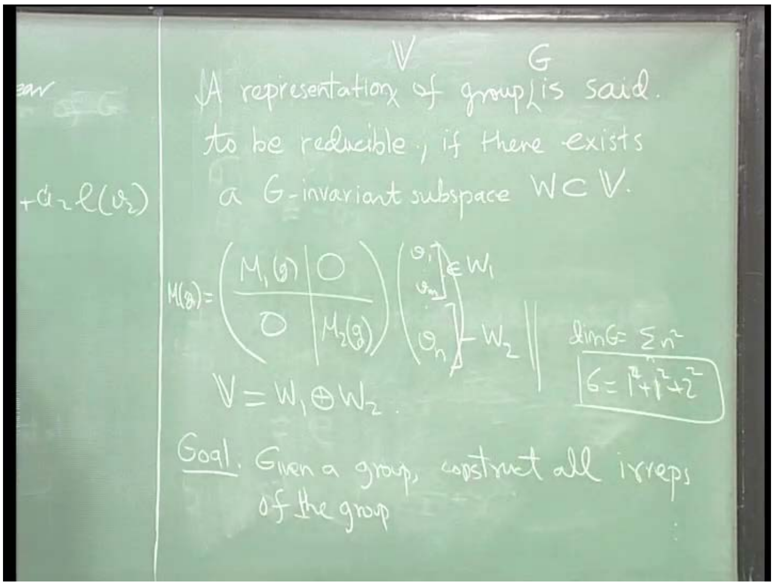

About the idea of if a representation is reducible. So, we say that a representation, actually before I discussed this you notice here I have already I am looking at Lie groups and you can see that if you pick your group element in some representation, the Lie algebra guys also come as some matrices. So, there is a direct connection between representation of a Lie group and the representation of the Lie algebra. So, if you give me a three dimensional representation of SU 3 then by considering infinite decimal group elements then I can get the analog as Lie algebra value or we can be even more this thing, you take d by d theta a of this thing and set theta equal to zero then it will pull down exactly lambda a. So, the thing is that we can talk of even representations of Lie algebras is very obvious. I mean again for every Lie algebra element value the guy will you a matrix; again it is enough for the same argument which I had out here.

Again it is enough for me to give you how it acts for the basis of the Lie algebra. So, in this case the basis is the Gellmann matrices. So, a representation of a Lie group, not necessary Lie is not so important in this, of a group is said to be reducible if, so a representation v. So, usually the vector space itself is called the representation; the representation v of the group G is said to reducible if there exists a G invariant subspace.

If you give me some vector space that there is something of lower dimensionality in notice; I am not saying equal to, it is a strict subset, so something which is lowered in dimension. So, what that implies is that we could go ahead. So, let us say that we have a bunch v 1 to let us say we have a n dimensional representation, there exists a basis where let us say the first m of them they let say they span w; this whole thing spans b, what it says is that this matrix can be written as a block acting on this.

So I am just writing, M 1 of g M 2 of g and I am just fixing the zero's zero and this could be non-zero. So, what this tells you is that the first under if it were in this form M 1 of g would only mix the first M components for all G and not mix it with this thing but it does not tell you I mean you could have a bad situation where these guys actually mix with this, but quite often we will come across situation where even this star is a zero. So, the question is but of course it need not come in this form because I can always make similarity transformation and make it look messy. So, it is only a sort of this thing that you should I mean just because you find that in your presentation somebody gave you a bunch of matrices it did not look like it had this block diagonal form and then you should not let yourself to think that it is irreducible. If it does not have any gene variant subspaces, then you would say that it is irreducible. In some sense the irreducible guys are the building blocks.

So, we will look at only situations where it is truly nice, we will say something like this we will call this del x, then we can call this W 1, we will call this W 2. So, we write v as this is just the usual direct sum of vector spaces which is just the fancy way of saying you take certain bunch of guys put them and put the other things together. So, yeah we cannot write it as a direct sum; one of them is irreducible. It could yes, but we will always we will never look at quite often in all these examples we will not have the bad. You would not be looking at situations like this; we look at nicer things like this. We have actually seen this before once where we found that we could find some change of basis and that was in the case of orthogonal groups; does anybody remember that? We looked at second ranked tensors. Yeah exactly, so in that case when we took a second rank tenser in three dimensions it had nine components and I pointed out that we could be break it up into actually five plus three plus one, one was the trace part.

