Classical Field Theory
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Lecture - 21

Conversion of u, it is a theory which is local which has local u and gauge invariants the example which we looked at. And we will see that this, this model actually exhibits finite energy solutions time independent, finite energy solutions; obviously, it gets around Dirac's theorem. And the simple answer to that is, is that is that this there is more than scalar fields in the theory there is a vector field, but there is more.

So, but I will not for instance go back and redo the scaling exercise. You can go back and do the scaling exercise and see that there would not be an issue. But we will need I think a couple of lectures to setup the thing. And the beginning part of this lecture will be to sort of remind you of things, because there is a tendency to forget details and...

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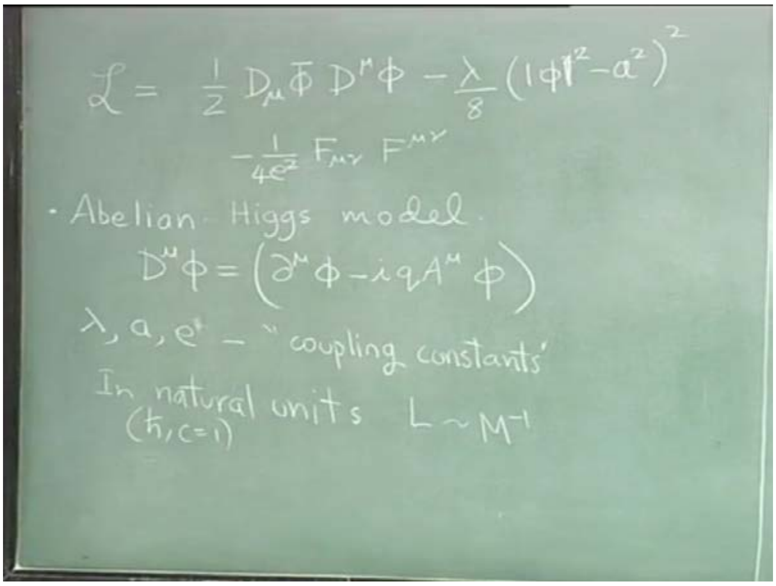

So, the Abelian Higgs model or whatever the way we did was had a Lagrangian density. And we will consider a case whether it is spontaneous symmetry breaking in the global model. So, we will look for something where the value of the phi vacua the, of this form. And I think I used 2 or 4 here I will put 4 factorial here to match some other thing or the factors are important and...

So, this would be the missing and then minus 1 by 4 e square F mu nu F mu nu. So, this model is sometimes called the Abelian Abelian, because u 1 is the, it is an Abelian group, so Abelian Higgs model, this is the name for this. And just to set notation or remind you of the notation there was a minus sign the way I did it and q was the charge and there was one more coupling constant. I want to do a little bit of dimensional analysis before proceeding we will. So, we have several coupling constants or whatever so para ampere is so we have lambda we have a and e. So, these are what we will call coupling constants, and we will try to determine their dimensions in e, no need to write e square. We will determine their dimension in natural units. And we if you recall in, in natural units, length is like mass inverse. So, both h bar and c are set equals to 1.

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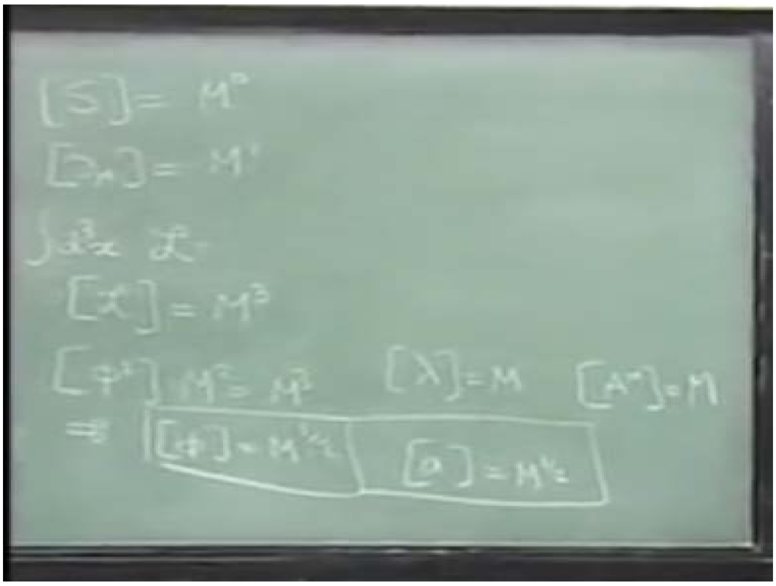

So, usually one says that action has dimensions of h bar, but if h bar is set to 1 you could say that in natural units action is dimensional less. So, this is something which we will assume and we will start assigning we will assign scaling dimensions to everything, all the all the various objects such that dimensions to each of these things such that this is true. So, you just start usually with. So, let us lo at this D mu would be like length inverse L power 0 which is 1. So, this would be like L inverse.

