Classical Field Theory

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Lecture – 22

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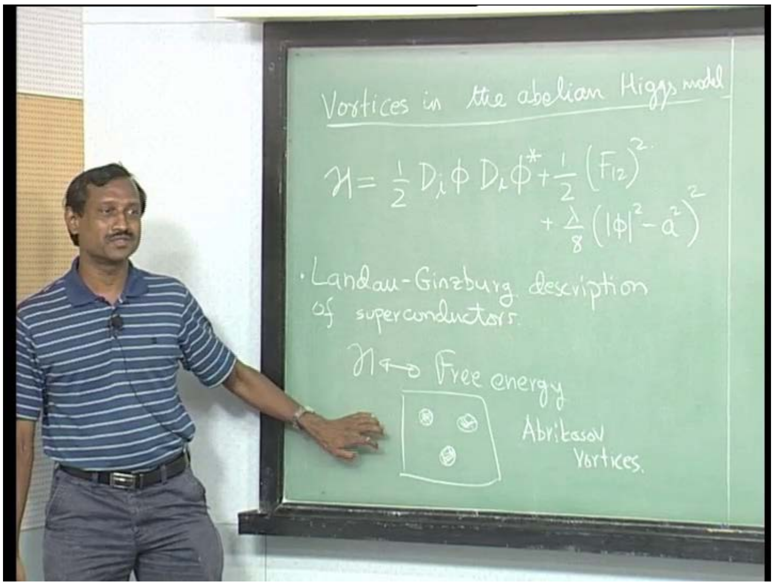

What we, I should give a little bit of historical inside into where these things came up. First, recall the Hamiltonian density. We were looking at time independent configurations in the gauge; I guess a 0 was equal to 0. So, we had something which looks like this. So, I just need to fix factors of e square instead of like that. So, it turns out that this similar thing appears even in super conductivity, in something called the landau Ginzburg description of super conductivity. Yes. So, the thing is that, now, super conductors are in three dimensions. While these, this problem we were looking at, it is in 2 plus 1 dimensions. So, the way you do, you sort of get to this thing is that you assume the things are independent of third direction. So, what we thing of if, if we think of a vertex as something like a point in a plane, then you can see that, if you extrapolate it in the third direction, it will look like a tube.

So, what we will call vertices in this context would end up being tubes, flux tubes. So, modulo that carve here, you will find that h gets replaced by the free energy. So, we will end up with similar setup out there. So, the question is, for instance, in type two super conductivity, you know that what happens is that, if you turn on magnetic field, it can, I mean, if it is completely, if you are below t c, it should be super conducting. So, there is obviously, no way for flux to go through the thing. But, really, I mean if it is, I mean it can go through, provided some regions become normal. So, a cross section would look something like this.

So, you would have flux tubes going out here and in this region, the material will be non normal. So, this is a slice in three dimensions. So, this would be like a tube, which is coming out and these solutions in this module, were actually constructed by Abrikasov. So, they are called Abrikasov vertices. In fact, beautiful; there are beautiful experiments, where they actually show how these vertices actually will form a lattice that could have interactions with that. So, depending on what kind of thing, I mean people have really, I mean experiments are really neat. I have not followed the literature, but, you can search for it and I am sure you will get beautiful pictures of vertices.

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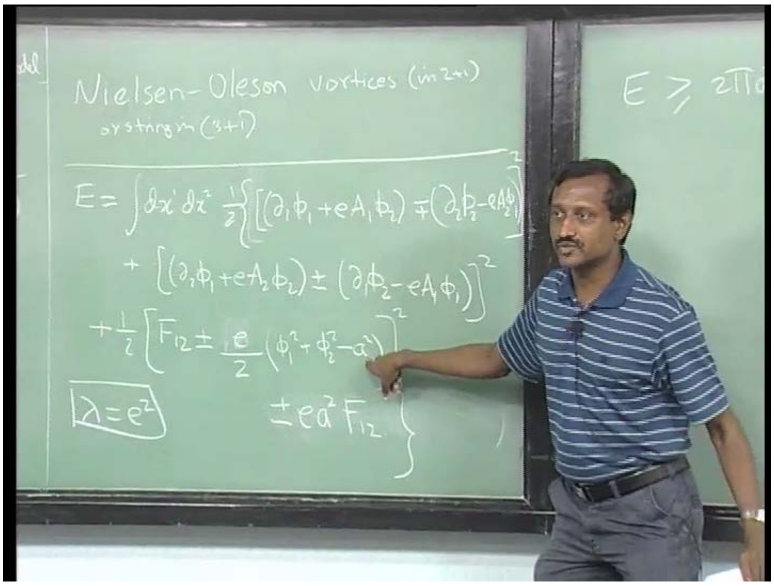

So, what, so, the extension of this, the realization that these two are the same sort of goes back to Nielsen Oleson embedded into a relativistic theory. So, you could even think of this coming from a 3 plus 1 dimensional theory, where instead of thinking of the flux tubes, we think of them as strings. So, they are sometimes called Nielsen Oleson vertices or strings in 2 plus 1 or strings in 3 plus 1 and the finite energy condition, of course, in the other direction will get replaced to finite energy per unit length. So, it becomes energy density in that extra dimension. So, these are the small changes that you have to do, but, from the view point of solving equations, they are really the same. So, let us get back to what we were doing. What we saw that, we could rewrite this thing as a bunch of terms and for now, what I will do is, I will not put lambda equal to any value. We will try to rewrite things like we did last time.