So, these never mixed with each other. So, that is the case. So, in general this kind of thing could happen and its all mucked up or messed up by the fact that there could exist a similarity transformation which would be this thing. So, the goal in general is to given a group, construct all irreps of the group. Irreducible representation that means it is something which does not have a gene variant subspaces; somewhere there will be one group element at least which will mix up everything, you can never find. So, you cannot break it up into a smaller part on which you have an action. So, this is the thing; for finite groups actually there is lot of nice structure. First thing is the number of irreducible representations is finite. In fact there is some nice number theoretic stuff you will have; for instance, yeah I think it is like this, dimension of the group for finite group this is.

It is equal to I think the sum of the dimensions squares of the something like this. So, if you told me that there was a group which was six dimensional like S 3, then what are the representations it could have whatever. So, you have to break it up into number of squares. So, how can you break up six into sums of squares? It can be one square six times, it can be one square plus one square plus two square, it cannot have three square. So, that would tell you that so six equal to one square plus one square plus two square. This is some nice stuff you can do and you can prove and so in fact for S 3 this is what happens. It has 2 one dimensional representations and 1 two dimensional representation but we are not interested in finite groups. So, we will be looking at continuous groups and this is not true. So, like I told you long ago that finite group are fun because you can do a lot of explicitly things with them and here is an example of that. It does not matter if there is a group like the monster group which has 10 power 53 elements but people have been able to construct its character table.

In other words they are able to construct all representations from that because as somebody put it is 10 power 53 is a finite number. I think the number is correct but so even though it such a huge thing, you can still sit down and do; it in fact it has only 26 irreducible representations. So, but coming back to this we are interested in finite not infinite groups, we are interested in Lie groups which appear a lot more in physics. Though in condense matter physics if you are doing crystals and stuffs like that finite groups do appear in that context. So, the idea here is to sort of see how to go ahead and construct this and we will sort of do this for SU n and then I will wave my hands for the orthogonal groups and then leave it as a homework exercise for you to work out for all the other groups. So let us revisit; we start with SU 2 which we think we understand because we see it in quantum mechanics all the time.

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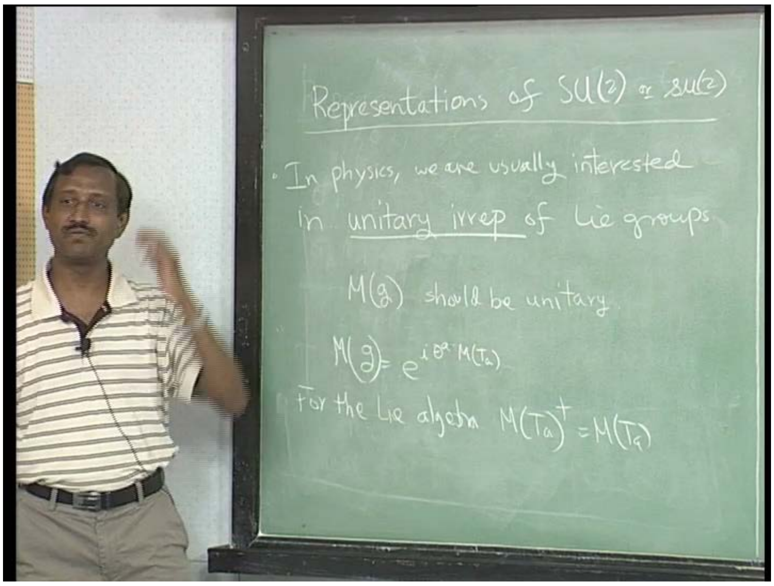

So, let us look at representations of SU 2 the group or you can even say the Lie group SU 2. Yeah, there is one more bit I forgot, I mean so the term which we are also required that the linear mapping unitary that has been extra condition, it is not arbitrary generally linear transformation where it is a unitary transformation. So, in physics we are usually interested in unitary irreps of Lie groups. So what we mean by this is M of g the matrix should be unitary. It also is equivalent to saying that the Lie algebra element should be omission if you put i out here.

Also if we write g as some e power i theta a T a so rather m of g is this and this will write I will use the same letter T a even though I should write M of T a, maybe I should. So, if this is unitary and we take the parameters to be real, then this will be condition for Lie algebra. So, this will be the condition that will if you put in unitary, irreps is already something so we need to classify these things. So, the answer is known for us. So, what we will do is we will solve a lesser problem which is to actually look at Lie. The reason to look at Lie groups is that it is enough for us to look at the Lie algebra elements and that is all we will do.

