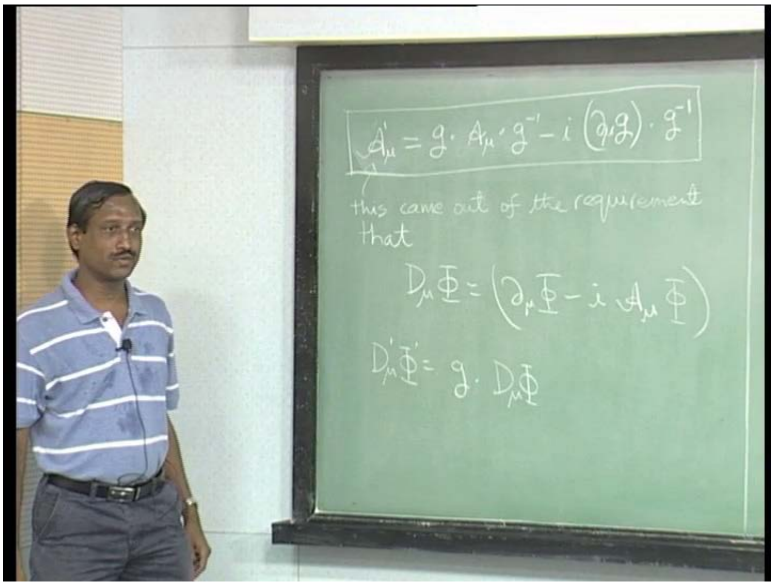
Classical Field Theory
Prof. Suresh Govindarajan
Department of Physics
Indian Institute of Technology, Madras

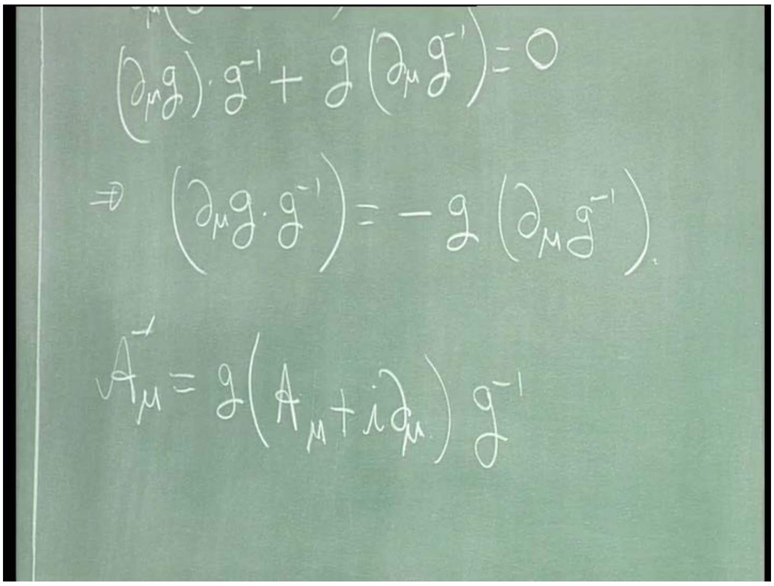
Lecture - 24

(Refer Slide Time: 00:12)



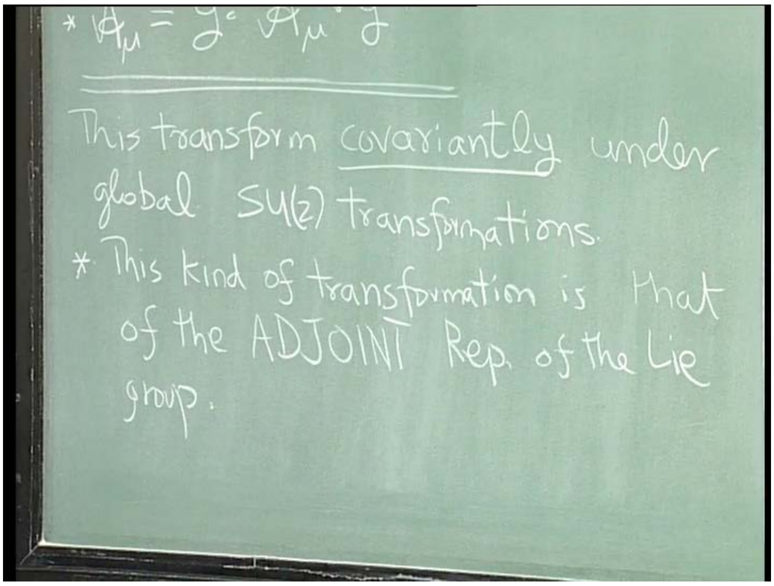
So, this was the transformation we worked out by demanding, so this came out of the requirement that the covariant derivative of phi which we defined to be D mu of phi minus i A mu of phi should transform nicely. So by that what did we mean; we meant D mu prime of phi prime should be and this sort of gave us something very nice transformation for A this is how this has to this. So, and the key point to remember is I am using script A to remind you that this is a matrix valued object, so that is one thing and this can be rewritten in slightly different forms. For instance I could try to put the d mu on top of this.

