Classical Field Theory
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Lecture – 23

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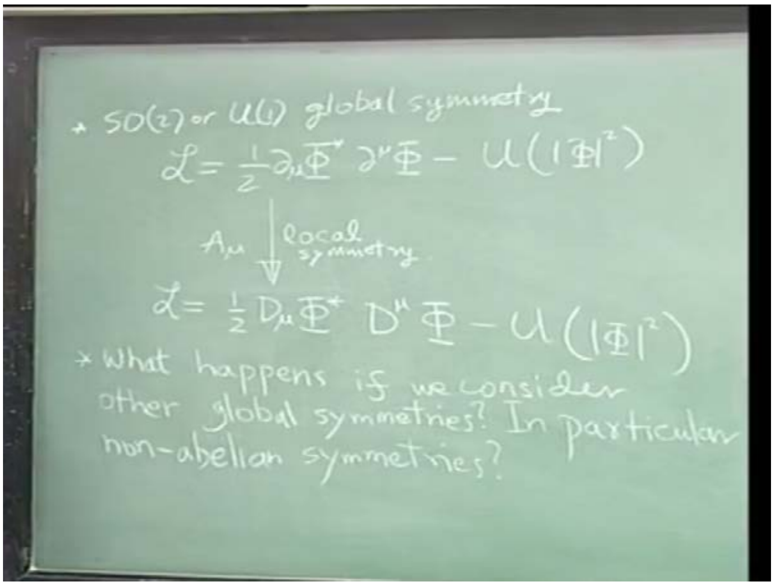

Looking at what, what we started out, we started out with a theory which had S O 2 or equivalently u 1 global symmetry, and so we had a typical Lagrangian which have this form, phi was a complex scalar field minus mu which just depended on the magnitude of these terms. So, this is a typical Lagrangian true derivative kinetic energy term and an arbitrary potential which has global u 1 symmetry. And then the idea was to convert the symmetry into a local symmetry and and that we did by introducing the idea of introducing an extra field, gauge field which had a specified transformation property such and you defined a covariant derivative. And so this theory was automatically now invariant under local u 1 symmetry.

So, now the question is can we what happens if we consider some other I mean more importantly other global symmetries in particular non abelian once. And keep in mind that the, the symmetries we have in mind are the continuous symmetry not discrete ones. And so the so the canonical and the first not trivial example of of non abelian symmetry is the S u 2 group S u 2 so....

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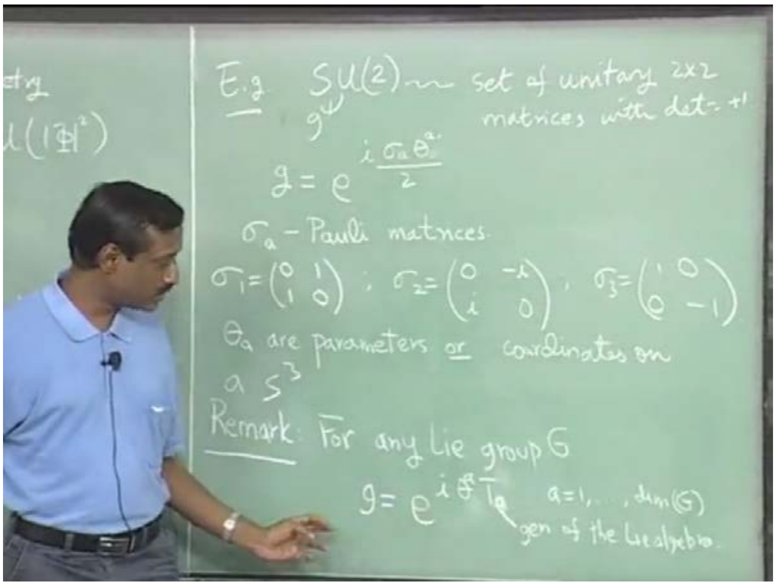

So, we will work with this example which is S u 2. So, we have seen this definition of we, we saw that we could write this as the set of unitary matrices with the, terminate it could be plus 1. And in your, in your one of your assignments we are actually shown that a typical element of this g which belongs to S O 2 can be written as exponential of. So, a this was sigma a here are the Pauli matrices, and just for completeness let me just write out all, all of them is a minus here this is correct I always make a mistake with respect to this.

So, this actually shows you that n theta is, so theta a is a parameters, parameters, but we also saw that the condition on these thetas is that they have they, they live on a sphere 3 sphere. So, theta a is a parameters or we saw that coordinates on a 3 sphere. So, all these features are actually kind of generalize to any lie group as we, we saw, so just a remark for any lie group we could write g as sum e power I, maybe I should be little bit more pedantic and write upper and lower out here. And but a will run over the to what you will write is dimension of G.

So, this is an example. So, the dimension of S O 2 in this case is 3, just roughly the number of parameters that you need and one more thing we saw I said lie algebra. So, these are the generate T a's and T s are generators of lie algebra so coming back in mapping, looking at this.