So, let us look at a term like this and for today, we will restrict things to 2 plus 1 dimensions, because I want to get the answers for 2 plus 1 dimension. But making this arbitrary d is kind of trivial and I leave it to you. So, I write d cubed x for d square of space and time put together and into L, so just combining this with this will tell you that what we need for L. So, this is l cubed it should be l power minus 3, maybe I should put it as M so that things do not we will do mass dimensions.

So, this is L cube so this should be M cube. So, look, look at this and say what can. So, we get so phi square phi mod I mean phi and bar phi bar will have the same dimensions. So, I am just putting them together times D mu square which should be a mass square should be equal to M cubed. So, this implies that the dimension of this should be M power half. So, this is like your standard thing first you take your field and if you look at it is kinetic energy term and you fix that, that will give you the dimension of the field and now after that you work through and get the...

So, now let us go to the potential term you see here phi phi square and a square will have the dimension so a also has mass dimension half, a has same as this thing. Now, we need to determine lambda as well, so this is now this is a square as m, so this is M square so this should be M. Now D mu and A mu have to have the same thing. So, A mu you can see as dimensions of mass this field. And you can see it is dimension independent this statement, because it always comes like this. So, the gauge field, but pretty soon I will change the definition and then it would not be true now we have done. So, we have we have got gauge fields So, we know this thing so this is D mu A mu So, this is like M square M square so this is M power 4.

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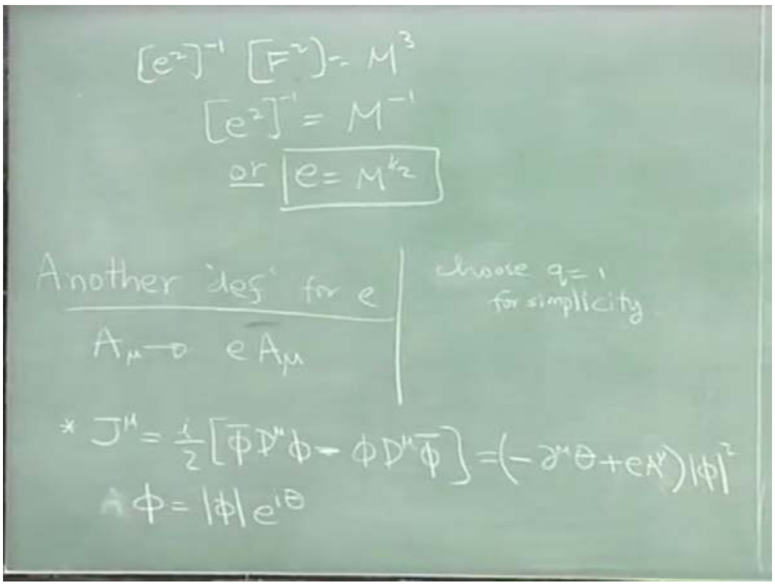

So, what we get is e square power minus 1 times F square should be M cubed , but F is D mu A mu, D mu is an M, A mu is an M. So, it is M square so this becomes M power 4. So, so the coupling constant e will be dimension full, it will have mass dimension half, but if you had done this in 3 plus 1 dimension, what should have got is that the same and I think I would have told you l should be M power 4. So, in 3 plus 1 dimensions, this is our physical dimensions, the coupling constant, the electrical electric charge kind of thing coupling constant is dimensional less that is special to 3 plus 1 dimensions.

So, in that sense 2 plus 1 is better it gives you something quite different. Now, now I am going to change things a bit that is because it is not always conventional to put an e square out here, there are situations where this is true. But what you would like is to have your kinetic energy code and code for the, for the gauge field to have standard normalization and that is done by just. So, another definition for, for e which as follows you just take where ever you see A mu in the previous expressions you replace it with e A mu.

So, what will happen is, so there you get 1 factor of e coming here 1 factor of e coming here, so this becomes standard minus one fourth F square. But so one you are saying it is that you write this as e A tilde and then gets aid of the tilde, but I am not going to I am going to jump that step and just right it as A mu goes to A mu and you come back here this is normal. But what changes is now this A mu becomes, so you get e is out here.

This only happens, so when when you see an e here then there will be no e in the kinetic energy. You can see if you did not put a kinetic energy term for this, there is nothing to compare there is no coupling constant, there is nothing you could have always absorbed this into the definition of e. So, this is more conventional in perturbative quantum field theory where you put these things in here. But there is something call large n stuff which you do in the young means, there you would going to you, you kind of do this.