(Refer Slide Time: 01:31)



And the way you do that is by just taking equal to identity which I write as 1 and take d mu of this. So, what this tells you is that d mu of g dot g inverse plus g dot because identity is a constant matrix. I think I wrote dagger here; I will keep switching back and forth, but in unity representations g and g inverse and g dagger are the same. So, this tells you that you can see that this is exactly this quantity but this can be rewritten. So, you can go back and forth between these two things but there is something nice about writing it in the second form which you will see pretty soon. If I do that it will just correspond to putting the d mu on this and changing the sign; that is all that it corresponds to but you can something nice happening that you see that A mu prime is equal to g plus I; it is kind of something like this, maybe it is useful pneumonic to remember things. Otherwise, I mean there is nothing special about this over this. Now coming back to this object the thing is suppose we looked at things under global transformations.

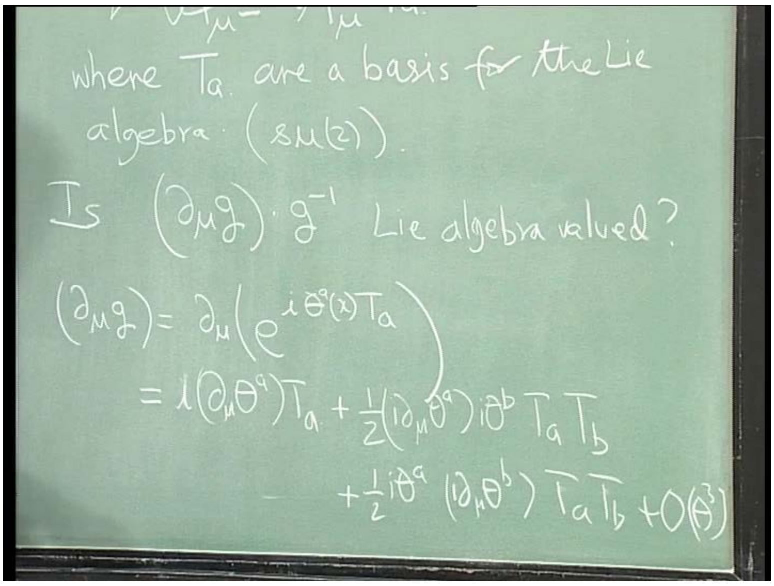
(Refer Slide Time: 03:25)



So let us assume what happens. So, the question is how does A transform? The nice guess would be I mean you would have said oh, it should be invariant. So, what do the global imply? g is derivatives so g should vanish but this term will not go away. So, it is very important to realize that A mu prime is not invariant but covariant. Unlike the case of the abelian or u one case what happened there it was invariant. t a mu prime was equal to a mu and that you can see happened when things are abelian or communicating, g can go through and cancel the g inverse; everything commutes so then it is invariant. So, this is the natural generalization. So, what one says is that this transforms covalently. This is generalizing what we know for normal transformations on space and time; we say if something that transforms nicely, we say it transforms covalently. It does not have to be ah covariant vector, even a contravariant vector transforms nicely; you do not say contract variantly or whatever, you say covalently for everything; that means it transforms nicely under global SU 2 transformations.

In fact the key thing as I mentioned last time is that will you write this only thinking keeping taking g to be SU 2 in mind but actually if you look through none of the calculations I really needed anything very deeply I mean which made use a particular properties of SU 2. So, all these things actually go through. You do not have to believe me; just sit quietly in your room or wherever and convince yourself that that is indeed true, just go back through the alternates. So, in fact one can show that actually this kind of transformation is that of the adjoint representation; we haven't defined this but I am just mentioning it of the Lie group. So, the adjointable representation has one property; it is dimension is the same as the dimension of the lie group, not the rank, so the dimension of the lie group.

(Refer Slide Time: 06:58)



So, another point which may not have been clear is the following thing is this was a claim and I leave it as a claim because I know I have given enough quasi proof but not a real proof. Claim is that A mu is Lie algebra value; by that what do I mean? Remember Lie algebra is a vector space and the generators of that thing provide a basis. So, what that says that A mu can be written as. Since we are doing we are discussing SU 2, it will be. So, coming back to this then we have to look at something like this and ask is this also Lie algebra value. Of course, it follows that if this is true this has to be true. It is like saying that let us look at this transformation; this is Lie algebra value this is also Lie algebra valued so this should be. But the thing is but this is something you can check explicitly and I sort of indicated last lecture how this is done to first order and said it in word but I think it is worth seeing of what happens to second order and so let us do that.

