Classical Field Theory
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Lecture – 20

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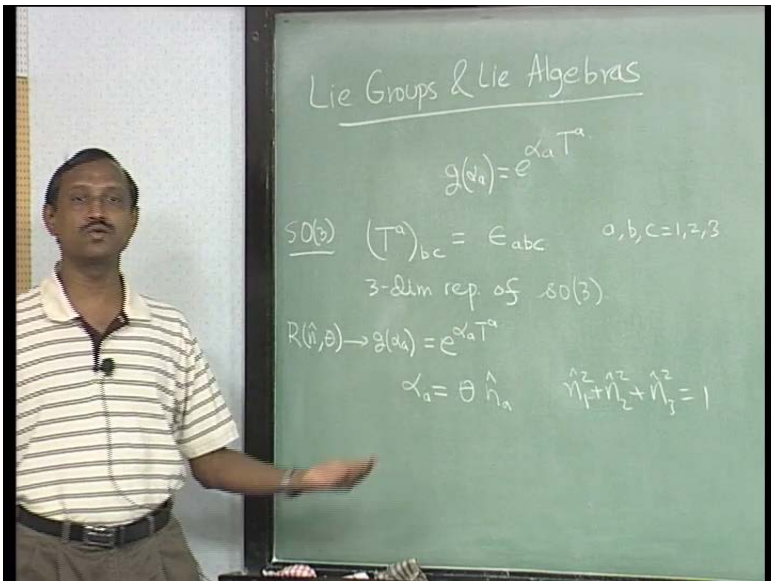

And we saw that Lie algebras made life a lot easier, because we start working with linear vector spaces rather than some horrible nonlinear thing, and so the key was that you get a group ellipse that is a nice relation between group elements and exponential of… So, let us say that you have generators T a and some parameters alpha a's. Did I use upper case or upper index for T a in last lecture; then let us stick to that. So what this tells you by looking at it in this form what you realize is that this is a group element, so you can actually write the parameters in this form.

And towards the end of last lecture, I told you that you could I asked you to consider something like this for SO 3 or SO 2 or like SO3 and T a b c, you take it to be epsilon a b c with a b c everything running from 1, 2 and 3. So for instance, so this would tell you that you could write a group element of SO 3 in terms of something like this so for g of SO 3. So, again like I pointed out this is in some representation, so this is a 3 by 3 matrix.

So, this is in the three-dimensional representation of SO 3, and one more bit on notation, when you write the Lie algebras we use lower case; let us use an upper case for this. So, the group SO 3 would be written this way and here I would mean the Lie algebra. Some books use some kind of different script some other font, etcetera but I think for writing at least you can have upper case for group and lower case for the corresponding Lie algebra. This standard notation and so we are in a three dimensional representation, so the group also. So, if you through the exponential map so I will just write the same thing alpha a T a; I am not putting I out here, I am using the math convention just for now. So, this would give you g of alpha a and if you just go ahead and rewrite alpha a as follows, I write this as some angle times n hat of a where n hat of a is such that. So then you can rewrite this, so it is still three parameters because if I just wrote alpha a it is alpha 1, alpha 2, alpha 3.

So, all I am doing here is calling theta the magnitude of alpha a in some sense; if I define magnitude to be alpha 1 square plus alpha 2 square plus alpha 3 square and then I just write this. So, now we know that this SO 3 is just rotations in real space and so this you can see now I have written it in a particular form. So, we know in three-dimension any rotation can be written as a rotation by some angle theta about some access and n. So, now you can see that and I think I believe I have given this expression in one of the assignments I asked you to work it out. I think so if I have not it does not matter.

But now this is what we would have called in our earlier notation; we would have called this r of n hat and theta. So, we see a nice connection between what we had seen in instead of in more elementary courses or even earlier in this course and something much more starting out from a much more abstract setting. The advantage of the abstract setting is this kind of formula holds for Lie groups for arbitrary Lie groups.