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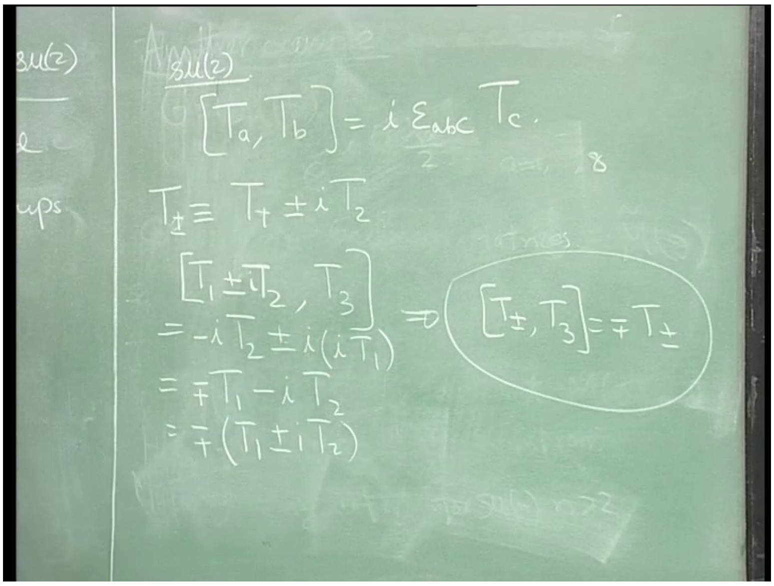

So let us remember certain things about the Lie algebra SU 2. So, this is what it satisfies and we have to find different matrices which satisfy this sort of a thing and so the way we will do this is to rewrite this thing as follows. Let us define t plus minus; so these are what we would have called the raising and the lowering operators and we have T 3. So, if we write out the various so let us look at for instance T 1 plus minus i T 2 with T 3. So, T 1 with T 3 will give you minus i T 2 and T 2 with T 3 will give. So, this just becomes minus or plus T 1; is this correct. So, I just wanted to work out this. So, this implies because I never remember this. So, this is the nontrivial commutative and so this does not make use of any details of what matrix we are using; this is true for in general.

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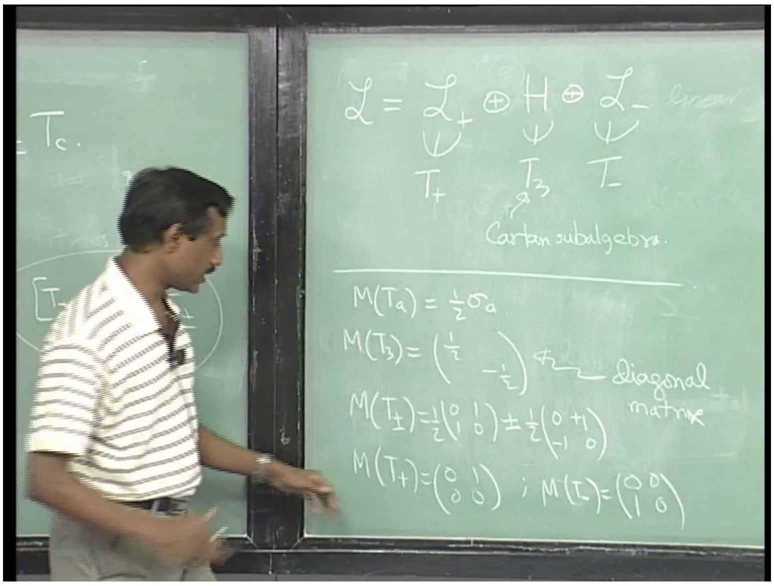

So, the reason to change into this is what we are doing is taking the Lie algebra and writing it as bunch of it is a direct sum of three parts which is exactly the separation. So, in this case I will just define T plus belongs to this, T 3 belongs to this and T minus belongs to this. They are all subalgebras; in the sense that if I take T plus and take it is commutator with itself I get zero. It is a trivial sense in which it is a subalgebra and T minus is also sub algebra and T 3 each one of them are one dimensional. So, they will all they form closed sub algebra. What we will do in general is to break up all the generators into what like the raising and we will call this a raising part lowering part and this we will call the Cartan subalgebra.