And so first thing is we already worked out what g mu would be g mu d mu of e power i and we expand this and we get first term is if you notice I have put t a instead of half sigma a because again it is not so important. So, this is what we get to second order keeping terms up to second order and so now next thing I need to do is to multiply by g inverse. Right now this T a T b is not Lie algebra value or anything but we should see;

what we should see is it would almost be that if this sign were different one of them because I can always re-enable things and make it look like this and that is exactly what we will see will happen. Again it is useful to put this lambda kind of book keeping parameter out here and the key point here is that d mu of g is actually order lambda and if you want to keep terms order theta square it is sufficient for me to look at g inverse only to first order.

(Refer Slide Time: 11:15)

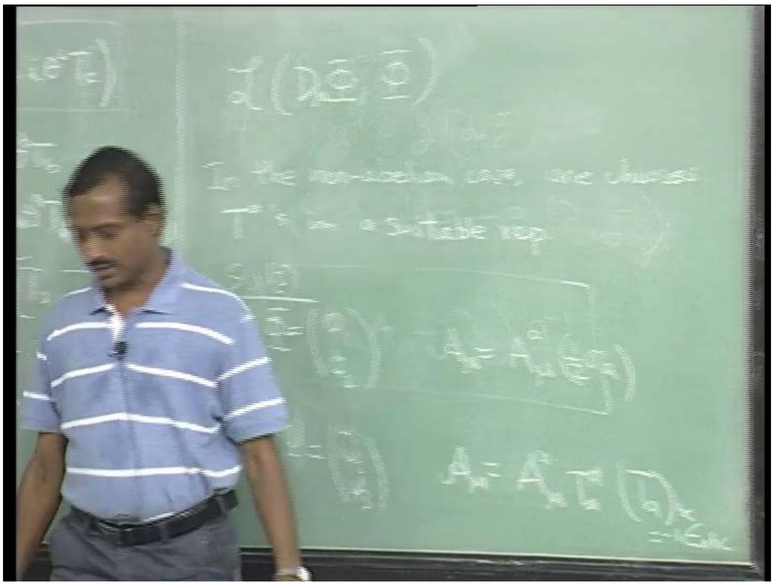
So, whatever I have written at the end of the board times minus. So this will be equal to, so now I will be going back and forth from this end of the board to the other end. So, this term will give you the first term into one, not a problem. Now so this is multiplying by theta c from the right side. So, coming back here that would give you a term with the d mu theta here with theta on the other side and that is exactly like this term but has a minus sign. So, the plus half minus one will make it a minus half. So, I can just put everything together minus half mu or there will be i's; I am just awful with these i's no more terms. Now I have to do some juggling which basically corresponds to saying let us go ahead and write, call this guy, call this b and call this a and then you can see that this is equal to i.

So, it is the magical combination which is the commutator comes out and this is now again in the Lie algebra. So you can see at least, so this implies that all these terms are in the Lie algebra. In fact you can also see that this is the complicated function, so this would be some function of theta d mu of theta theta some a and t a and there will be only one derivative but it can be nonlinear. So, the next term will have one d mu of theta and two thetas and you can see it and I can replace this by the structure constants and you can massage it look and make it look like something like that. So, this is in some ways just like the analog of the b c h formula which again you can only check order by order and of course the thing is that you know that there exist a full formula and the same thing holds out here but there is something very geometric about this thing.

So, what this actually tells you is that we know that g is an element of the group and if you take SU 2 it is like every point, every group element is like a point on the S 3 and what this is doing is looking at small variations. So, a derivative is like doing a delta g. So, think of this as your S 3; I pick any point it does not matter and then sort of look at a neighborhood of that point epsilon neighborhood which is all you need to do a first derivative and you sort of linearize things, it is called the tangent space. So, this is like looking at the tangent space at that particular point and so Lie algebra is always related is valued in the tangent space of the group manifold; that is the statement that this is actually valued in the Lie algebra. It is a very, very important point.

So, what it says is something it need not have been if you just did some delta g dot g inverse for any delta g so by delta g, I mean so point here is given by some value for thetas and you change your thetas by take theta a to theta a plus delta theta a but this is variably taken to be small and so you work out what delta g would be in that for that kind. So, then this object would be valued in the Lie algebra; that is what it says. But you can check I mean if you feel up to it you can check that the third order piece also works out. I do not know, often I do not know of a neat way of algebraic way that is to show that it is always Lie algebra the algebra valued. So, we have actually accomplished the goal which we set out which was to start with a Lagrangian which was globally invariant and we made it locally invariant.