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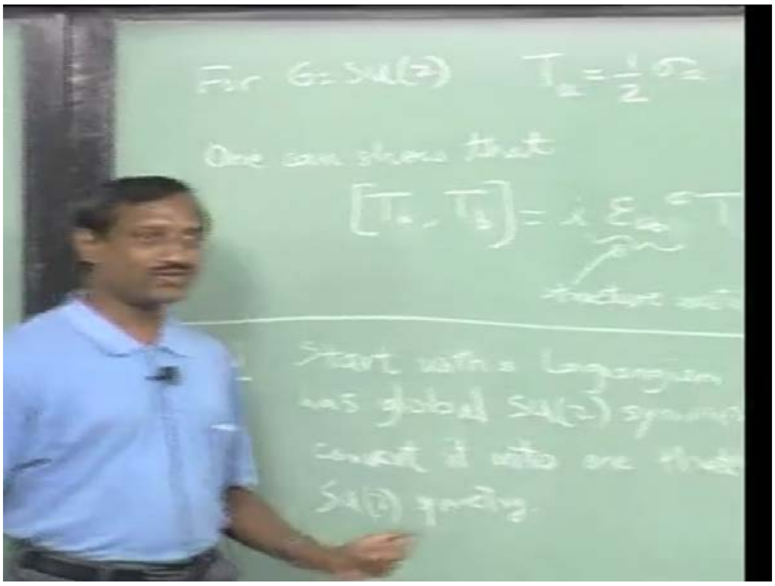

We can see that T for S u 2 the T a, that we have chosen here is this half sigma a the half is part of the normalization. And it is easy to show from one can show that lie bracket, but in this case we saw that ones you write it in terms of matrix the lie bracket is nothing but a commentator. So, the lie bracket of these guys is equal to i epsilon a b c epsilon. And this is an example of the structure constant, so I have to be really precise if I if I want to write things correctly, do I write with an upper a so let me just need more correct and write. This is I do not know I was not consistent from here to here, so let me be consistent so theta a is upper and the T a's are lower, so then I need to do less work here.

So, this is what you get and so we know quite a bit about this and the nice thing is our S u 2 is that it captures the complexity of all non Abelian gauge forms, this is a nice thing. So, what we will do now is to you write something which has S u 2 replacing u 1 and just push this through and see what we will get. So, so the goal is start with Lagrangian that has global S u 2 symmetry and convert it into, into 1 that has local this is the goal and more or less we can follow these thing these things through. But that is some nice thing, one nice thing to see is a suppose we considered a particular only a situation where we are looking at the sub group its only one theta turn now let us just.

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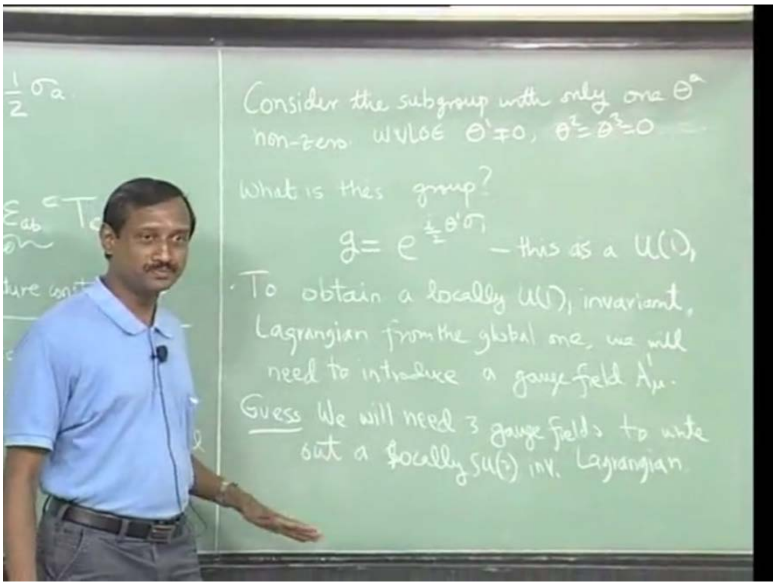

So, just for simplicity consider the sub group with only one theta a non zero so with no loss of generality I can choose that to be theta 1 I mean dim g I just choose one of them. But in principle I I can do that for every one of them, so I am just write now choosing one particular example. So, first thing is what is the group generated? So, the question is what is the group? So the group a so it is a bunch of elements of this form, let me so in this case it is why I am writing this. So, let me the even more so let me choose here theta 2 equal to theta 3 equal to 0 could be 2 general. So, I have theta 1 sigma 1 what is this group look like, what is this group? S O 2 it is just a rotation it is a u 1 or an S O 2.