So this is the standard, the reason I am sort of writing this out is to show you that this generalizes to all other Lie algebras. We will see it in more detail in other cases but the key point here is to do this and so coming back to let us look at some details; for instance if we look at let us take the case when M of T a is just half sigma a. So, if you look at M of T 3 it is a two-dimensional thing it would be plus half minus half. So, this is a diagonal matrix and what is M of T plus? It should be what sigma half of. So, you can see that M of T plus is upper triangle, the other one is lower triangle. So, this is something which we know but I claim that we can recover this in a very nice way without even by using this kind of decomposition and the way it goes is the following.

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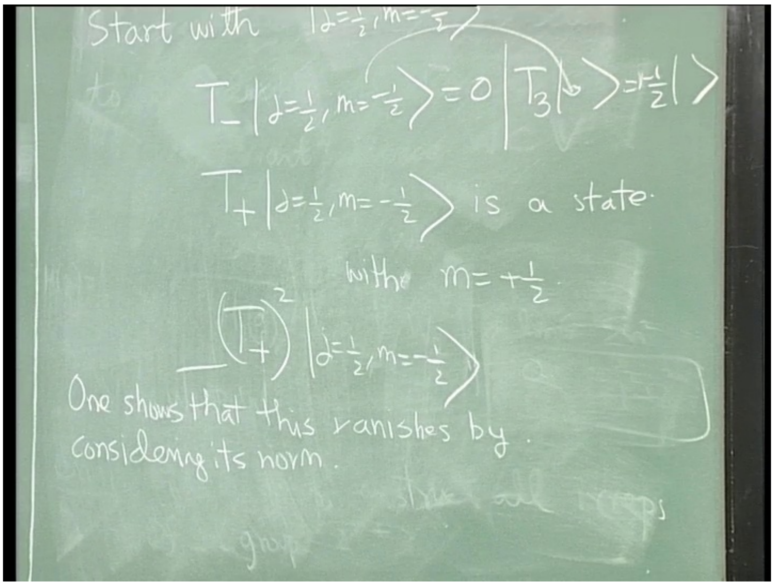

So we will start with, so what I am going to do here is to construct for you a vector space and the way we will construct the vector space is to start with one basis vector and keep constructing the other guys and at some point I will get all the guys in the all the basis vectors of the whole thing and it may terminate, it may not terminate, it depends. So, we start with something. So, we start with an element. So, let me start with so in this for this example we will just start with usual state which is m equal to minus half. I will use the direct notation the ket notation and just for completeness I will just write this. This is the j equal to half, m equal to minus half say and what is the property of this; if I try to lower it further. So, for the rest of this thing I will just not use M of T plus, I will just use T plus for M of T plus. So, what we know is that if we try to lower it further, what happens? It is zero.

So, we start with some guy which is such that T minus on that is zero. Then we act on it with T plus and this m I can value is defined to be T 3 of the same thing but this is not going to be zero, but then we can use we can also show that T plus T minus is T 3. So, you can show that this m equal to plus half. Now but nothing tells me that I cannot go on further, so I should proceed further; let me take one more. So, I do T plus square is this possible, no. So, but this is supposed to be zero. So, how do we show that something is zero if you have a unitary system a nice positive definite vector space; what would you say in such cases if something is zero that is equivalent to it having zero norm. So, what you can do is you do not prove that; so you prove one show that this vanishes by considering its norm and then it implies that you will you will get zero norm and that implies that it terminates. So, I need a little bit of definition. So, first thing is that I should define for you, what is the Cartan subalgebra?

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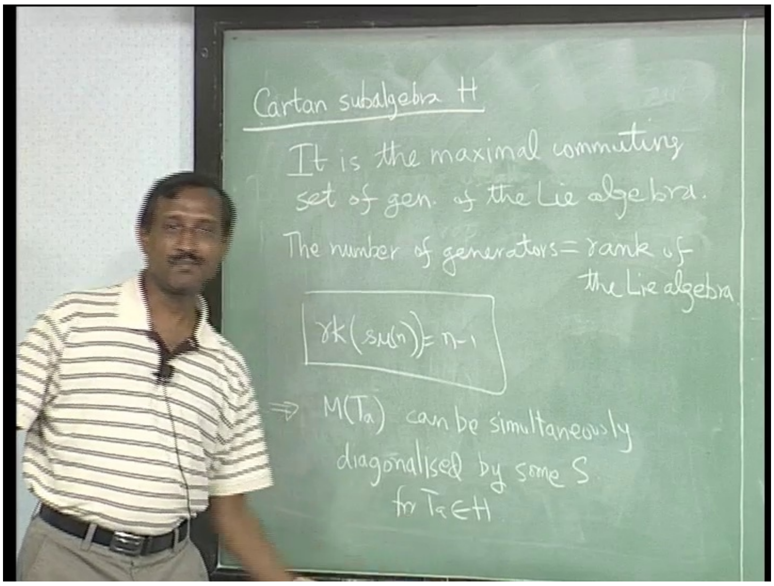