(Refer Slide Time: 16:56)

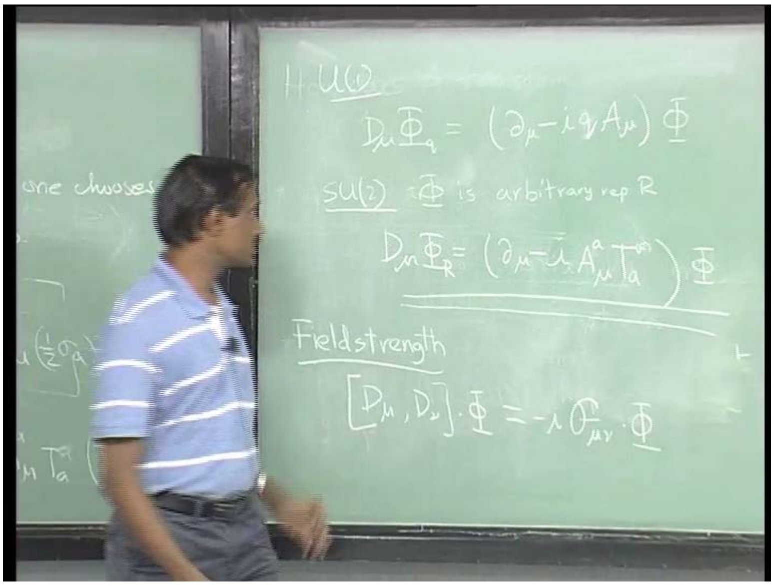


So, the rule now is very clear, you just do this and this takes care of it. And in fact if you have several fields again for each field every time there is a covariant derivative they are suitable depending on how that transforms you write the corresponding this thing a covariant derivative. It is similar to what we did even in the u one case, if the charges were different you put a different q but in the non-abelian cases what you would do is you would put. So, in the non-abelian case one chooses what plays the role of q, any idea? q determines the representation how it transforms; what determines in the non-abelian case, what is it I have to change? So, the thing is that suppose you have two fields in the u one case let us say with charge q 1 and charge q 2, the covariant derivatives depended on the charge. My question to you is in the non-abelian case what is the analog of charge? No, thetas are the parameter even there it is thetas is the parameters. So, it is just a representation.

It is a representation one chooses the T a's in a suitable representation. So, suppose so let us go back to the SU 2 and yesterday we looked at the case where phi was phi 1 and phi 2 in the two dimension representation and then we wrote the gauge field A mu to be A mu half sigma a. So, this is in the two dimensional representation or spin half representation but I could have chosen a different one. I could have chosen phi to be in the three dimensional representation; this is a different phi, let me call it psi just to in the three dimensional representation. And then the A mu's it will be the these fields are the same but I have to change the t's because these are two by two matrices I need to write the corresponding three by three guys. We know what that is, it is a minus I epsilon a b c or whatever.

So then so we would write A mu, I use the same symbol but really you will see what is different on this side by T a of b c. I think it is minus i epsilon a b c. So, what will remain unchanged is this is the gauge field that these guy's the three guys here and three here they are the same. So, it is similar to if you remember what happened we wrote q a mu except q gets replaced really by the generators of the Lie algebra. The reason is it is just a number in u 1 is because all represent an irreducible representations of u 1 or one dimensional. So, you would use a one by one matrix which we will not call a matrix; we usually call it a number or the charge. So the point here is that if you go back to the covariant derivative which I will write again so you can compare.

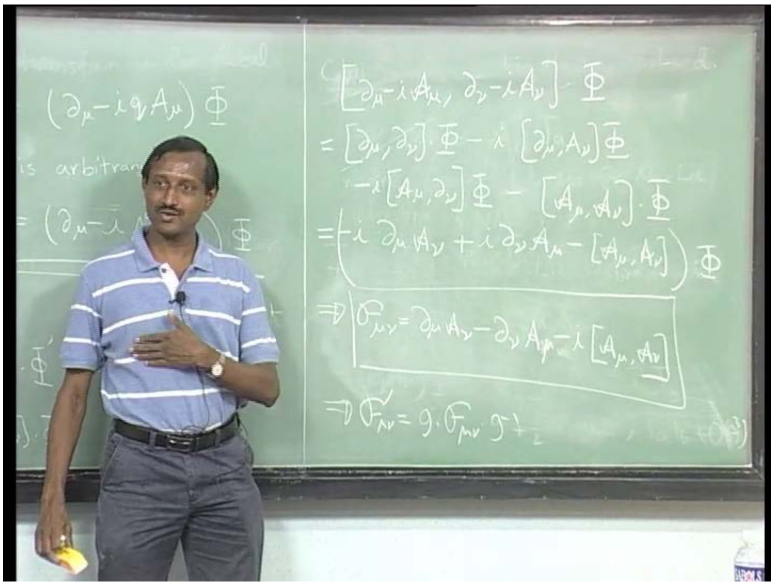
(Refer Slide Time: 20:59)