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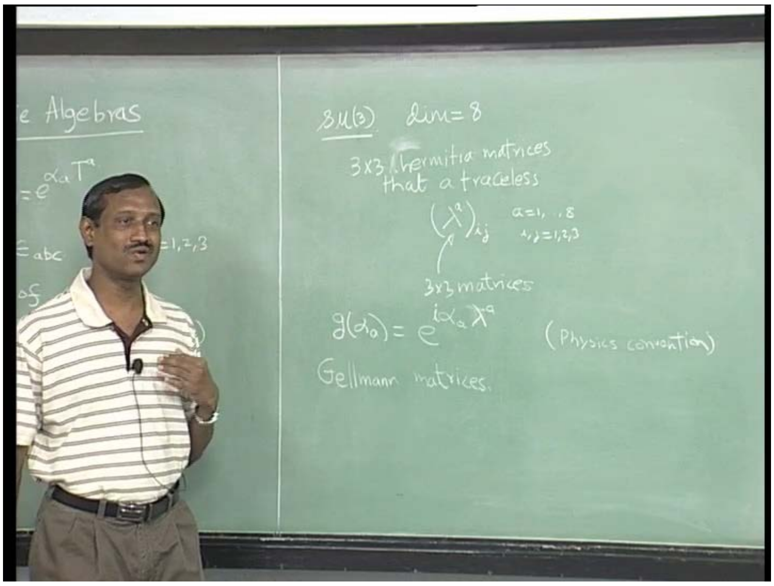

So for instance if I take SU 3 we know nothing about it. At this point we accept for the fact there it is a Lie group and so what about Su 3. So, first thing we can check is that the dimension will be 3 square n square minus 1, so that will be 8. So, it is the set of n by n so the natural representation if you look at three by three Hermitian matrices that is traceless; that will be the Lie algebra similar to what was antisymmetric. So, the dimension will be 8 and you need to choose some bunch of three by three matrices and so let us just call them instead of calling them T a’s I will just call lambda a; a runs from 1 to 8 but there will be some i j index, i and j run from 1, 2, 3. So, these are some three by three matrices that satisfy this property which will form a basis for these things. So, the statement would be if you wanted to write an SU 3 arbitrary SU 3 element, just one second I will not say I should say skew-Hermitian or antihermitian. Because I am not putting i or we leave it like this and yeah, let me stick to the physics convention now I put an I out here.

So, this is physics convention while this is the Math’s we are writing it here. So, you can see that i t a, sorry not T a lambda a. So if lambda is Hermitian i times lambda a will be antihermitian in a trivial fashion. Now you can see that so in your assignment you will see that I have given you a bunch of a particular representation of this eight matrices, These are called the Gellmann matrices; Gellmann is the very important name in this high energy physics for one of the reasons is that he actually brought about the idea of using groups to classify the particle zoo. So, what happened is that they were used to find what are called resonances every day and it looks like should we call them new particles, etcetera because people did not know about coax or anything. This was before the quack model and so what he did was he realized that there is some underlying SU 3 symmetry and it goes by the name of he coined something called the eight fold way. This eight I mean it is related to this and he introduced this particular set of basis vectors or basis vectors for the Lie algebra Su 3.

But you do not need I mean the point I making out here is that we do not need to remember anything. You can just write out the group elements but there is one tricky part in this whole business. If you are asked what is the values that what is the range of values alphas can take such that you get a single valued this thing; so out here its easy, I will tell you the answer theta goes from zero to 2 pi n and its identified but if you go to SU 3 it is not so easy. So, local properties you can capture but if you want global properties which means which tells you things like what are the ranges values these various alphas can take, then you will see that things are more involved. But at least notionally you can see that you get you are able to write out group elements which satisfy all the product rules which are required. So, now coming back to the original problem we want it to parameterize find a nice way to choose fluctuations when you have symmetry breaking.