So, let me define that. It is the maximal commuting set. So, in other words any Lie algebra element could be taken and that will commute with itself. Then you look for among the other generates is there some other linear combination which will commute with this; keep on doing it till you get all the possible generators. So, you come here I chose T 3 to be in it. So, now you can ask what else remains; there is T 1 and T 2. T 1 does not commute, T 2 does not commute with it and no linear combination you can convenience yourself commutes with T 3. So, it just tells you that for SU 2 you will find that there is only one such generator. If you go to SU 3 in this example which we looked at if we look at the Gellmann matrices, one of the natural things is lambda 3 and the other one is lambda 8. It is only conventional by the way; I could have chosen some other thing. So, even here there is nothing which tells me that I should have chosen T 3 I could have chosen T 1. There is nothing it is not unique but what will be unique will be how many of them you can choose.

So, the number of generators is called the rank of the Lie algebra. So, what we see is that rank of SU n turns out to be n minus 1. So, for n equal to 2 which is SU 2, there are two generators; for SU 3 there are two generators so on and so forth. So, once you have maximal commuting generators you can see that you can simultaneously diagonalize them if you have some matrix. So, that acting on implies M of T a can be simultaneously diagonalized by some S. We already saw that if you make a similarity transformation it is a same representation. So, I can always adjust such that I go ahead for all T a in belonging to the Cartan subalgebra, only those I can. So these become diagonal but in other way of saying it is that we can label every element basis vector of the vector space by the Eigen values.

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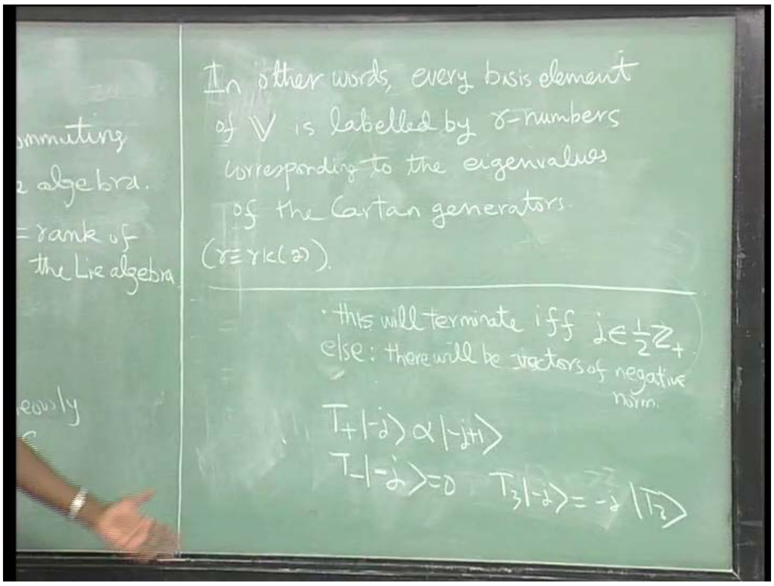

In other words every basis element of v is labeled by r numbers corresponding to the Eigen values of the Cartan generators; r is just rank of the Lie group. So, what it means so in this case coming back to this thing here T 3 was diagonal, I told you that this way in this of course you have to work in the way or you have to figure out what the S is and diagonalize it; only then you get a natural basis which is one zero the natural one to that this thing. So, here you can see that the vector once which is one zero plus Eigen value half and the vector zero one is Eigen value minus half. These are the two m equal to plus half and m equal to minus half states; is that clear. So, obviously if we have n minus one of them there will be n minus one labels but remember the representation is the full vector space but each of the these elements will pick up some labels.