So in the u 1 case we wrote D mu of phi which was in some q dimensional representing and charge q we wrote as d mu minus i q A mu; this is what we wrote. Now for SU 2 some arbitrary representation phi is in some arbitrary representation R, then you would write D mu of phi R where I am putting this superscript R to indicate that I have to choose, whatever is the matrix. So, I gave you two examples but there can be many many more examples. Are there any questions? The thing is that what I am saying here the point here is that for u 1, there is only one T number one and the Lie algebra is very trivial t t is zero. So the only way and it is one dimensional. It is a one by one matrix which is a number. So, instead of calling q I could have called it just the t; I could replace it by t if you wish but I am just calling the number so that I stick to my notation.

So, it is just to make the parallel. So, really how this reduces to this is more what I am getting at rather than the other way. So, now we are ready to discuss what the field strength would be and we just repeat what we did. What did we do in the u 1 case in the abelian case? We worked out the commutator. So let us see, so the analog of field strength. So, for that we need to compute something like this and what does this become, what is this equal to? This is I just need the factor of i. How did we define that? Was it i times f mu nu or minus i times. So, I will define again some matrix valued object dot phi. Was it minus; there was a q that is okay but the q is replaced by a t here. So, I am writing this, yeah t inside the f, so that is perfect. So, again this analogy is now useful. So, we just have to compute this and the calculation is quite simple but there are some the factor things do not commute give you extra terms.

(Refer Slide Time: 24:28)

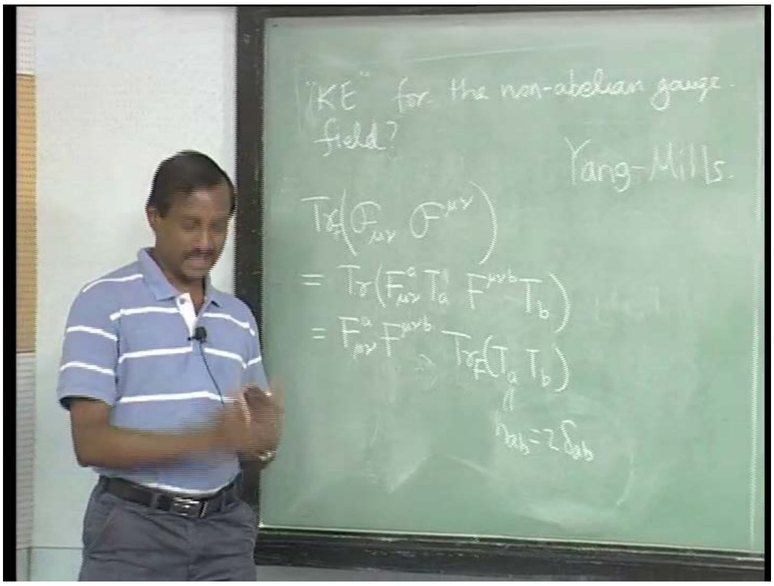


So, let us look at this. So, this would be d mu. I will do things all are quantum mechanics now and rewrite things because these are matrices. So I can rewrite this as that is this, last thing is minus, minus i into minus i is minus one. First term obviously vanishes because derivatives commute once acting on smooth functions so this becomes zero but what does this give you. So, let us look at this d mu of A mu of phi minus A mu of d mu of phi; this cancels with this only when it acts on this. So, these are all script A's. I am going to erase it in a moment but in your note books you can. So this gets this and this is just minus of the other thing. So what we end up getting, so this term is equal to.

Now I have to pull out n s, so this implies that f mu nu. So, I am pulling out a minus I so that should get a minus. There are several ways to see how phi transforms; one is to go back, is this clear? Yes or no; you can say no also it is fine, I can try to explain. See there was something wrong out there. So, the hard way of getting the transformation of f mu nu which I would heartily recommend that you do is to we know how A transforms; go and plug and charge but I will do it in two seconds by using this definition. We know how phi transforms and these things act nicely. So, phi transforms with a g out here and so what is it we have to write, so I start like this put primes on all these guys and so this will be equal to this is just g phi. This is what I get but you look here and since these things transform nicely what this will tell you are that this whole object on this side transforms like g times.