So, this is in fact, if you expand it out you will see that it has cosine theta so and so I will not it will be some 2 phi 2 by 2 matrix sigma are this thing so it is just a u 1. Now, you can see that now, so what I can do in my mind at least to say first step I will do is to may I consider this global symmetry and try to make it local. And then it then this looks exactly like the problem we have already solved, we already solved this problem, modulo some details of working things out. So, the thing is what the, what does this tell you that for theta 1 I should introduce 1 gauge field now I can repeat this thing by saying I do not look at now the thing is invariant under theta 1.

Next step I doing is let me ask can I make it invariant under theta 2, theta 2 is not switched on and theta 1 and theta 3 are 0, again it is not hard to see that is also u 1 or S O 2 so again it is a same story. So, you can see for every such parameter even in the most general case, we have every one we can start looking at different u 1 sub groups and you can scan through the whole set of these guys and it so what does it tell you? It roughly tells you that if you you need to so the conclusion by looking at this to so let me write this to obtain a local e u 1, let us call this u 1 1 u 1 1 invariant Lagrangian from the global 1 of course. So, if your theory had S u 2 full S u 2 symmetry; obviously, any u 1 sub group also it is invariant. From the global one we will need to introduce a gauge field or a vector field which we can we will call A mu 1, so the 1 is to remind us that this has to do with the theta 1.

So, at least a guess based on this is that we will need 3 gauge fields to write out locally S u 2 invariant, what is the constraint? That is a constraint yes, but but the key point here is let us let us look at let us assume that the theta is very small just for moment. So, what that is telling you if you pick up a particular point and you are looking for changing theta’s in that neighbor it will there are 3 independent parameters that all that matter’s s 3 s 3 dependent parameters and this is also s 3, because usually what you do is write it as theta a. So, you write it as theta 1 square I mean this is not the way to show that it lives on a sphere these are not the these are not the coordinates for which, which square to 1 these are not the once something else so we showed it in a different manner.

But what I am saying is that does not matter that is why I just call this theta as coordinate these are not the coordinates you get by starting with that you would start with 4 such things s 3 comes by taking 4 coordinates and putting a constraint. But here so these are exactly the same number of parameters as you have coordinates on the things. So, they so I why it is not over kill even that way it is, so it is consistent with the constraint. And I am not saying I am this is only a guess, but this is a good guess and we will see that more over less this goes to.

And I will not follow this route I will follow the route that I just straight ahead go ahead and turn on all theta’s, but the intuitively even before you start you should realize that you will have as many gauge groups as the dimension of the lie algebras is, is 1 for every theta. So, let us let us go ahead and first step is to write the most general global, globally invariant thing we will use the same set up. So, except I will redefine what I mean by phi.

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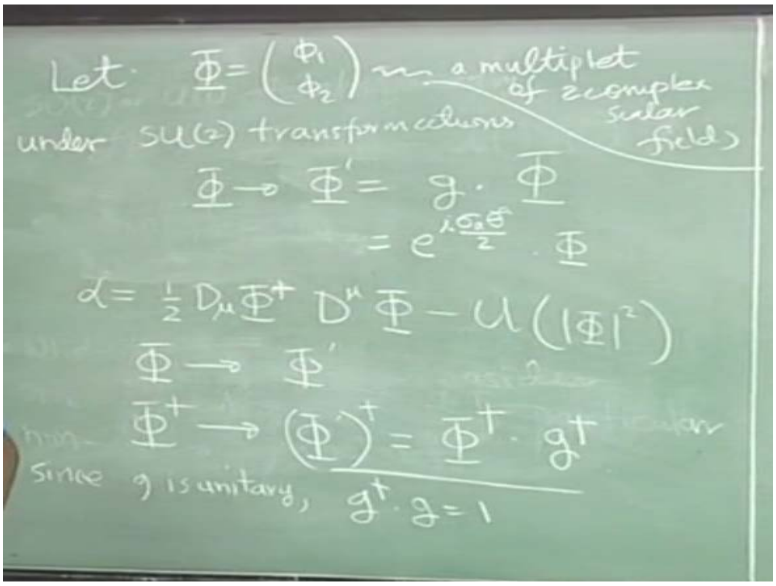

So, let us it is important to realize that here what i mu g here, this is a unitary realization of these things. And we will always assume in, in most physical literature we always assume that we have a unitary realization of a group. For S u 2 itself you could have instead of writing this I could have written a 3 by 3 representation that would be a normal rotation matrices which is the same as an S O 3 matrix. But if I could have also done something like 4 dimensional one which is like spin 3 half that is right 2 j plus 1 yes spin 3 half would be 4 dimensional. So, here I have just choosing something very very special.

So, this should act on something which is 2 dimensional. So, I will just choose phi to be 2 component complex of g. And under S O 2 S u 2 transformations phi goes to phi prime which should be just g dot, where g is what I wrote out somewhere this. So, this is equal to and so this is multiplet. So, the correct the one states that one says that phi transforms in the 2 dimensional representation of the lie group S O 2 it has many other representations. If it was in if it was in the 25 dimensional representation, then phi would have 25 such things. And we would not use the poly matrices we would use the one suitable for that it gives some 25 by 25 matrices which satisfy how a satisfy the same, same lie bracket the bracket structure.