So, let me look at that expression. The reason is that, I claim that lambda should be equal to a square. But, I think it should be equal to e square. So, we will sort it out in a few more minutes. So, the energy, so, for the rest of this discussion, we will be back to 2 plus 1 dimensions, with the obvious this thing. I think there would be a nice description of this in the Nobel side as well, because he won the Nobel Prize early in the century. So, energy was; you need not note this because, you already have this in your note book. Probably, I need one more bracket. So, let us use this bracket for that. So, you can see that, that is just rewrite of this term. So, it is really, what is going on is that, we should combine these two guys and try to make them work as one. So, then we just write this as plus half F 1 2, pull out the half, so, it would be plus or minus root lambda by 2. So, if I square this, I will get F 1 2 half F 1 2 square with this term and if I square this, I will get lambda square by 4 into 2, that will give you this term.

But, there is the cross term which we have to write, so, that should be minus or plus. Actually, it is little bit more involved because, I am wrong in saying that these terms only come from here because, there will be, what happens is that, let me write this and then, you will understand what I am saying. Then, so, let me write this. That would be 2 times this. So, root lambda into F 1 2 into phi 1 square plus phi 2 square minus a square. Good. So, the point here is that, these two terms should actually be absorbed out here and now, this is where you have to, you can see that, you will need to, need this value to be something very special. So that, these two terms are already accounted for that here and as you can see, there is really no A going around in this place.

So, this, so, the error now is very clear. Lambda should be e square, not a square. So, I will just put that in. I will get root lambda out here and I will just erase this particular term from here and then, I get again plus or minus; minus sign, I absorbed here, so, I get plus or minus e a square F 1 2.

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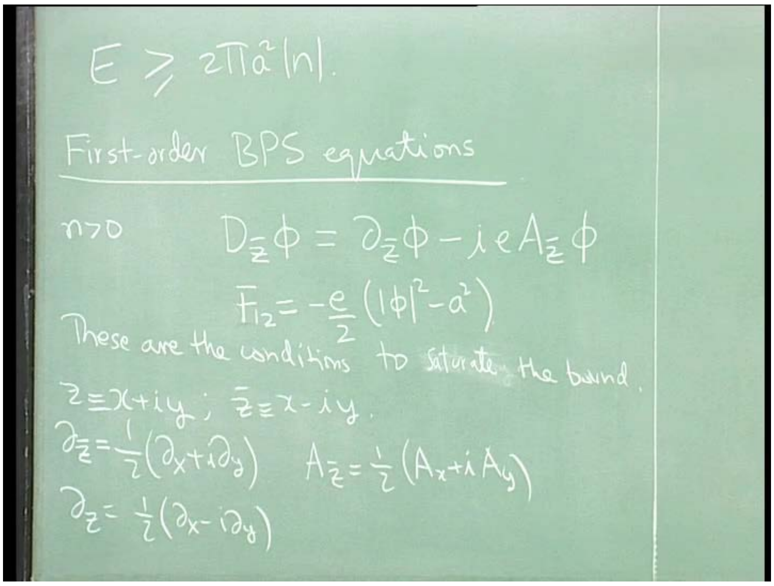

The energy, which we worked out should be greater than or equal to 2 pi a square mod n. So, there was a divided by e, those things cancel out. Good. So now, we have to get back to the equations. The equations correspond to setting; these guys in the square bracket equal to 0. So, the first order BPS equations, first order BPS equations, so, we will consider the case when n is greater than 0. Then, I have done some redefinitions and the other one would be, so, these are the two the equations that you would get.

So, these two plus these three, the square, setting them equal to 0, that will let you saturate the bound. So, these are the conditions, or saturate rather, everything will satisfy the bound. So, I need to put in my definition z is just x 1 or x plus i y, if you call x 1 as x and x 2 and I need a little bit more, d z bar should be equal to, there is a half I think, the weight of fix, this half as the d z bar of z bar should be 1. Similarly, a z bar.

I always like to work with a lower case, lower index covariant because, that usually, it goes with the derivative and anything you may, any change or any change of variables that you do, you do the same thing for this. Then, your formula will look kind of nice. That is a natural thing to do. So, we need to solve these guys and what are the boundary conditions of course.