So, putting things back together this implies A transforms nicely. Again so, this is exactly how the gauge field transformed under global transformations but this is true even under local transformations. So, again unlike the u 1 case the field strength is not invariant but rather it is covariant. Is this clear and the most important part here is that it also has this kind of a piece A square piece it is not just a derivative piece. So, if you try I mean so in some ways this definition is a much nicer definition of the field strength because if you try to say let me try to start with some gauge field and try to construct something which is invariant. You can do it but, then it will be messy figuring out this thing; you can I mean I am not saying it cannot done. So, now that we have field strength the idea would be to go ahead and try to write an action for the field strength and so for this.

(Refer Slide Time: 31:00)



So, what about a kinetic energy for a non-abelian gauge field, any suggestions? f mu nu square you would say. So, we have to you would write something like this but we need to, so but this is a matrix valued thing, so it says we take a trace. So, now while the thing is that so again this is also Lie algebra valued, so you could write this as trace. So, these are the only matrix part. So, I can pull these things out but now this what number you get; what you get here depends on the representation. You can show that it is always proportional to these things but they will differ by some factor. So, then we have to fix the normalization, so you need to choose some representation. So, what is usually done is that you choose to define it in what is called the fundamental representation. So, if you take SU 2 the fundamental representation is the two dimensional representation, for s u n it is the n.

So, all the continuous groups which we discussed earlier we actually it was like chicken and egg; if you remember we defined the group through a particular realization of it; that is the fundamental representation for all the groups the basic groups that we looked at, the orthogonal, the unitary and the syntactic. That is this in some sense they are also the smallest representation that you have for this thing. So, for SU 2 the smallest representation is two for SO 3 it is three. So, what one does is to, say, choose to fix normalizations you choose things to be in the fundamental representation and in most cases the normalizations are such that trace of these things you call this some this is called the cutter metric; it is symmetric of course from the property of the traces and this is the definition of this thing and if you just take the case of SU 2 you will find that h a b is delta a b.

In fact for SU n you did the Gellman matrices also, for there also it was delta a b. It was two delta a b, thank you. So, this is two delta a b, yeah; probably it is true even for the other thing for the probable matrices, all of them it is the same. So, up to this normalization factor you can see that this is the definition which you would have for the kinetic energy but now comes if you look at this thing it is f square and coming back to what we have out here, if you take f square it has terms like this with square and these look like kinetic energy terms but it has it has cubic and quadratic pieces because when I square this I get one term which would be d A A square which is cubic, then there is a quadratic piece which is A square and this is no interactions nothing; it is just a non-abelian gauge field. It is highly nontrivial. It is equation of motion is nonlinear unlike Maxwell's or the abelian case where it was linear. There were no interactions, nothing.

So, this already shows you that non-abelian gauge theories are very different in characteristics and so the thing is that it is not there is no analog of a free theory but what you do in quantum mechanics or quantum field theory. It is that you break this up and treat this as interactions and do them perturbatively but in terms of gauge invariance or if you take only this combination this does not quite transform nicely. So, you have to be quite clever about how you go about doing things and maintaining gauge invariance, etc because this combination is not invariant nor is this combination; it is only this particular combination which transforms nicely. I should not gauge invariant gauge covariant. So, that that itself shows. So, for instance q c d is a theory where the group is SU 3 and so SU 3 the dimension of it is 8; it is 3 square minus 1 which is 8.

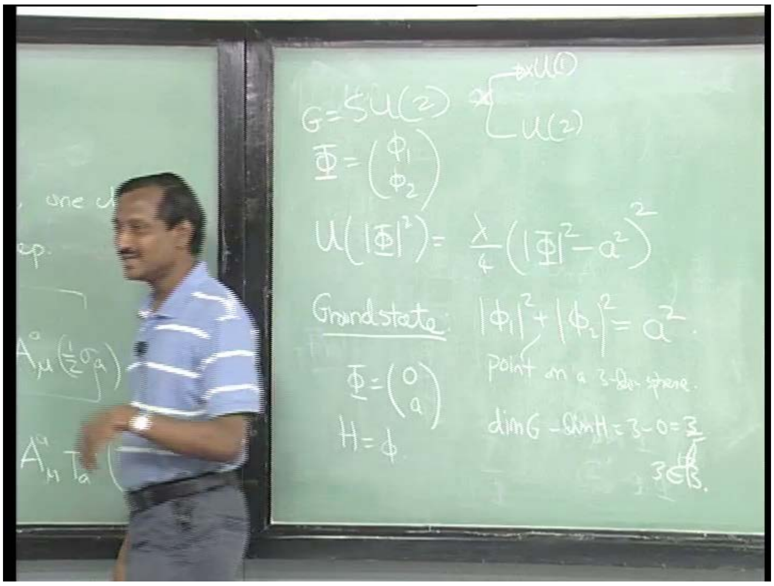
So, that will tell you that it has to have eight gauge fields and in quantum mechanics for every particular gauge field you will expect one spin one particle so that would predict that you should see eight spin one particles. They are called gluons and except that we do not see any gluons in nature free gluons in nature. What happens is that quantum mechanically this theory is like I said there are interactions are they interact such that they are confining that you never see objects which actually carry this charge. So, there are quarks also which transforms like which are analogous to this; they transform in something. You could write some Lagrangian but that cannot describe low energy physics as we see it. But there are high energy regimes where you go to accelerators, etc where you break things apart and then available to see I mean evidence of existence of this object.