We will see there is more structure in that case, here it is kind of just anything is it looks like. So, this is so books will have both this convention or this convention, so it depends on the application. And for the rest of this lecture I will follow this, this lecture and maybe the next lecture I will use this notation, so in context it will be very obvious, what I am doing. So, are there any more dimensions to fix? So, we have fixed all of them and so these two are M power half and this is M. So, now the, what we would like to do is to.

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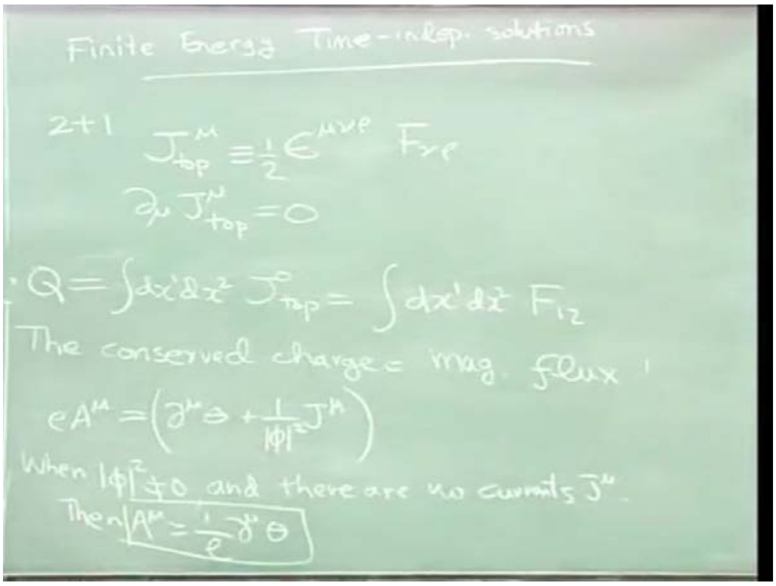

We would like to lo for finite energy solutions, finite energy time independent solutions I need one more bit of this thing. We had discussed also we saw, we worked out these couplings we expanded it out and we saw you know that the, the linear coupling with respect to a was the Noether current for the global symmetry. But then I also pointed out if, if we in the presence of the gauge field the conversed current is where you replace the derivatives by the covariant derivative kind of.

So, you take so the conversed current is something like this i by 2. Now, this has A mu stuck in here and you can sort of play, play with this thing around and you can write out A mu so let me expand this. So, let us first do phi we can write this as mod phi times e power i theta, so mod phi and theta are some fields. So, this will give you one term which will looks like this. So, let me focus on the derivative ordinary derivative part. So, that will only the only which will contribute is when it acts on theta.

The other one will cancel out you can check that. So, you get minus D mu of theta and A mu term will give you that will here you will get a minus i into I, you get a plus A mu mod phi square oh I forgot a mod phi square here also I can write in this, did I eat up some things, yes I ate up A Q and an e. And just to make things less, less messy I will just say Q equal to 1 I will just choose the field to have charge 1 so that we have only 1 e. So, choose it as a nod set. So, this, this formula I just get back to this. So, before we get to the finite energy time independent solutions, since we are in 2 plus 1 dimensions there is a there is a nice converse current which always exist when there is a vector field.

So, you consider some J topological, something like this. Now, you can see that if I where epsilon mu nu rho is symbol in 2 plus 1 dimensions. So, doing at this thing is trivial easy or because this will become the, what is called the banker identity? It is a trivial identity, because if can see that why, because epsilon mu nu rho D mu. And you can see that this can be written D mu 2 times D mu a rho using the dummy indices. So, this is symmetric and this is anti symmetric so trivially true. So, this is not there is no symmetry, continuous symmetry associated with this, but it is because and we did not even need to refer to any action. So, we can ask but nevertheless for a conversed charge this is all you need, it does not say that you would has to come from, necessarily come from a symmetry you can ask, what is the conversed charge associated with this object?

So, let us look at J 0 of topological and we have to integrate this over the spacial coordinates. So, the charge conserved, conversed charge would be integral d x 1 d x 2 times J 0 of topological , but what is J 0 of topological let me put a half here. So, you can see if you put a 0 here this can be 1 or 2 or 2 or 1. So, this just gives you and I put the half here, because I wanted to get rid of by a factor of 2, what is this object? What is F 1 2? It is a magnetic field, since we are in 2 plus 1 dimension, the magnetic field has only 1 component which is F 1 2, electric field has 2 components. So, again you can see that there is something special happening in every dimension.

In 3 plus 1 dimensions both electric field and magnetic field both has 3 components and so there is this nice electric magnetic duality which exchanges these guys that is not there quite here, because 1 has 2 components, the other has even counting wise it does not work. So, this is the magnetic field. So, what would you call this object, we are integrating magnetic field over some area, what is the thing called flux. So, the conserved charge equal to magnetic flux. So, this is this is always true in this model that there exists a conversed charge, and for instance we can go back to the vacuum solution that we looked at in this theory. So, a vacuum solution which was the lowest energy solution which had zero energy was to set you know for instance phi equal to a, mod phi equal to a now actually phi equal to A i fix the constant equal to a.