So, the first step here is realized that in a matrix realization this is diagonal; you achieve that but what about L plus and L minus. It is really this; L plus you can think of them as upper triangular guys and L minus as lower triangular guys and you know that under matrix multiplication upper triangular matrices remain upper triangular, lower triangular remain lower triangular. So, it is really that is the idea in this separation. These are like upper triangular diagonal and lower triangular. So, I think some of you would have done numerical method course. It is called some u d l or something decomposition. So, what one is doing is the u d l or the lag digit is it called l u d, l u d decomposition. So l how does it go, l u d. So, this is really glorification of that it is really the decomposition that is what you are doing but doing it in the abstract first. You have Lie algebra with structure constants given to you, you can always break it, you can always bring it to that.

So now let us see how we would do. So, let us now we hopefully we understand this decomposition of the Lie algebra number one and we also understand one simple example of this thing. So, now the question is what about the other representations? We know that there are these pin j representations which have two j plus one dimensional representation. How would we go about constructing that? Actually the procedure is really the same. We start with something, we will call it the lowest weight vector and since its lowest let me put it the lowest weight vector; that is the lowest of my thing with such that T minus on that is zero and it has to have some label. So, we will just call that label; I will call that label I will just call it that is the m label. So I will just call it minus j. At this point j is just a number; I just say that T 3 of minus j is equal to minus j times T 3. I do not have to assume that j is integral half integral or anything.

I do not assume, make any assumption. I just say that this thing is that way and I have to introduce a norm. I assume this in normalized to one, it exists. So, then I start raising it. So, I will do T plus of minus j. We know that this is proportional to minus j plus 1. You keep on doing this, you will keep raising, what you are doing is raising the m Eigen value. So, it is a like a raising operator and each time it only gives you proportional to you have to work out the norm. So, you need to check out the norm. This will terminate if and only if j belongs to; at some point you let zero. Now you can ask suppose j is not this let say it was irrational, then you will find that this will never terminate; if this is j plus one whatever it will keep going, it will go off to this thing. If you work out the norm you will start getting states which have negative norm. It violates the fact that you want a unitary representation.

So unitarity else, so here something nice happens it hit zero and if you still say I do not care about zero you ask what happens about the next guy, it will be negative. But at least I can consistently truncate it by saying that whenever it hit zero it is I do not proceed any further; that is it stops. But if j were not belonging to half integral values, then there will be states or yeah vectors rather of negative norm. If you do not care about unitary irreps, you can construct these guys and there is no reason for it to terminate because it is never hitting zero, you will get end up getting infinite dimensional representation. So, there exist infinite dimensional; in fact I have given you an infinite dimensional representation, if you take j to be anything, by hand you can construct an infinite dimensional representation of SU 2 which is not unitary. So, this is not something I am going to prove but this is a theorem. So, what you do in general is analogous to this you will have a bunch of Eigen values as many as like as the rank of the group.

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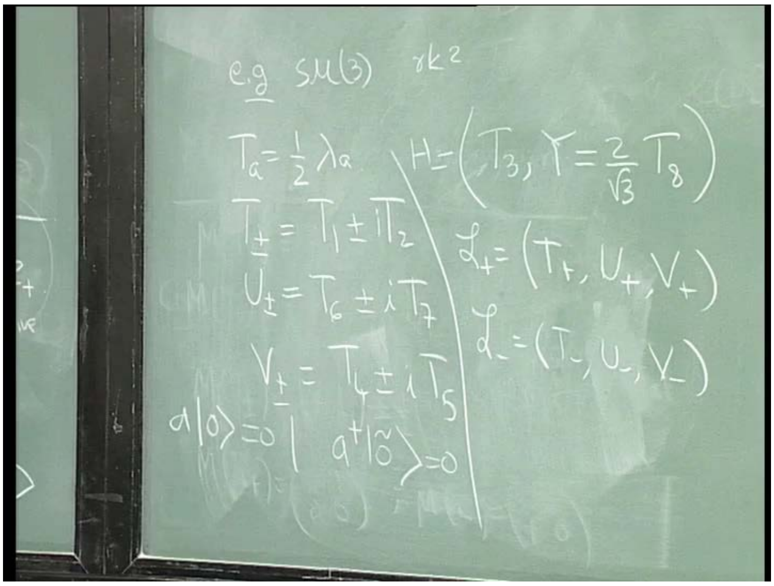