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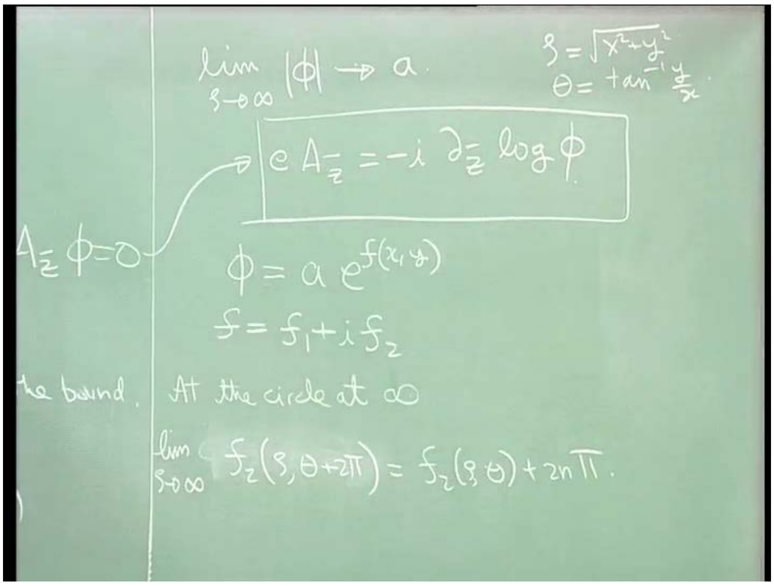

So, limit as, let us define rho tends to infinity of phi, mod phi, should go to a. It should go to its web, so that does not fix this thing. So, this is what we need to do. So now, coming back to this, we can go ahead and see we can simplify things. First thing we can see here is that, this equation along with its complex conjugate equation, lets you solve for a z, a in terms of phi. This should be equal to 0, I forgot to write that. So, for instance, this implies e a z bar, I just written 1 by phi, this thing as log of phi. This is what you get. So, we can also, keeping in mind that this is what happens, we can put in an answer, that is what a as some phi, a e power f of x or y, whatever it does not matter f of x and y, where f is some complex number.

So, we can write f of x y. I am just getting around to simplifying this equations and the other thing we know is that, if n is greater than 0, one thing we know is that, as you go around in the circle at infinity, so, one more thing we know is that, at the circle, at infinity, when phi of rho is just plain polar coordinates as just and the theta as y inverse. So, phi of rho. So, one thing we saw is that, the flux was nothing but, how the phase of phi changed. So, that should, so, we can go back and write in terms of the phase of phi, which is f 2. So, this is the definition that it carries that much because, that was the flux, right. So, this is what. So, I have just translated that because, f 2 is really the phase of phi and this is what I get. This is the, this is also another condition that we have to solve. It is not enough to just say this. So, we just go ahead and so, we have this, so, we can work out what.

So, this equation becomes another, so, this equation will, so, this corresponds to, well let me repeat myself. a, we have written in terms of phi. So now, it is this equation, which will give the equation of motion of which we need to solve. So, let us, simple algebra, which I will skip.

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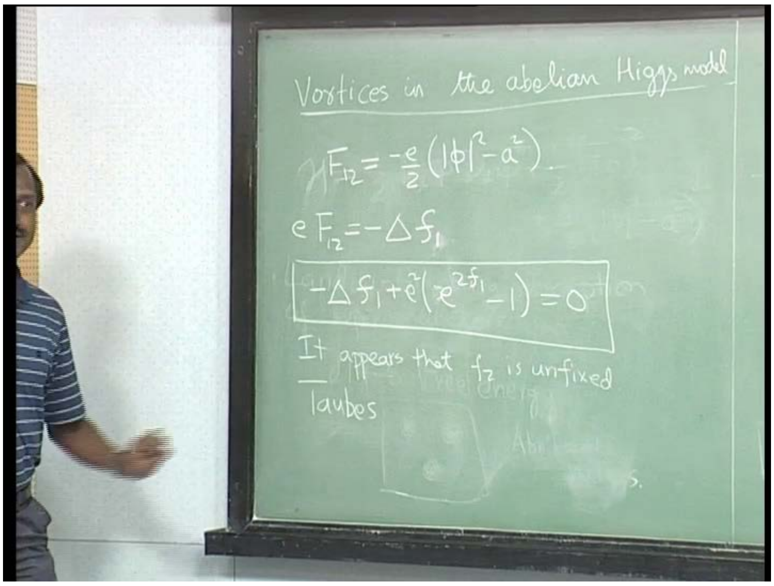

So, you can show that F 1 2 with that thing is equal to; that is just a repeat of that equation. I am terribly sorry. So, but now, what do you have to do is, write out what F 1 2 is in terms of, so, you need to take a derivative of this expression. So, you can see this already has one derivative. So, you get, for this you have to take d z of this thing. So, you end up in seeing that you are getting the Laplacian active on log phi. So, what we get is, the equation that we get is the following. So, if you compute using that, you can show that e F 1 2 is minus the Laplacian of F 2, F 2 or F 1. So, if you plug these things in here, what you can see is that, the equation of motion becomes minus. So, F 1 to (()), this is what you get.

So, this is the equation that we need to solve. But, surprisingly what we see is that, there is no F 2 in the problem. It looks like F 2 is unfixed. The question is, is that a problem? Should we expect intuitively whether S 2 should be fixed or not? So, the equations are, we solve the equations now. So, it says that, this gives us F 1, but, it does not give you F 2. But, we do have some condition on that. This thing on F 2, which is that, but, that is an asymptotic condition and while this as I said, we have to solve this equation. Any ideas?

Exactly. So, you should not expect F 2 to be fixed, because you can always make a gauge transformation and make F 2 look like something else. So, what we did was to fix the gauge freedom by choosing a particular, this thing. So, it is really not, it should be expected. So, it is. So, really what we are getting is F 1. So, we need to go ahead and solve these equations. But, we will not go ahead and solve these equations. I mean, there is a nice the work by Taubes, where he discusses the solutions to these equations etcetera.