So, that minimize this thing and so such a solution clearly has, what is the flux for that solution; 0. In a short well we will see that there is a nice argument which will show that this flux has to be quantized, and the quantization is kind of a kind of need and it comes from this observation out here so. So, first step is to look at this as so I can use this equation and try to write A mu in terms of, in terms of J mu and this derivative. So, let us write that first so this will tell you that e A mu should be equal to D mu of theta and plus. Of course, this requires that mod phi square should not be 0, and we will see that later, let us hold it for now.

So, if you are in a region where mod phi square is non zero, and then there are no currents where and there are no currents no J that is no sources there. Then we see that A mu in that region looks like this. We are not saying anything deep I am just saying that this term vanishes, and this is just to it is like a legal statement that that I can divide by phi square. So, we will see that this will kind of hold and for finite energy solutions we have this with modulus will follow.

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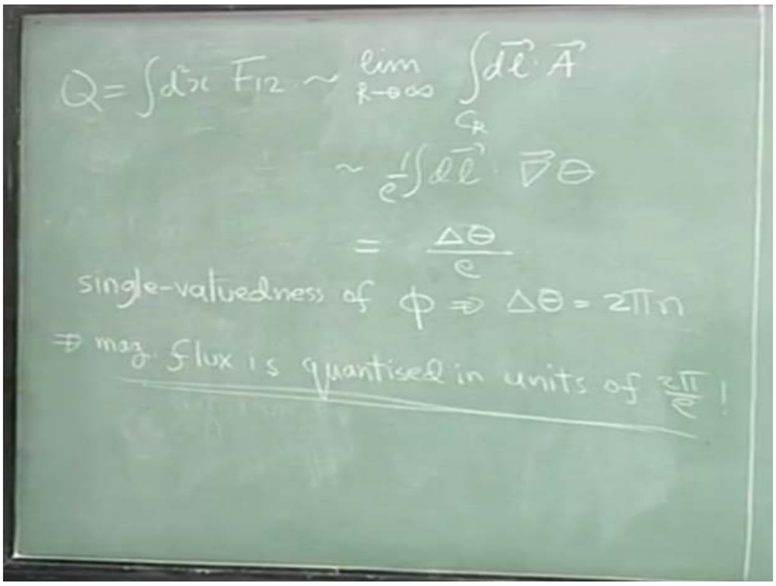

So, let us, let us now work out what Q would be now, now the the nice thing is that you can use what is this called stokes’s theorem or something in the plane. And you can write this as the same, this is the same as an integral of over a circle of radius R, limit R tending to 0 times A dot d l, it is infinity oh it looks like 0 I said 0 and not infinity fanatic I thought my infinite look like 0. So, now we have an expression for A and we have if we assume things are time dependent we will later see that the A 0, you can forget, so there is only a this thing. So, you can see that I can write this out 1 by e, very good. So, then basically what we will see is that will becomes, if you plug that in you get integral of I am having some dimension probably dimension that is.

So, this will so this is going to be so this looks like d l dot d, signs are not important for what I want to say, the identity of theta I am just writing the spacial gradient of theta. So, and 1 upon e so if you just work through this carefully, what you would see is that this become, this is equal to I mean in somewhat cavalier about some factors of R you may worry about rho. But I will give you a quick answer why this is correct and this is got to do with dimensional analysis F 1 2 and it is M square dimension. We just saw that this has dimension of l square so this is dimension less and so this is also, there is a problem e is dimension full and power half so, so I am missing something that is what, is this correct? This cannot be correct yes yes so, good good so yes so wait, wait no no no no e will not change its dimension, e will not change its dimension that is a wrong statement. So, let us let me just redo this argument.

So, F F 1 2 will have, it will have a dimension of e I think the dimensions are correct. So, I did a double, double so it should be dimension less in the whole thing, then you put the e into the thing and you put it here very good so this is what you get so following this this argument. So, I I had to make this assumption, but usually this is kind of true that you assume that at infinity your currents or there are no currents. And even this is this holds, because if you want a finite energy solution, you know that the at spacial infinity your phi fields better go to its vacuum expectation value and mod phi better go to a which is taken to be non zero, because we are looking for symmetry. So, those conditions are actually hoped, and so this is true so and I am using the finite energy condition I am also implicitly I am there is some time independence part.