So you will write something like this. So many labels and there will be this each one of them will be the Eigen values of some Cartan generator and then you just say that the analog of T minus equal to zero is to set all the generators of L minus should be equal to zero. And this is consistent because if you give me two elements if I take its commutator that is also in l minus; it is a close subalgebra, so it is more than one conditions. So, example would be in SU 3 it has a rank two. So, conventional things is to choose for instance let me write out in terms of the. So, one defines T a’s to be half. So, I am writing it out in the three by three representation but it is true in general, I mean you can abstract a way in terms of the Lie algebra what it does. So, let me define T plus minus to be. So, there if you look at this, this is just exactly like you did for SU 2 because it is SU 2 subalgebra. So, these are standard conventions which we could have. So, T plus u plus and v plus and you choose T 3 and usually there is some Y generator which is some factor which I never remember 2 by root 3. So, this is H, T, U, V. So, eight generators I have broken it up into three parts.

You can go back and look at in terms of the Gellman matrices, you will see that these become upper triangular, these become lower triangle and of course these you know already that they were diagonal. This 2 by root 3 stuff has to do with Y being related to something called hypercharge; some of you were doing particle physics right, you would have seen that may be. Even if you have not seen it you are going to see it in this course later. So, I have to be a horribly explicit here but you can see that these generators this decomposition holds not just when T a equal to half lambda a it holds in general because this is a faithful representation of the Lie algebra. Now coming back to this, so this condition L minus would tell you that T minus u minus and v minus will annulate this. But now comes the problem; we have to raise things but for raising I have how many possibilities? I can raise with T plus, I can raise with u plus, I can raise with these things. So, it is like now a tree and they do not commute with each other. So, you have all these weird things which you can do and in principle it is the same as this and every time you hit a zero you terminate things.

So, you put a cap and the nontrivial statement is that the representations of unitary reducible representations of SU 3 are finite. In other sense whichever way you did I can start first with T plus and then operate with u plus, etc but of course using the Lie algebra you can prove that something else they could get related; it could be two different path could lead you to the same guy. So, you have to work with all these things and you will get this; you start with this and you terminate it and so you get a whole bunch of representations and this is just an algebraic way of saying how the representation is constructed. But you can see that beyond SU 2 that is not the way to go; I mean it is pretty hard to do it and this is not the way we will construct it but conceptually it is important to realize that it does not matter if you represent your thing is finite dimensional or infinite dimensional; it does not matter just about all of this.

Whenever Lie algebra admits this decomposition, you can always do this thing and put this condition. I could have equally well have started with. So, this is like starting with a bottom and showing that there is a roof but I could have started with plus j defined my highest weight state as T plus annulating everything and doing the lowering with T minus. So, similar things can be done out here but when you have a situation where your both roof and ceiling it does not matter which way you start in some sense. But there could be situations where there is only a roof or only a ceiling and they will give two different kinds of representations. But if you are looking at just the normal orthogonal groups, unitary groups and these things, all these things they end up with situation where unitarity constraint will give you only finite dimensional representations.

Of course you could say that j going to infinity but in some sense you know it is a limiting case but typical representations for a given j are finite, so it terminates. In some sense there is one more example we know which fits into this sort of a setup which is the angular momentum algebra a a dagger equal to one, a square is zero. So, you can think of a; so you would say that a on zero is zero. This is how you define it and this one as norm and then you keep raising but this is a case where there is a floor but no ceiling. A dagger will keep going all the way through. Now you may think, well, there could be one more state which is dagger acting on something equal to zero. Why do not we consider such states in quantum mechanics? Like I said you know I could have considered something like this. So, let us call this. You work out go back and you know this thing in terms of realization in terms of x and p, you solve the differential equation, you will find that this thing is it is x space representation would be if this is Gaussian this would be e power plus x square by 2.

So, it is not in L2; it is not in your Hilbert space, it goes out of the space. So, that is why you do not consider this but in the abstract setting which is the way we are doing it this is also possible. So, now you can see there are other things which will come and so unitary reducible it is a pretty interesting this thing and you can see that this also unifies just the harmonic oscillator and the angular momentum algebra. These are the two algebras which we use quite a bit in quantum mechanics and the neat stuff is this actually covers not all the Lie algebras require only these kinds of things and that is why I am sort of giving you this abstract thing which you can never ever really use it to construct things but it is conceptually useful thing. So, next lecture what we will do is we will see how to actually construct things and it will be this tender method or whatever; there will be nice ways of constructing irreducible representations of SU n.