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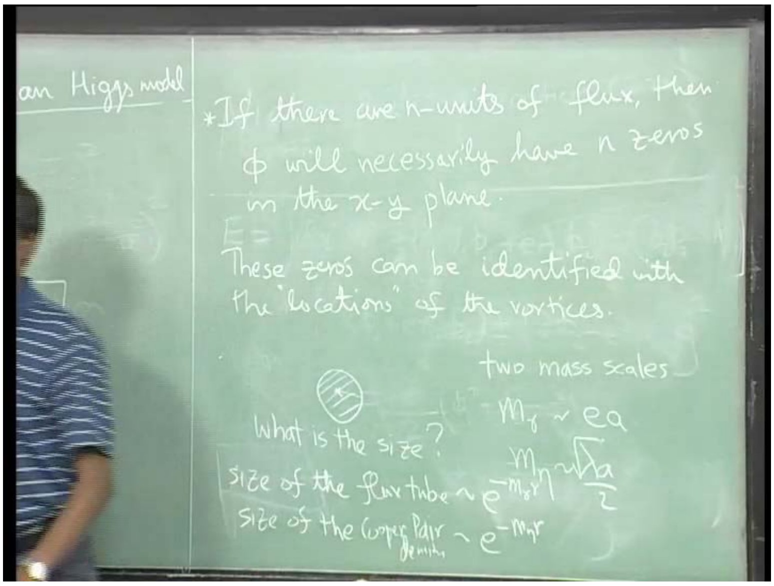

But, what one can show is, by simple sort of mathematical arguments is to show that, if, I will not, I will only state these things, if the vertex number, if the flux is, if there are n units of flux, then continuity conditions, then phi will necessarily have n 0s. The 0s may not be distinct. They can actually be co incident on top of each other etcetera and roughly what it says, sp, intuitively what it tells you is that, what is happening when phi at a point where phi vanishes. Your full symmetry restoration or in the, if you go back to the super conductivity story, where phi is the density of, mod phi gives you the density of the cupper pads, it tells you that there are no cupper pads at phi equal to 0 and phi vanishes. So, that is roughly where you would expect a normal state, when there are no cupper pads.

So, these 0s can be identified with the locations, if you wish, of the vertices. In fact, you can also show, that is again is a mathematical statement, you can ask how many, what is the moduli space of this solution. In other words can, I deform what are the number of parameters. Remember, when we went to the kin's solution, we got something. We found that the location of the kin's was a free parameter. We also saw the velocity was; so these are. So, you would say, so, the location of the kin's would be one parameter. So, you can show that, if you have n units of flux, the number of parameters that, your free parameters that you will have, you can make a count of it. It is called an index theorem computation. It will be 2 n, which is exactly in synchronization with this idea that you can identify it, like just tells you it is the freedom. So, if I have a solution, when I have two vertices say, located out here. I can keep; I can move these things around, such that. I still have the same solution.

So, this is completely in synchronization with that, that in that picture that one has. So, but now, the question is, what is the size of a vertex. But, these vertices; obviously, there it will be 0, but, it would not. So, what you will find is that, you will, so, let us say that this is a 0. You will find that it has a particular size and outside which to exponential accuracy at pi equal to a, but, inside this location, in this region, it would be deviating from a.

So, what is the size and this actually, by the way, you can pose this question not necessarily for lambda equal to e square, you can pose it for any arbitrary value of lambda. The thing is that, but, there are, what is the size? But, we can now go back again to length scales in the problem. There are two length scales in the problem. Let us go back to the Higgs mechanism. There were two mass scales, if you wish. One was n gamma, that was the photon mass and the other was, the eta. That is the radial direction, his thing. So, this was determined by the second derivative of the potential at this thing. So, this would be, this is, let me just look up my numbers here, m, eta would be, was lambda a by 2. Factors are important, but, not really. While the other one will be, was e times a, There is a dimensional problem. There is a square root. Now we are fine.

So, we have two length scales in the problem and we can ask what these two would correspond to. But now, I mean, but again, you can now, again there is no need. We can look at the details of the solution, but, we can see that the photon becoming massive. So, that will have something to do with the magnetic field. So, that is, so, you can ask there are two sizes in the problem. The first size is, what is the size of the flux tube and that would be determined completely by this thing m gamma.

In fact, you will find that the size of the flux tube will go, if you look far enough, it will go like e power minus n gamma r. In natural units, remember m gamma is 1 meter. While you can ask, also the other thing is, if you are looking in terms of cooper pairs or whatever, you can ask how far, where is the density of cooper pairs really and what is the size scale associated with the cooper pair density. That is the eta t. So, size and that will go like e power minus m eta r. This equality is roughly, when these two length scales become the same. Almost approximately you can see a modulo. This factor of 2, it looks like lambda equal to e square.