But let us hold that for now so you can see that this magnetic field flux here has as to be this way. But now we know that phi is a single valued field, single valuedness of phi implies that delta theta can only be multiples of 2 phi, because this is the phase of phi. So, what you doing is a going a circle at infinity a circle at infinity it can only change by 2 phi so implies that magnetic flux, what we will see is that is it is finite energy solutions we will consider. So, what we will see is that this n here will have some meaning, it will it will correspond to the so in other words the smallest guy you can get which has non

non zero thing is 2 phi over e. And let us say we call that a vertex an object which carries that thing and so it, it will be again localized, but localized in the plane now as appose to a point.

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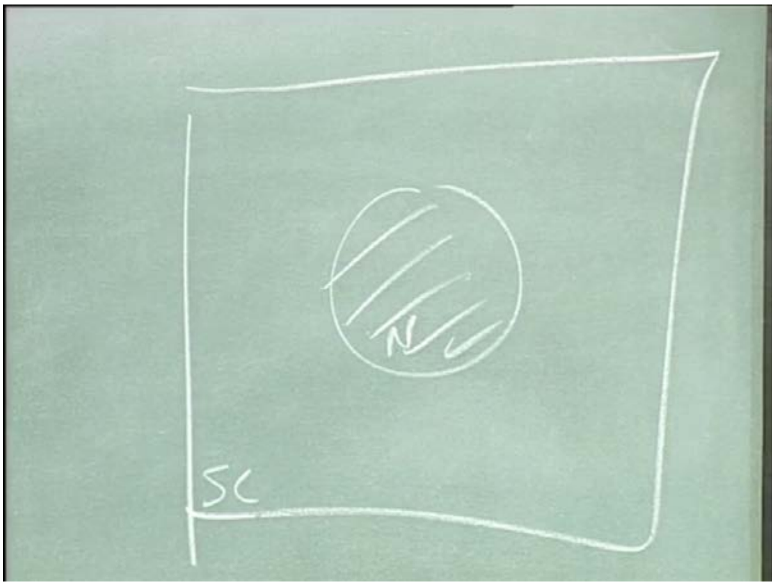

So, the picture will have, if this is the plane, we will have some region which will carry that flux, it may have some size, we can even estimate what this size would be far away it should go to the vacuum solution otherwise you would have this thing. Except for the thing is that what we did is if you had 2 phi over n the field phi as it goes around should get wrapped in some sense, and so that has to be that so in some sense it is not like it is gone there is a what like a branch card there is something knowing which you will remember the factor you have to make this make a jump of 1, 1 unit.

So, an N would be so you, you would think of this is what you would think of as a magnetic vertex. And this exactly the picture you would have seen even in in type 2 super conductors where this would be interpreted where mode phi would be the cupper para density. And the, this would be the, so this would be the normal super conductor in mod phi non zero that means there is a finite non zero cupper pans, this would be super conducting and this would be non in that thing. So, so we get something which is very nice there is no quantum mechanics or anything in during this, it is just topology in some sense just tells you that these conditions which has to be context. So, now let us start's the, are there any questions?

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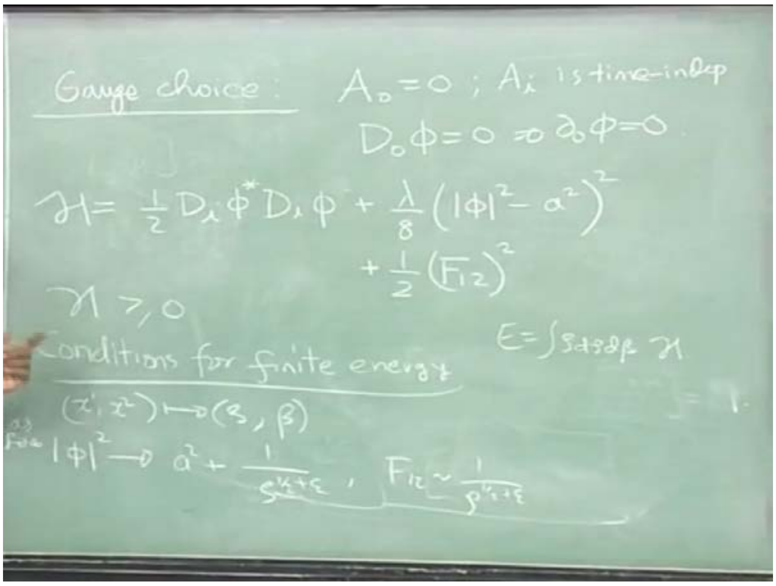

So, where so, so we need to know what the what the Hamilton density for this thing is we have worked out the Hamilton density without in the absence of we have done it for F. And we have done it for this thing in different times or at least in your assignments so let us and so this also additional thing, so we have to make a gauge choice and the gauge choice I make is to set zeroth component to be 0. And time independence for this in a covariant fashion would be this, but now it is also implies the 0 of phi is equal to 0. So, I am just starting here I am just looking at the, this is this current is conserved there is nothing deep about this.