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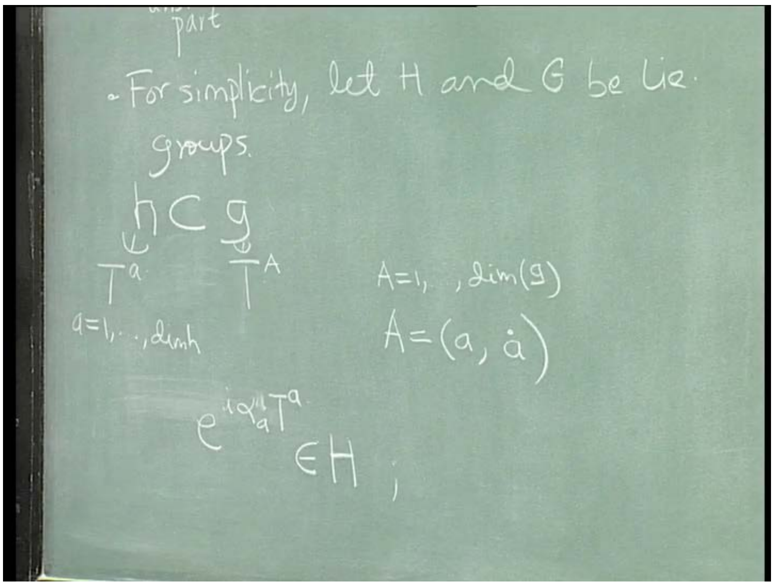

So let us assume, so let us return to symmetry breaking and let us say that H is some take subgroup of g and let us assume. So, as usual G is the symmetry of the thing and H is some subgroup which is preserved by the classical vacuum and we will assume so this is symmetry of the theory and this is the unbroken part. Further we will let us assume for simplicity we assume both are Lie groups and now one more point here is the fact that H sits inside is a subgroup of G sort of tells you that the Lie algebra of H will also be sub algebra of Lie sub algebra of G. What is the statement what you will say so when we looked at when we said subgroup, if you take two elements H 1 and H 2 which are in H you take their product the group composition it will stay within H. So that guarantee so that via the exponential thing you will see that the Lie bracket or the commutator of the Lie algebra elements should stay.

So, it is just the closure part you have to check but the closure at the Lie algebra level is done by taking the Lie bracket or the commutator of the Lie algebra elements and while in a group it is just the composition part. So, in other words what you will so let us first go ahead and write let us say that let us use little g, so we will write something like this, h is a sub algebra of g. So, this I am replacing this statement by the Lie algebra. So, now we can go ahead and let us say that we choose T, let me choose capital A and let us say that A runs over something's a trans over dimensions of g and let us say that this has a bunch of generators. So, this is a basis for this thing; similarly I can have another t let me use a little a index and a will run over dimension of this the Lie algebra h.

Because it is a strict subgroup obviously this dimension will be less than this and we can be clever enough because this sits inside this, we can always adjust it such that the first h elements of this. So we can also write a, I can break up the indices or the generators a and a dot where a runs over 1 to dim h. So, that is the set of generators which are in h and a dot is the set of generators which are not in h. So they belong in some sense morally speaking to the co-set the directions along the co-set. So, elements in h rather group elements. So, now we can see they are very easy things lot of simple things can be done, we can write a group element. So, an element like this e power let me put i strict to the physic rotation alpha a T a. What would be this group element where would it belong to? It will be an H and suppose we wanted elements in g mod h.