So, in your assignment you will see things done in more generality, but I always find it useful to keep this S u 2 example in the back of my hands whenever I get a bit confuse I just say you know put it out where and then it sort of explains this. So, now so we have a field like this so we just need to make sense of this. This is where the mathematicians we have thinking of star as dagger helps, but since we are not this we are physics people, we write let us it replaced by dagger.

So, now we need to know how does phi dagger transform phi phi dagger actually transforms in the complex conjugate representation, so phi dagger so so you do it this way so you are given that phi. So, phi goes to phi prime so phi dagger should go to phi prime dagger which will become phi dagger dot g dagger. And since g is unitary, what does that imply? g dragger dot g equal to it is just a unitary property. So, now we so the idea here is to go ahead and construct objects which are invariant under, under the S O 2. So, the first invariant we could write.

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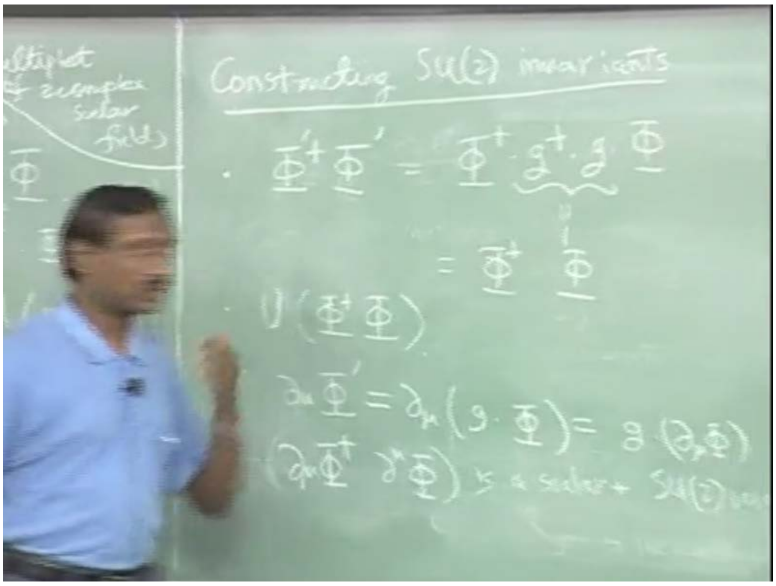

So, S u 2 invariants let us constructing S u 2 invariants. So, I want you to give me some examples of objects which are invariant under the S O 2 starting with the fields phi of course, you are given this. So, can you give me some examples phi dagger phi so first obvious guy is phi dagger phi, why is that? So, let us put primes out here and look at how this transforms these two things comes. So, this is equal to phi dagger dot g dagger times g dot phi, and the unitary unitary property of g implies that this is equal to 1. By their this is true for any, any all representations like I said we will always choose them to be unitary. The group may not the unitary group, even this could be S O 3 it does not matter it could be S u 10 it does not matter.

So, this kind of objects will always be invariance this prime is correct so g so this is equal to phi dagger phi. So, in fact you can see that any function of this, so more generally any function of phi dagger phi, let us call it u itself is obviously invariant under S O 2 nothing needed. This is kind of nice you can see that with this structure it looks exactly like what we did for the u 1 except for the u 1 which was phi to be 1 component. So, it becomes a special case of more general things, what about now looking at derivatives, how does D mu of phi prime transform, D mu of phi prime this is a local I mean this is a global things this is equal to D mu of g dot does not act on anything else.

So, what this tells you it is a D mu of phi prime transformed exactly the same representation as phi same thing exactly like phi. So, now question is can we construct Lorentz range so Lorentz scalar from this object which are also S O 2 invariant. So, can you give me something can we construct invariant from this. So, the kinetic energy kind of term you can check again the same thing goes through goes scalar plus S u 2 invariant, we are only talking about global, so there is nothing. So, now you see that now the whole Lagrange which I just raised out here works with just one change that I did write phi dagger instead of just phi star.