So, then I, I just rewrite it and pulling out an A mu and so this expression from here to here is, is just a rewrite of this. So, the next statement is that suppose I am in a region like we are when you doing this thing is really at spacial infinity, at spacial infinity I know that mod phi square has better go to A A square and there was their will be no sources at infinity this 2 are nice assumptions. In that situation A mu is just the A mu just may is equalize proportional to equal to 1 by e D mu theta, because J mu is 0 here and that is it I think this is. So, here I did not say that I just that point had not come I said when this is true, but that is true at spatial infinity which is where I am using it out there. So, the so this, this conditions are trivially satisfied here with the, with the factor i required finite energy.

So, now you can see that time, now time independence in this gauge is very nice, it just meets the usual statement that phi does not dependent on time. Now, the Hamilton density would be this thing plus I was just passing, because I wanted to check the one fourth will become I think half F 1 2 f 2 1. So, that exactly 2 so so square. So, this is just there are too many stars out here now we are fine, there is no e in this problem, you would have that so we are just saying that the fields are this thing, and so this is the same as this and A i is time let if like saying there is no electric. So, in that thing I am just looking at, so the energy so one trivial thing observation is this sum of squares so; obviously, it implies that e is greater than 0 or equal to 0 rather, let us say this way h is greater than or equal to 0 e could be infinite also whatever.

So, now the conditions for finite energy so let us do a little bit of work let us, let us write phi let us choose let us write x 1. In polar coordinates x 1 and x 2 I will take this to row and no I cannot use phi right what, what angle should I use? Alpha I used alpha for the gauge transformation, so keep running out of Greek letters, let us use alpha for now, it I was used for the radial fluctuation beta. So, this is just rho radial rho or whatever so first thing we knew know is said mod mod phi mod phi.

So, let us look at so we need mod phi square should tend to so as rho tends to infinity, mod phi square should go to a square plus rho power what? So, we have so E would be so say rho power half plus epsilon, it will do. That is from just looking at this particular term so for I get and what about F 1 2? I am just writing for some of them F 1 2 should go like any way you can that whatever it is a magnetic flux at the infinity by the F is 0, I mean other F shall not get. So, these are all kind of conditions that would have.

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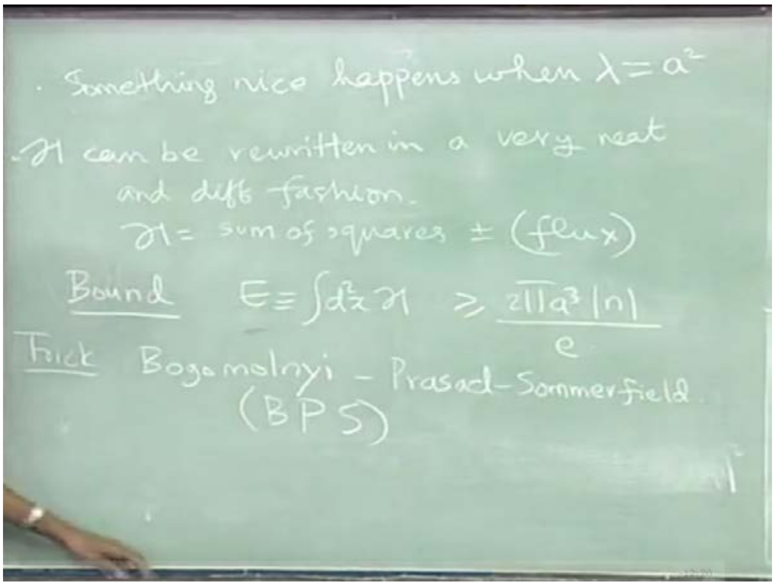

And so in general it is sort of difficult to go ahead and solve, but something very need to happens when… Something nice happens, so this lambda in a were independent parameters to start with and in they can be in general, but when lambda equal to a square, what happens is that we, we can we can when lambda equal to a square what happens is that this term can be rewritten slightly different form there is some of different squares. And so h neat and different fashion, so basically it is a same so what so here for instance this says that the energy is less look at we have sum of squares. And if you ask when is this bound saturated? So, what you would you will lo at this and say every term has to be 0 so that is the vacuum solution. But what, what we will you do here is we will rewrite this in this thing as again as a sum of squares which almost is sum of squares plus a term which should be like the the vertex number. And what we will see?

So, we will see once we do this thing, so it is still some of squares plus or minus flux with some factors. Now, you will see that so the, the bound will be stronger than this thing. This just tells you that the energy h is greater than or equal to 0 we will get something stronger so the bound we will get bound we get will be E which is defined to be greater than or equal to, are the powers correct? I never get these powers right, all these numbers are just to get the dimensions everything correct. But the key point here is to remember is that tells you that if, if I have a solution is magnetic vertex 1 number 1 with or flux 1. Then the energy has a is bounded not by a by 0, but it has to we bounded with intuitively that make sense, because that that object has, has to a 0 h it something greater than the greater than the ground state solution which has 0 energy. But the key is that it actually bounded from below by a large number.