And so this is even classically it is a complicated theory but even quantum mechanically obviously it is much much more complicated. So, non-abelian gauge theories are also called Yang-Mills theories. This is after two persons Yang and Mills who independently kind of came up with this proposal. I do not know when this was may be late fifty's or early sixty's and another point here is that you cannot write any mass term for this. You might think I mean that the mass term would be a quadric term. There you cannot write a gauge invariant mass term. So, exactly like so the thing is that, yeah, you cannot write a mass term because there is not gauge invariant and so the predication would be if you took some gauge group; let us say you took SU 2 you would say that you should see three massless spin one particles.

So, in the early sixty's when Glashow, Weinberg and Salem, they were the first people to actually use this setup and they actually said that there exist some SU 2 cross U 1 whatever that is; we will see in more details of it later but then the predication was like I said for every vector field we should see the thing but nobody saw any massless spin one particles. The only massless spin one particle we have seen is a photon and so to be honest I think it must have been very courageous of those people to actually propose things and actually push it and they may have been a laughing stalk of people at that point in time because they are saying look these guys are writing out these theories and then we have not seen any spin one particles and so it lead to this issue of how do we understand this. And today we know that these particles are seen but they have masses and so the question is, how do they get masses and so I will just discuss that in the next ten minutes because we already have the structure for it and we already have seen how to get a massive photon.

How did we get a massive photon? Higgs mechanism, so what happened in that case? We started out with the situation where you had a global symmetry which was under the vacuum spontaneous broke the symmetry to some subgroup h, u 1 broken down to nothing; that is what we looked at and so there was one Goldstone boson but when we went to the local theory that goldstone boson disappeared and what happened? We ended up with a massive photon and as I explained to you a massive particle has one degree of freedom more than a massless particle; excuse me, sorry. So, the framework is very clear to us. We should look for a situation where we have global symmetry which is spontaneously broken. So, let us go back to our SU 2 model and ask these questions and see what we get.

(Refer Slide Time: 41:32)



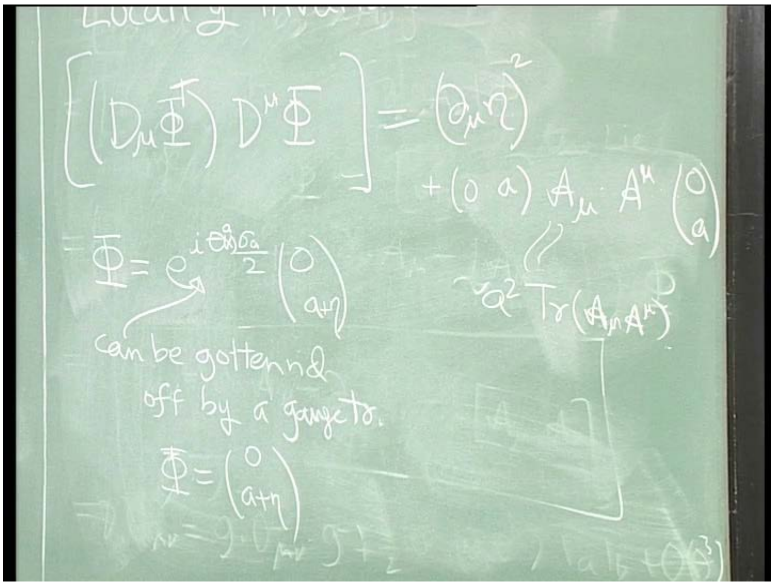
So, we will be very very specific here; we will take SU 2 the group to be SU 2 and phi will be exactly the example we have been looking at in the two dimensional representation and what is it we have to do? We have to choose a u of phi with such that the ground state has a nontrivial value. So let us choose u of phi. As we all know this is the only potential I seem to know. So, the ground state would correspond to, say, remember phi 1 and phi 2 are complex fields. So this is like a point on a three dimensional sphere and so let us say that we go ahead and do the following. We choose a particular solution; so let us choose something like this, choose phi such that it is zero for this and a out here. Now the question I want to ask you is what is the unbroken symmetry, what is the analog of h, what is the symmetry of this solution, is there any subgroup which is preserved? Yeah but that is not a subgroup of this. So yeah, so there is a good point. So, somebody is saying there is a u 1 I could have just changed the phase of A for instance but that is not an element of this group.