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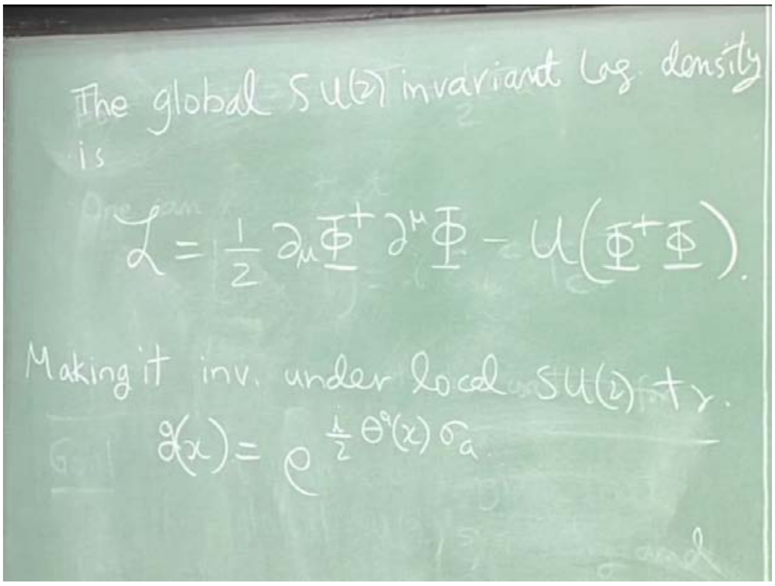

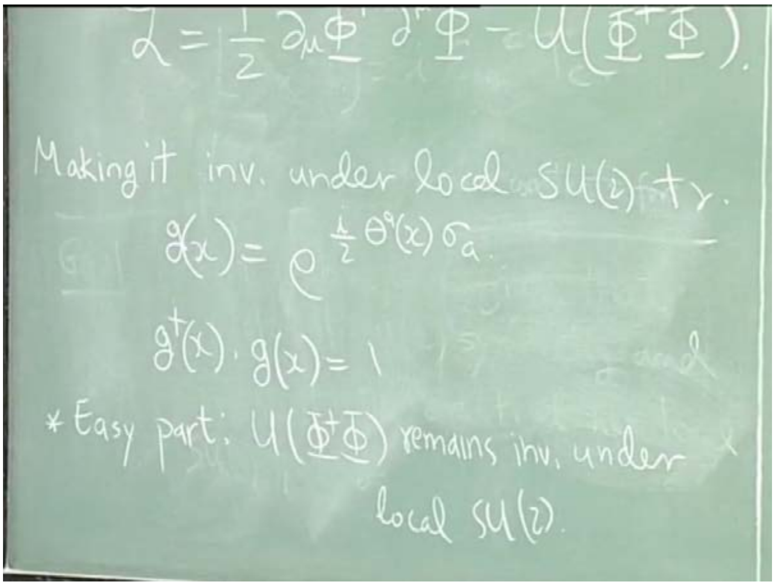

Global S O 2 invariant lagrangent density, hence and it is invariant by inspection we do not need to do anything. Now we have to clearly do the next step. So, now making it invariant under local S u 2, so now we have what happens is that g is a function of the coordinates. So, the way we make this depend on space time is to require that the parameters become x dependent. The sigma is poly sigma matrix is just scope for the ride I mean they just they keep track of the matrix of the group I mean the group theory part of it, it ensures that it is indeed unitary matrix without and this continues to hold g dagger x dot g of x is always equal to 1 the identity still, it does depend on where the theta is constant of it depends on this thing.

So, in other words what this tells or one more thing we can go back now and look at this Lagrangian and ask what happens now under local transformations. Again this is very easy to see that the potential term is continues to be invariant just for the easy part it remains invariant under local S u 2. So, but the problem will be we have to re visit how this transforms now. So, we need to work out so the...

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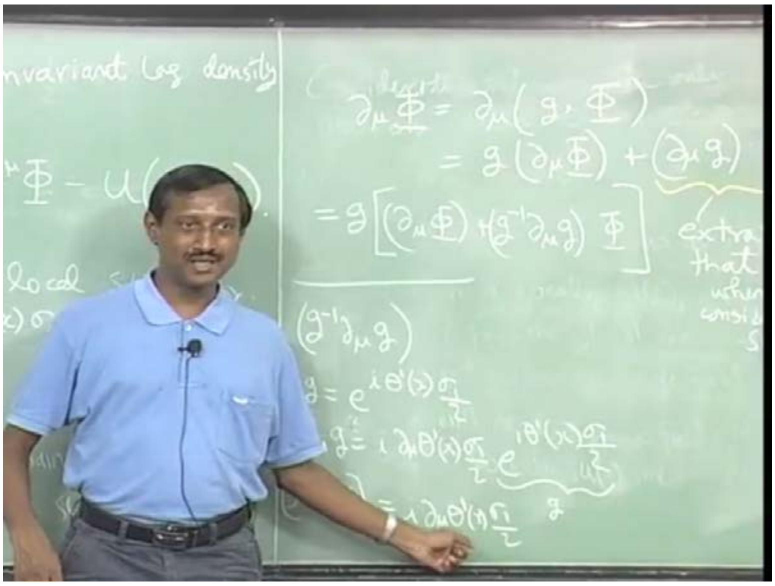

So, let us look at d mu of phi prime, again that transformations are exactly as before except we will put for local we just say g is g of x, and similarly out here phi prime transforms like g dagger of x, so there is nothing more. So, we have to write this, this is d d mu of g dot phi. So, it has one part which is nice which is when d mu acts on this, this is as before, but this is the bad news of the back part. This is the extra term that appears when we consider S u local, let me just do one thing. So, the I will just pull out the g so that I can write out something I have done nothing I have just write a written a g g inverse this is the reason for doing this, because I would like to study the structure of this object, what is this object?