So, again if you ask for when is this saturated again it becomes like saying there is a sum of squares, you set the squares equal to 0. Now, to so the this thing this, this trick I think goes back to Bogomolnyi, Bogomolnyi I am sorry Russian names the, they have many different spellings this is one of the more common values. And and the same author I have seen ever in papers write their names in 2 different ways so I guess when it come from translation from Russian to English names change. So, this is also this kind of trick is goes back to Bogomolnyi, I think, so you write things as sum of squares, you have to massage it, it is it is not the so factures that important everything. But you may think this is just a curiosity why do I have to go and take lambda equal to a square looks very, very artificial.

But it turns out that, this kind of conditions have very, very natural in super symmetric theories. And one great thing about this is at the end of the day you are solving first order equations you are not solving second order equations. Here this is D i of phi equal to 0, this thing equal to 0 so and so for that kind of things we are doing. So, you are solving first order differential equation I suppose to euler range equation which have second order. So, whenever these things happen you are solving and it is always easy to solve a first order equation. Of course, it does not give you all solutions of their second order equation, but it necessarily follows that whatever we get by extremizing this sort of a thing will give a solution. But you are always solving for these cases the first order equation implies that you are solving something simpler, many cases you can write the solution.

So, in fact, this is the first I mean, and this was also used by 2 other authors Prasad and Somerfield, Somerfield I think Somerfield and they also converted in a different context they, they solve for this problems. And so this goes by name of BPS so, so whenever people are talking BPS in some sense what people have in mind is you are solving first order equation instead of second order equations and it is very, very natural out there. So, if I so if I take this thing and embedded into a super symmetric theory which we cannot do, because these 4 we will not discuss fermions in this course so and so because of that.

Then you will find that that theory super symmetry will force conditions of this kind, precisely these kinds of conditions. And so let us, but it is not that we cannot solve, we cannot solve these things when lambda is not equal to a square. But this trick which I am doing definitely requires it now I this again one of the few times were I do not want to waste anybody's time.

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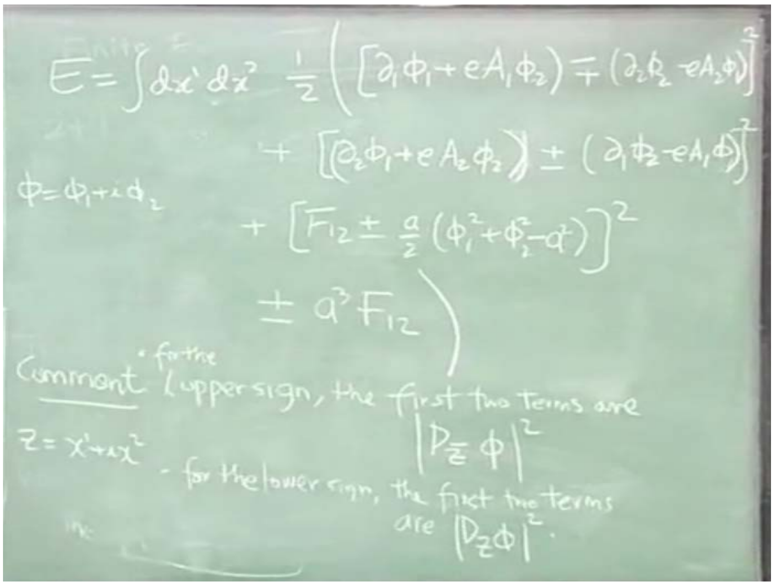

So I will write out the sum of squares in a it is written in a very horrible form, sum of several several terms, so half, half you know like many things I will give a simple way of looking at it. But I have to write it in this form, because it is because it represents 2 things minus sign and a plus sign there are 2 ways of writing it not 1 way. So, phi 1 and phi 2 so phi is just. And there there is some confusion here right this 1 and 2 are nothing to do with space in this is, but this A 2 A 1 are related to this thing. But there is some where it looks like they are getting mixed up and this is again a trick which is called shouldering, you shoulder some internal index with some, some special index it is called shouldering like in electrical engineering.

So, people come up with interesting names I mean it does not make this internal index as a space time index, but in solving that solution it can. Nothing says that phi 1 cannot be x 1, so that can that called shouldering right if we write phi 1 equal to x 1, because you are making that one direction of this thing and phi 2 becoming x 2 that is one example. But any way hopefully, there are no errors here unlikely, so here is the promise sum of squares, then as you can see I needed to look at some sheet of paper with this written, because I cannot work it out without insulting your time. Now, we so this is sum of the squares 3 squares, but that is extra piece out here. And now you can see that e integrated this integrated over this is nothing but the flux.