So, there are two length scales in this problem. In fact, if you go ahead and solve the other equation, which is the or normal Euler Lagrange equation, you will find that you will get naturally, I mean, you can linearize your equations. They are non-linear equations; horrible to solve, but, you can linearize and then, you will get Bessel functions, the k functions with arguments, which are exactly these. Initially, I thought I would work out this thing in the class, but, I think it is very tedious. But, I will show you the idea of linearization of equations because, that is a very important trick. So, we will, so, in your assignment, you will work out the full Euler Lagrange equation and I will show you, I will write out that equation and then, I will show you how to simplify it.

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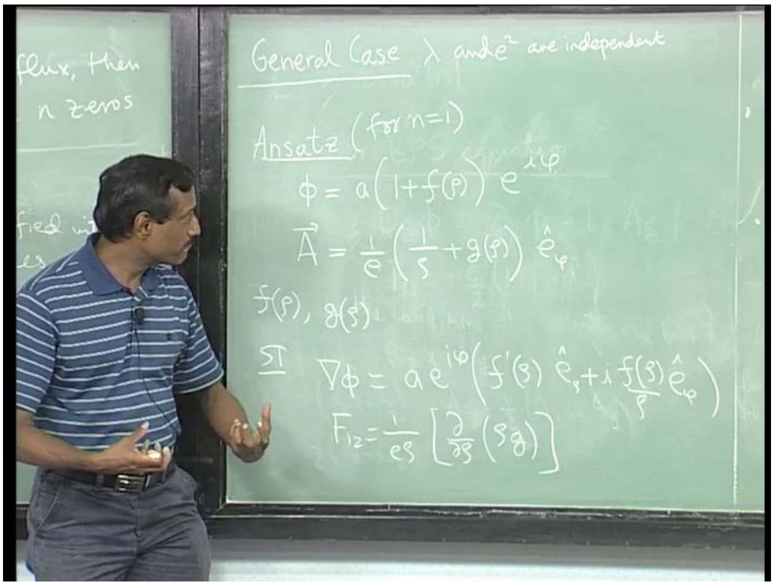

So now, we will, we are going to a general case. So, lambda and e square are independent. So, we need put in some ansatz etcetera. I will do that in a moment. So, let us choose an ansatz. I will choose something for n equal to 1. So, I am doing this for a single vertex. You can do it for more vertices. So, you can see that, but, these ansatz, I have taken care of the factor winding number is 1. This is the simplest one you could write. If you wanted winding number 2, you could have put a 2 out here and then, I am assuming that it is, I assume a radial ansatz, symmetric ansatz. So, I put f o rho. That is this is function and of course, f of rho should vanish as rho goes to infinity, so that, it becomes, phi becomes a mod phi becomes a.

Similarly, the vector field a. Assume that it as a only a e phi component and again, you will see that this is consistent with the flux, be there is a suitable flux. This will give you what you need. So, really what you have is f 2 functions, two unknown functions f of rho, g of rho. Unlike the other case, we had one really, one function phi because, everything was determined in the (( )). Now, we no longer have the freedom; f and g are independent of each other. So, it is, I mean slightly painful exercise to go through, the fact that you are in plain polar coordinates and gradients stuff like that. We will require, I mean, actually I need to taught the formulae correctly and worked it out slowly. So, for instance, you can show that gradient of phi. So, you get this piece because, your polar coordinates and F 1 2.

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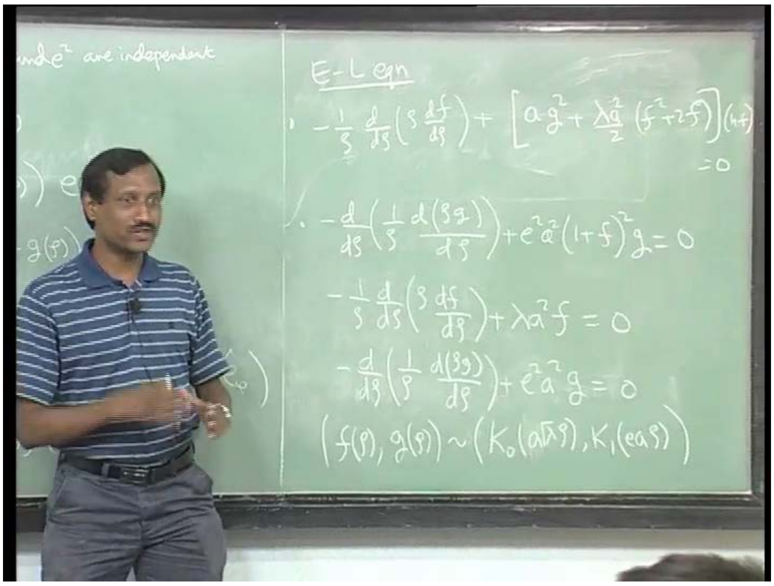

So now, I will write out the Euler langrage equations and it will be horrible; non-linear, f and g everything will mix with each other. This Euler langrage equation for phi, you can see that it appears with two derivatives acting on f. But now, the interactions take over. So, you have a. See, you see you got a non-linear piece, g square. So, let us look at the non-linear terms. We have, this is non-linear, in fact, this, this, everything here. It is little complicated because, there is an f out here. So, f with anything will be gone. So, we will just get rid of that. So, I will not underline things because, it is too complicated. This is one equation. The other equation.