So, we need to understand this, so consider g inverse d mu g. So, now again we will do it like we will repeat what we did with our, with the u 1. And so let us so let us study this guy and let us say g to be just e power i theta 1 of x sigma 1 half, we take we take this and so we need to conclude d mu of g. So, let us do that if you are cavalier about this, you would have thought; you would write something like this well this will work this will work. So, this is what you would get by 2. So, this is what we get, and if you work with and this is nothing but g and if.

So, we need to work out what g inverse d mu g is so g inverse will come and cancel this, and I end up with i d mu theta 1 of x sigma 1 by 2. So, you can see this exactly what we saw in the u 1 case except the, the structure constant sigma 1 is floating around sigma 1 by 2, not the structure constraint, the generator of the lie algebra. So, this is where the non… So, there is now the if I did the same thing with theta 2 there will be no problems again the same thing will go through, if I did it by theta 3 again it will go through. But if I do with all of them together then things do not the non commutativity comes in... So, let us see what happens. So, let us now so the so this, this looks nice this looks exactly when I turn on only one of the thetas as local it looks exactly like the u 1 case, except we learnt 1 more lesson that you have to put this thing. So, we are fine in that sense and so roughly you would have thought.

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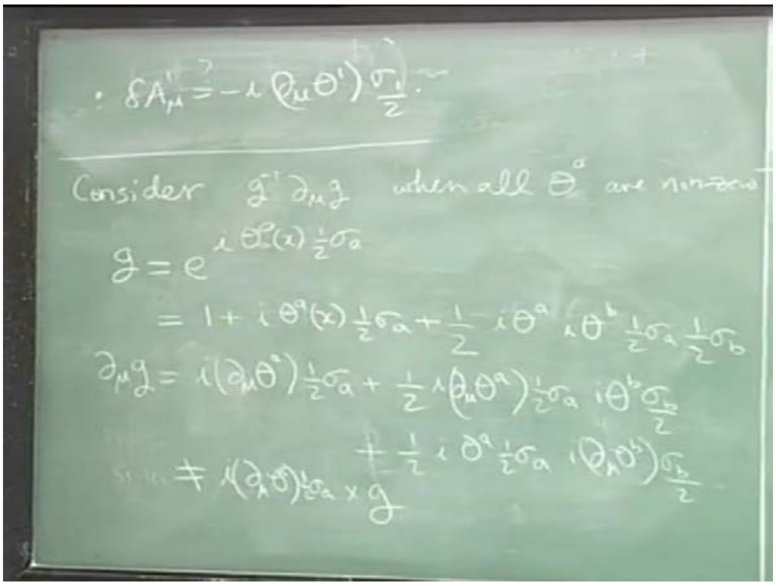

May be delta A mu of 1 should be minus of whatever that thing is right minus i d mu theta 1 we should write something like this is what we would have thought correct. But this turns out to be not quite correct the reason being that things are non commuting. So, so what we will do now is instead of taking these special examples, we will take the full g inverse d inverse. So, let us let us now consider the full thing that was just a toy exercise, So, I will put a question mark consider when all thetas, now we have to write g equal to e power i theta a of x. And we go back to the basic definition what was the basic definition which has expand power series.

So, what I need to do? I need to work at least to second order that is where things would run into trouble. So, I am pulling out these parameters, but sigma a sigma b is not generically equal to sigma b sigma a so I have to be careful on that, but these theta parameters of course, there is they are just numbers they commute with each other. So, now let us now let us take derivative of this with respect to g. So, the first term of course, will lo exactly like what we wrote out here, but the next I want to keep track of the second order piece as well. So, we get 1 goes away i, but now I I have 2 ways of doing the derivative. And I get 2 terms and let me keep these next to each other just for the, the temptation here would be to say that look, these 2 are the same and I can get rid of the half, but there is a problem, the problem is that is that or equivalently i you might think.

So, the question is can I let us do something maybe I can full out this d i d mu theta of this thing I will try to pull it out. But then first thing is this is the half this I can pull out, but this I this I have to if I want to pull this you may think fine I can re label this I can this is dummy index I can call this b and I can re label this as a, but it is on the other side. So, I cannot pull it through so that is not a simple way of writing this guy out, it is a whole bunch of terms you can see if you go to the cubic part you get you get 3 terms and it keeps getting worst and worst and worst. So, what I am trying to say is that this is not equal to now you can I mean, but but it is true if only one of the theta 1s were turned on then they commute any sigma 1 commutes with itself. So, then these things can be added up and this is this becomes true. So, now what it tells you is that you have to live with this object there is no simplification, but now can you make sense of this object is the question. First thing is this is, but the whatever you say you can see that this object here is, is object which have always comes with the sigma a. So, one thing you could say.