So but I could either think of it as I could do in two ways to add that u 1 into the story; one of which is to say that it is just some cross a second u 1 which would act in some phases on these guys or go to u 2 if I choose the global symmetry is also u 2 it could be.

U 2 would be a special case of u 1. So, you can get SU 2 cross u 1 and SU 2 cross u 2 but right now I am not looking at that part. Eventually we will in the Weinberg-Salam model you will see that it is an SU 2 cross u 1 will have this same thing. This is called a Higgs doublet or whatever; does not matter but right now since that is not part of it, there is no so it is not so complicated. What you have to do is you just look at that you know t 1 t 2 t 3 or sigma 1 sigma 2 to sigma 3 we will see which of them kills this; that means it is invariant. So, there is nothing. So H is phi; it is broken down to nothing.

So, what does Goldstone's theorem tell us? We should expect three Goldstone bosons. So, implies three Goldstone bosons. So, what we want to do next is to gauge it and what do we do by gauging? We have to follow the same procedure that we did a little while ago and convert this global SU 2 into local SU 2. So, coming back to this generalization you see that if I want to convert these guys also into Su 2 cross u 1 or u 2 into a local these things then I have to add one more gauge field. I would have four gauge fields but in our setup here we have only three gauge fields because I am focusing on the s u 2. But one thing you will agree is that even if I go to these groups the Goldstone boson theory number would be the same because the number of generators will go, this will increase by one, this will increase by one. So, the count of goldstone bosons will not change.

(Refer Slide Time: 46:30)



So now, we go to the locally invariant theory; and actually coming back to this you can see that these Goldstone bosons are exactly the flag directions on this three sphere which is going along the sphere three sphere. So d mu of phi, so this is what we were looking at and so we look at what we did; what we can do is to just expand things to quadratic order and so we just look at phi, and write out plus I can put an eta here. So, what I am here is parameterizing fluctuations and like I said what we should do is write the broken generators out here. We have already seen how to do this. And the key point here is this theta guys can be completely eaten up by just making a gauge transformation and a local gauge transformation. So, all I need to really do is in this not even worry about the thetas; assume that there are no thetas in this thing and just write out take phi by a gauge transformation.

So, for practical purposes I can just write phi equal to zero a plus eta and that is it. So, now I can just go there and plug these things in here and expand to quadratic order in etas and a's, etcetera. So, this is not as hard as it looks. So, d mu of this is just d mu of eta, so I will get one term which is d mu of eta whole square. But here eta is a real field and what else, yes and then but now the point here is that there are terms this thing d mu has also a mu piece. So, there will be one piece which will be, I am not writing all the pieces so there will be one piece which will look like a mu zero a. If you put all these things together you will start seeing, so this term will look like a square A mu trace A mu square. So, what does this tell you that this mechanism is exactly like what happened in the abelian case except that this now has all the guys all the three it is there and you see that they all getting masses which are again exactly like what happened to the photon.

The photon mass was proportional to a and it is exactly what we see here also the photon mass is indeed proportional to a. So, there will be n square so not the photon these three guys. So, this is called the Higgs mechanism. This is not the Higgs mechanism which is used in the standard model; it is an example of a non-abelian Higgs mechanism. After some about four, five lectures down the road we will actually discuss the standard model where we will look at SU 2 cross u 1 structure with a similar doublet and there will be some unbroken guy; that unbroken guy will be identified with the electromagnetism. The H that will be your u 1; so it will not be this u 1 nor will it be thing, it will be some linear combination of them.

So, this u 1 charge is called hypercharge. So, one usually writes that with a u 1 y not q. So, hopefully you are able to see the structure of these things and so you can see that you can get masses through this mechanism. Just one more bit, it is sort of interesting; if you take SU 2 and you take a doublet you can see that you can never you can never get a u 1 unbroken just with plain SU 2, because it is like saying if you pick a direction in two dimensional space there is what remains is nothing; roughly speaking but now so the question is suppose I want to get a unbroken u 1, what should I do? Any ideas; I want to start with SU 2 but I would like to get an unbroken u 1. I will let you think about it; may be you can back and tell me; that is a very simple answer to that. So, I will stop here.