So, of course, one of the exercises you have to do is to derive this. But, another exercise is to ask how in the BPS limit, how is it that, how do f and g get related number one and the other question is to ask, whether given a solution of the BPS equation does it solve this equation. You can ask these two questions and it should work and it will. So, these are, this is non-linear. So, the way you linearize it is just basically throw away all the terms, which involve, which are non-linear.

So, let us look at this. This is g square. Obviously non-linear. Throw it away. f square, this is also non-linear. 2 f, but, 2 f into 1 is it ok. It is a linear piece, but, 2 f into f is again f square. So, that term is gone. So now, we look at this. So, there is only one term, which is plus lambda a square and the next equation is, again look here. This is g. This is linear. There is nothing in this part. But, out here, you can see, there is g out here. So, this f is a non-linear term. Throw it out. So, you get e square a square g. So now, let us look at this in the linearize limit and we can automatically see what is going on out there. We see both these things appeared. These combinations appear. So, this looks, if you see here, we get this is m eta square and this is n gamma square and what is happening out here? f is related with the phi part and g is related to this. So, it fits in beautifully with what we except and in fact, these two equations are something which you can solve and you can solve for it easily. f of rho and g of rho go as. So, both of them are Bessel functions and the asymptotic should be such that f n g should vanish. So, it is not the j. j a for large values of argument, it oscillates. While what you want is exponentially dying guys, so that, so, the boundary conditions are imposed, are taken into account by which Bessel function you choose. So, f 0 goes like k of; I will just write it out.

Now, you can see that these two length scales are equal, when lambda equal to e square. Just two length scales become equal because k ‘s, this one the 0 and 1 does not matter. The exponential part is still the same. The change, pre-factors, 0’s etcetera change. So, you can see that this is really when these two length scales happen. Now, we also have to check, what is the region of validity of these things and so, the region of validity of this, of the linear thing, would be when f is closer to, f and g are close to 0. So now, it is sort of like chicken and egg. You put it back and you say that this solution is valid at some distance away. Is that clear? But, you can also go and ask, can I solve for the solution near a 0? Now, that would be a completely different problem and then, you have to take care of the full non-linear. But, this is enough for you to tell you how the asymptotic behavior is.

Another exercise would be to do it for not, for n, what is this etcetera. If you want, please go ahead and do these things. So, are there any questions? So, I have got 14 minutes. So, in this 14 minutes, I wants to give you some ideas from topology, which actually are implicit in this thing and we will look at it from the view point of topology and we will try to understand the vertex number. So, we got a typological current, but, there is something more to the story.

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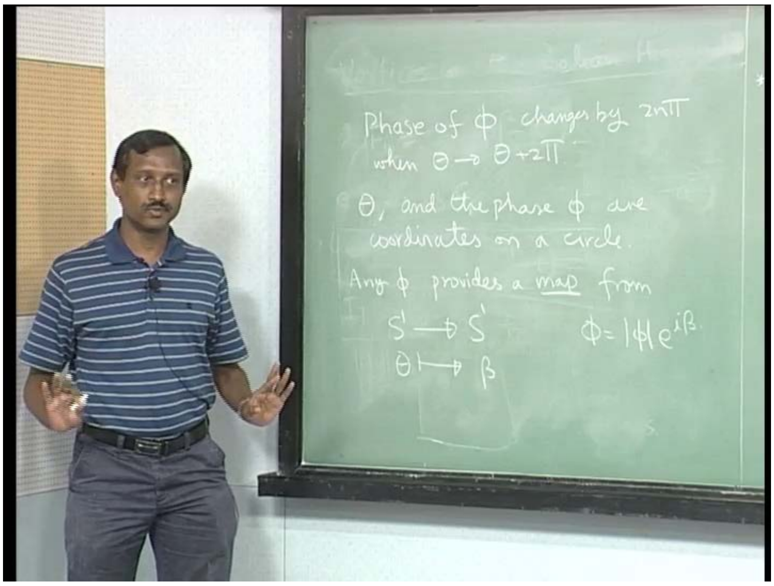

So, what, so, what did we see, we say that the phase of phi changes by 2 n phi, when theta goes from theta plus 2 phi. So, the thing is, theta and the phase of phi are circle valued objects. In the sense that, they are periodic or coordinates on a circle. That is much more obvious. So, if you give me any phi, I can look at its phase at infinity and ask and I can get a map. So, any phi provides a map from S 1, from one circle to another circle. The first circle is theta and which gets mapped to the phase of this thing. So, let us use, last time what did we decide we will use for the phase of phi? Beta. So, this time, where we write phi as some, so, you can ask; so, suppose we are interested in maps, which are continuous, smooth maps. That is all.