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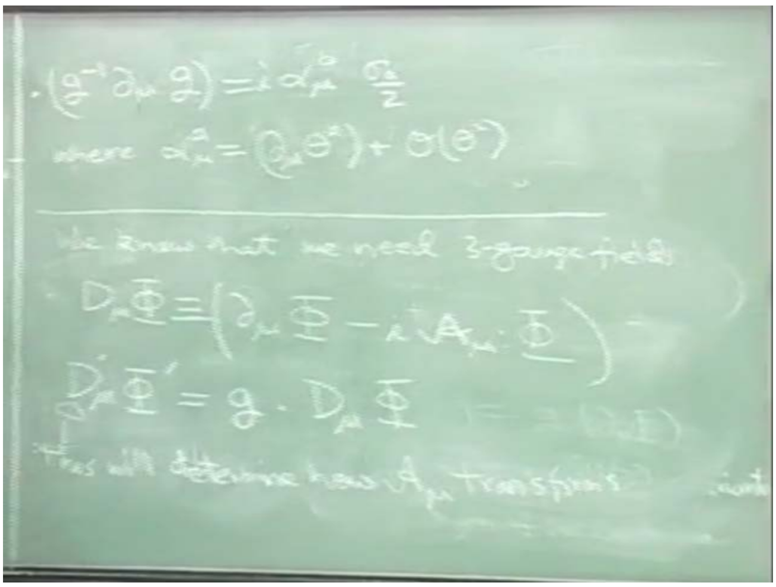

You can say is that g inverse d mu g will be some object which can be expanded as some let us call it alpha mu which is some complicated where plus theta square, because the problems came only at theta square, yes you can always see this you can always write it this way. So, the, the question is now how do we go ahead defining the gauge filed? So, what we do is so the, we know we know several things, we know that we know that we need 3 gauge fields. And the key is that what we have to do is to so what I will do here is try to write an object which I will write A mu I will explain what I mean by this which has the following property that it changes delta A mu rather no, no. Let us write it this way A mu prime is equal to I will write something and then we will sort it out, and there might be factors of i and I may also have to invert these things.

So, instead of writing inverse I should write dragger, because we are assuming that these are. Suppose we have something like we have a right hand transformation. And and we define D mu of phi, actually let me not even let work it out I think that is the best way let us write something like this D mu of I should be covariant derivative of phi and similar to what we did and I will write something like this A mu dot phi, this is just definition. Except unlike the u 1 case this will be a 3 by 3 matrix valued thing, 2 by 2 matrix valued object in fact, it will be a lie algebra valued object we will fix that. So, what we want and this should have the property and D mu of it should be such that D mu of phi prime. The prime here is a can also we will transform exactly like we did D mu of phi prime should be equal to g dot D mu of phi, this is our requirement this is our this will so this will fix or determine how A mu transforms. So, let us do that so we start with this guy and then try to equate it to so this.

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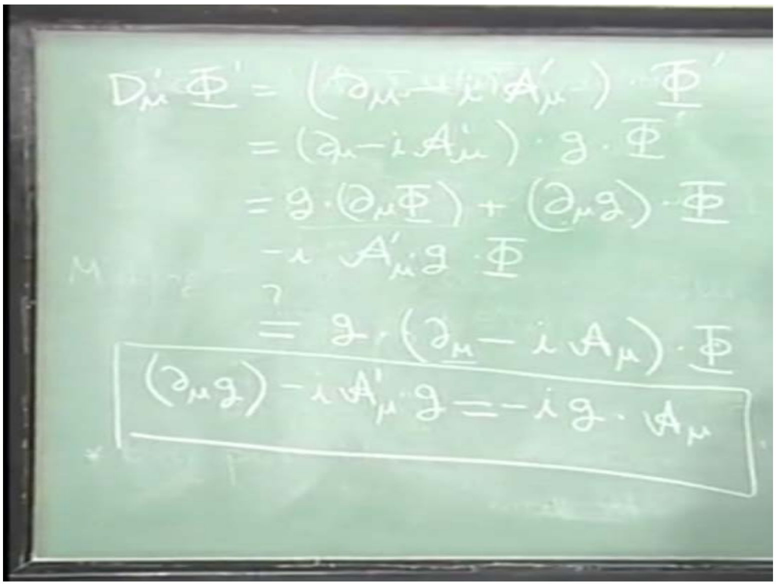

So, D mu prime of phi prime will be D mu and this will be equal to now there is D mu acting on this. So, let us write out first the good part which is g goes through g. So, this is the part you do the first, is it now thanks. And now this will be minus I, we do not know how this transforms let me just write that is all. And I am taking care that I should not there is no way I do not know if can take this through. Now the question is, is this equal to we need this equal to g dot, now we get the rules we can see now that so this term cancels with this, but all the others remain.