So I just need to comic comment on how to remember this for the top sign, the upper sign, for the upper sign so that is here, here the first 2 terms, the covariant derivative now Z bar is Z is nothing but going to complex coordinates. And for the lower sign see this is, so mod square is just sum of 2 squares, of a complex number so that is what this is, this is not so hard to figure out for the lower sign the first two terms are. So this exercise for you, go home check this, this kind of TDS, but I mean not that TDS I mean you could have done this in high school if somebody give you the definition. So, let us get the bound, so what does it say? So, this integral gives E, a cubed integral of that is n.

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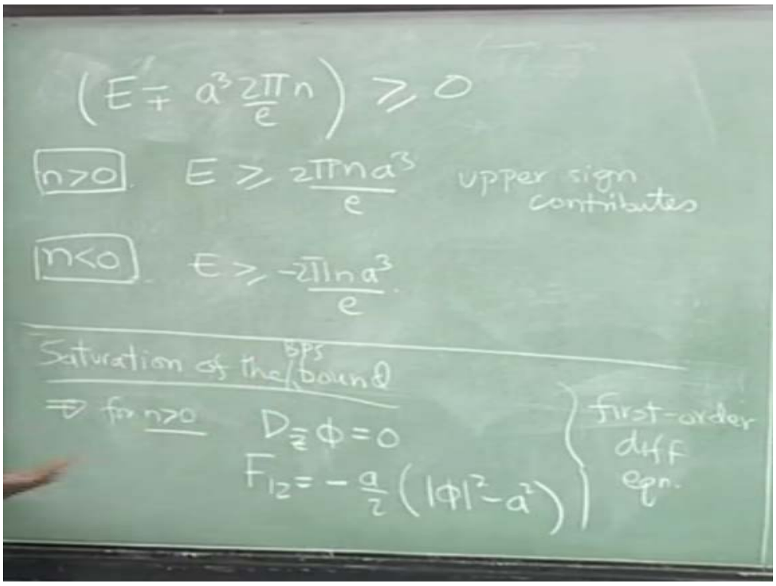

So what we get is that tells you that E plus or minus, no minus or plus or n was defined with half, phi no no, no is this correct? What is that? I have erased it what is it? So, a cubed and what is integral F 1 2 equal to sum of squares so this should be greater than or equal to 0. But I think let us I think that is not so important here, yes I did, but the quantization should not change, the quantization does not depend on. Topologic I know J mu topological I, that is the different, but the other that require the, J Noether, which half no J mu topology that is not relevant know, because the quantization came not from correct yes.

So, I forgot the half thanks so I missed a half earlier, not here so this is correct so two errors kind of compensated fine, now we are fine. So, what this tells you is that now we should take when n is greater than 0, you can see that one of them is weak, if n is positive e greater than the stronger one is upper this thing. So, so upper sign and n less than 0 the lower sign contributes and you put a minus 2 phi n a cubed upon e. Now, you can see that both of them can be put together by just putting a modulus that is in terms of energy, where in terms of solutions it tells you that the upper sign or D z bar phi the one which gives this goes with the positive this thing magnetic flux. The other one goes with negative magnetic flux and when is it saturated when these things so for instance so let us choose so, so saturation of bound.

So, we will call this we will loosely call this a BPS bound, so there is nothing it does not quite fit with what is a modern day usage, but it does not I mean the sprit is that. So, we can see that implies that so for n greater than 0 D z bar phi should be 0 and what else should be 0. So, that is the first two terms are taken care of and F 1 2 should be equal to minus a by 2, these are your first order equations that you need to solve. So, now and you can see that the same thing for n less than 0 which just becomes D z of phi equal to 0 and F 1 2 should be plus a by 2 this thing.

So, these are so I just have I mean what, what this kind of tells you as which intuitively you can see this since, since flux is conserved it is and quantized. There is no way on earth by making some jiggling things around, you can ever change the change n. So, you get to a distinct classes of, of solutions and they can never you can never go from one to the other, they are all independent of each other. And but that is that is actually nothing to do with ah with even BPS bound or anything, even when we are solving second order equations you cannot change them. The only way to change them is to take one vertex or something which carries that charge and take it away to infinity, but that violates our very first this thing, you are taking away to infinity.

So, so the topological conversed currents actually give you a way of understanding how there are these classification, it is similar to what happened even in the king solution it was interpolating between to vacua and once you fixed your vacua at infinity there was nothing there you were forced to have that solution so, in it is exactly the same way, but it has a different in this thing. So what we will do is we will try to see how one in the next lecture, how these things are solved? And we also discuss what happens in the other case when lambda is not e square why should it be there. So, so we will stop here for now. And we will continue that in the next lecture.