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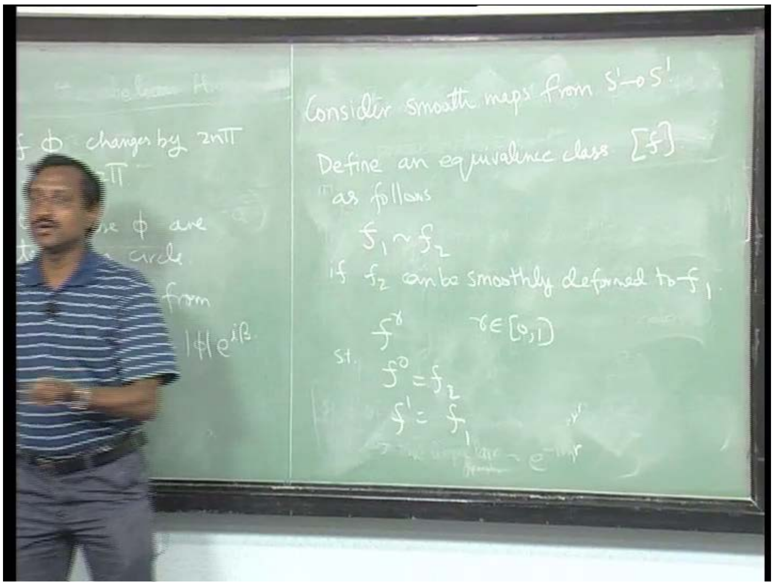

We look at, so, consider all smooth maps. Now, smooth here means just continuous. Now, even derivatives may not be there. It is just a smooth maps. So, let us call such a map, we will call it f, where f is nothing but, f of theta. So, we write this as beta equal to f of theta, smooth maps from and the question is, is it possible to smoothly deform this. No, this is not the correct way to state it. So, we will define, so, define an equivalence class as follows.

So, suppose you are given two maps f 1 and f 2. We will say that, they are equal; they are related to each other, if f 2 can be smoothly deformed to f 1. You can check that there is an equivalence relation. Now, the question is, what do we mean by smooth deformation? This is what we need to make it more concrete and mathematically that is the way of doing this. So, what you do is, roughly you look at, you consider some, so, the way to do this is to, suppose we have, we define something called as bunch of maps f of gamma, where gamma takes values, gamma belongs to 0 and 1, the interval, such that, f of 0 is equal to f 1 and f of, no, f of 0 should be f 2 and f of 1 should be. Each of this, for every value of gamma, so, in other words you can, this is what you mean, mathematically, precisely of making these two equal to each other.

So now, the question which Poincare asked was, can we actually, if you are looking at this simple example, can we actually specify what are all, I mean, if you give me two maps, how do I know if this is true? If you say, what classifies this thing? So, in this case, the answer is very simple. Suppose you have map with, so there, so, this case is just the winding number. So, if, as theta goes from theta plus 2 pi, if it winds twice and some other map, which is bound once, they will not, they can never be smoothly mapped to each other. So, in this example, the answer is straight forward. Proving it is not, but, the answer is straight forward.

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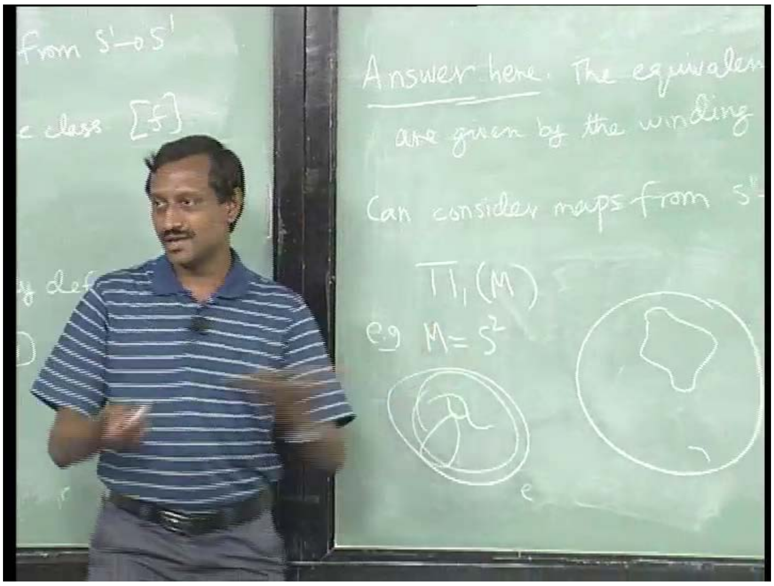

So, the answer for this, answer is that, two maps, so, this is, so, this condition is, they are called homotopically equivalent. That is a mathematical term. But, from our view point, it just means that they are mapped to each other. So, the answer here we already know this, is that two maps, rather the equivalence classes are given by winding number. I really mean, how many times beta winds the around when theta goes from 0 to 2 phi. The reason to use this winding number thing is as follows. Any circle, you can think, consider the following thing. Suppose we have a rope. We have this thing and we take a rope and we wind it around certain number of times with orientation of course. Negative means, opposite clock wise; anti clock wise, looking upwards, if you wish. So, if you, if something turns around twice and you are not and then, join the ends together, there is no way on earth you could change it from two windings to one, for instance. Unless, you can, sort of go out in the third dimension, which is not permitted in and un rapid. That would, that is not a smooth thing. So, really that is what is going on out here.

So, but here, actually in general, you can consider smooth maps from S 1 to any space. Not necessarily S 1. You can consider to any space, any manifold to be magnified. So, if you define the set of equivalence classes, you call them as pi 1 of m. These are looking at the set of equivalence classes from S 1 to any space. So, nice example.