So, let me just write it out and get rid of the phi from the story that is what I need to do. So, let us look at this so we get so what we need is d mu of g minus i and minus g should be equal to, this is not written in a nice manner, but we will massage things around and get it to what we need. So, to do that what I will do is I will multiply from the right hand side by g dagger. So, that I get rid of this i i g dagger, so the i is goes away and then I will take this to the other side. So, I just I am rearranging things I do two things in one.

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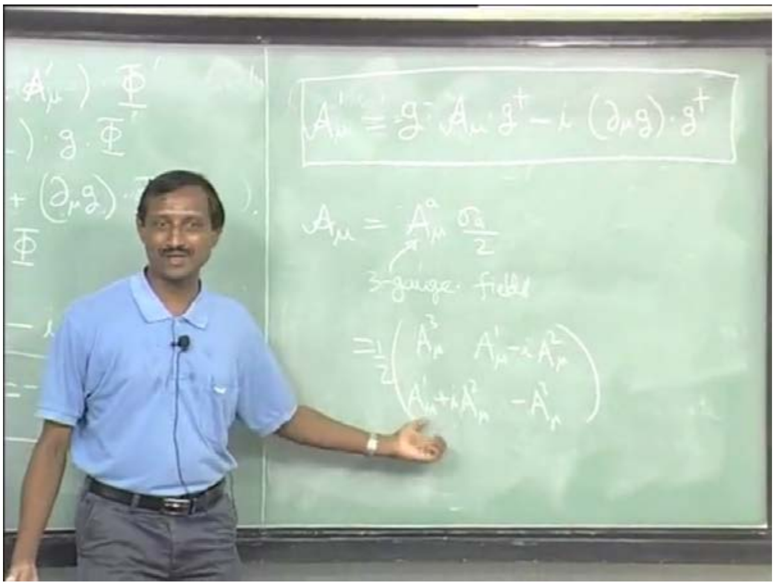

So, what we get from that is A mu prime will be equal to g A mu g dagger and this will come to the other side and I need it to multiply with an I, this is how it should transform. So, now let us check it this makes sense, first thing is suppose we go back to the u 1 example where g is just now e power something. Then g and g this will come here everything goes through everything is Abelian, so g goes through this a and g, g dagger is 1 everything commutes. So, this goes off and this becomes precisely I mean it, it becomes D mu of theta as we saw, but now you can say so this is a very natural generalization of this. Now this, but the thing here is we can simply things a bit more we can write, we can do 2 things. So, the thing here is let it remain this way. So, suppose we expanded A mu it is valued algebra.

So, I will write this as A mu a sigma a by 2, then this will you can rewrite so what you will see here is the, so this shows that you will have these are the 3 gauge fields. So, let us see how this matrix look likes, so let me write out. So let us you take the first the case of A mu 1. So the script A actually represents this, the 3 guys will come. And the simplest way to check is this correct is to do exactly you switch off if you switch off each one of these guys you switch off A 2, 2 and 3 and see what you get? It should reduce to the u 1 and you will see that all these things work perfectly. Now the only tricky part which I have not shown, but sort of you can see from even here you can see that. So, let us let me one part which I have not shown is that g inverse d mu g, here g dagger.

You may you also worried about why is it the D mu is acting on the first term and second term while here it is the other way. You can use the neat identities like this and take derivatives of this play with various things rearrange things you can. So, we will just write whatever is natural to our computations that is what I have wrote.

So, let us come back to this and see what is the, if I pulled out if I tried to pull out something like this, what remains on the other side. So, you see here I have to write a 1 here, but then I have to subtract a half. So, what I will get here is the commutator of sigma a and sigma b which is again an element of the lie algebra. So, all I am saying here that I am I can always write g inverse d mu g as an element of the lie algebra always. I am not proving it, but it is at least to second order I am proving it, but it it is true to all orders you can prove that.

So, in other words which is consistent with our intuitively observations that you require only 3 gauge fields, because if I to a arbitrary 2 by 2 matrix, then there will be 4 independent elements. But here that is not true and this is summation. Now one of your exercise will be starting from this, from this work out how these guys transformed that is one thing. And the second thing here is that I have no were assumed that g is small, this computation I have not assumed that g is small, if you assume g is small you should require this, again this is part of your assignment.

So, this is now we have a covariant derivative and which makes it transform properly and now it just takes it is a matter of going here and replacing the Lagrangian which is now not on this broad, replace all the derivatives by covariant derivatives, is that clear? And one more point I want to make here is these fields A mu are taken to be real vector fields not complex, just remember that they are real vector fields. So, when you are doing the star guy those things will work in a normal way. So, of course, and a natural thing to ask is what is the analog of the field strength? And we use the definition which we used it was the obstruction to commutativity. So, that is our next step, but right now I will stop here.