Second example, would be take m to be S 2. That is a sphere. So, you are asking. So, you draw a sphere. Now, the question what I want to ask you is, how many can you think of, what would be the kind of equivalence classes you could have? Do we have any non trivial winding which is possible? No, because anything can be shrunk. So, this whole picture here, of which I drew in this formal thing can be also understood. I can keep shrinking these things, I can actually make it completely; the only possibility is 0 winding. Nothing. So, pi, so, that is like saying pi 1 of S 2 is trivial. There is only one kind of map. All maps are the same. All smooth maps of course, acts the same in this sense of, same list given by this equivalence class. So, in some ways, you can ask the questions about any manifold.

So, this is actually kind of nice. Suppose you are a kind of blind you know. If you have really thick glasses, you know and you cannot see properly, but, you can actually quite a, I mean, so, you want to distinguish between two things. Whether one is a ball and then, there is another thing we could look at, like a torus, this has a handle. You can ask, what about here? In fact, I can do, I can wind around something called an a cycle n times. I can also wind around another cycle 2. So, the blind guy can actually feel his way and see that there is, you know, by just using this S 1 as a probe, can see that, can differentiate between a two sphere and this sphere.

In fact, so, in three dimensions, if m was a 3 dimensional space, then pi 1 the only manifold which has pi 1 of m, which is trivial is the three sphere and this was the Poincare conjecture, which took many many years to prove. Almost 100 years. So, in three, so, here this m here could be any dimensional manifold. So, you take m to be a, I guess there are some connected whatever, three dimensional manifold and if you are told that, the pi 1 of that space is trivial. Actually you can extend this to any essence, pi 1 of anything will be in S 3 also. If for the same reason that you can shrink it, it is always trivial. Now, the question is, are there any other three dimensional manifolds, for which topologically speaking, they have the same, which has pi 1 which is trivial and Poincare conjecture is that, it was not true. But, all these things should be equivalent to S 3. That is one, I mean, that is one part of the conjecture. So, in some ways, these are the, they are called topological because, they do not too much structured about the manifold, about the space, but, they tell you a lot about this thing.

Obviously, you can ask, can we generalize this further? Why should I look at pi 1? Why not look at S 2? Take sphere. So, pi 2 of s. So, you can define pi 2 of m. So, again you look at maps. From two sphere, to any space and with similar conditions. Again, that is a nice thing. Pi 2 of S 2 is z again and it is, is that a easy way to see it? The answer is yes. You can reduce it to how, you go to polar coordinates, spherical polar coordinates and you can write, trivially write maps, where the pi, the polar way angle, winds around n times. So, if you want a winding number 2 map of S sphere to itself, the way to do is that, to think of a globe kind of thing. When you cut open a slit like this and you kind of drag it round once and can do this back. That is an example of a map, which has this thing and you can see that, that is similar to the winding number for this thing.

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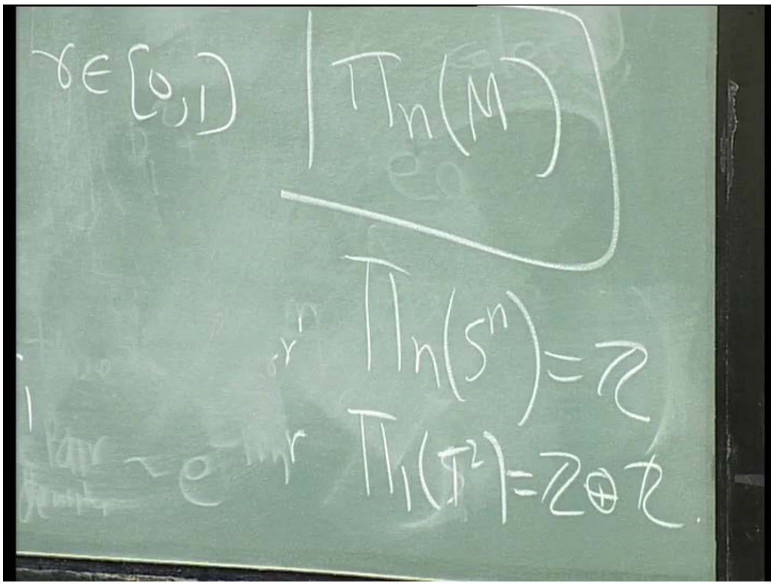

But in general, you will, so you actually study pi n of M. So, these are maps from S n to any. So, there is always one easy thing to remember pi n of S n is always z. Some kind of winding number. Pi 1 of torus is 2 z’s. There is some group structure associated with it. I am not getting into that. So, this charge, in fact has this whole topological structure associated with it. In fact, all the charges have some kind of topological charges. Usually have some mathematical under pinning and I just wanted to show you one example out here. So, what I will do next is, in your next assignment, you will have to work out the details of this particular solution. Work out the Euler Lagrange equation and verify for yourself that the BPS, the first order equations is indeed a solution of the second order equation. Fill in all that details.

So now, we will move on to non abelian gauge theories and next and we will also see how, before that, we will, let us, we will spend some time looking at how symmetry breaking, how to look at symmetry breaking for non abelian symmetries. Continuous symmetries we already know little bit of linear algebra now. So, we know how to, we know how to cleverly parameterize things; which means, we can, we can do the analog of the Higgs mechanism. We will see some interesting examples.