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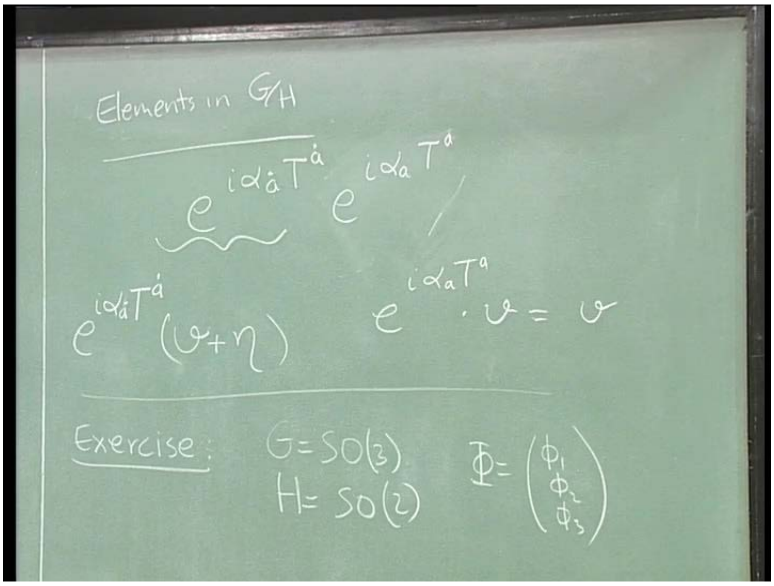

What about elements? This is what we wanted to isolate, right, we wanted to get the flat directions, etcetera, that is what we wanted. Yeah. So, H is symmetry of this thing and so we just want to write out. So, you can see that a natural choice will be to write an element of this form. So, this is the index generators of the Lie algebra not in H. So, that so you can see morally speaking, so how do you understand this. The most general group element would be something like this but now I can write I want to get the co-set but I can keep acting on it either from the right or left does not matter which way. I can keep acting on it and you can see that I can always adjust it such that this term goes away. So this is a representative of or equally well I could written the most general group element I could write out in some form like this; nothing wrong with writing it even this form. So, now you can see that these goldstone modes came from exactly these guys.

So, what you would do is you would write out these elements explicitly outside. So, if you have some vector v, so this is some vector in whatever representation it is its acting on this thing. So, what I would write is the fluctuations I will write as v plus eta where eta would be like the analog of the radial directions times e power I alpha a dot T dot, and the number of eta fluctuations will be whatever is the appropriate thing but this you can pull out explicitly and you can see all the nonlinearity, etcetera that you need is taken care of. So, this is what one means by choosing a clever parameterization and that is why the Lie algebraic structure is very useful in even just writing something like this. And then because you have chosen things properly everything will work out. So, we will get back to where we will see more general symmetry breakings, etcetera in this course and during that time you will see me explicitly work out examples of this.

But you could for instance ask what happens if I mean if I have three scalars in SO 3. So, it is one exercise for you. Choose G to be SO 3 three dimensional representation the normal vector thing and H to be SO 2. So, this is nothing but choosing one vector so that this would be the rotations and so you write out. I mean I think you should be by now very easy for you to write out something and just work out, verify Goldstone's theorem for this example; for an example you can construct a Lagrangian which has this invariants. So, and the scalar fields would be some phi, choose a suitable potential; you can even choose my favorite phi forth potential if you wish which has that invariance. So, this is a good fun exercise and I am sure we will look at this example. Yes, have I mixed it up? Yeah. We do not have there in the solution a is what is noticed? We rotate the solution by the unknown. No, no, I think it is correct because you just look at it right, if I put a out here it is symmetry of the this thing.

So, trivially so if I have something just forget everything. This is the statement that it is symmetry. So, you write the most general group element action on v and you see what you get but the v goes off. Okay, I think I am correct here. So, before moving ahead to more general things I would like to take a step back and like I said in my summary of what we have done so far, we are moving on to understanding vector fields and so I would like to complete the discussion of looking at the symmetries of just the electromagnetism and getting the various currents, etcetera. This is something you have done in your assignment but I would like to go over the solution to that and so that is what I will do in next half hour or so. And since you people have worked through quite a bit of the details I will just go over points which I want to emphasize, etcetera.

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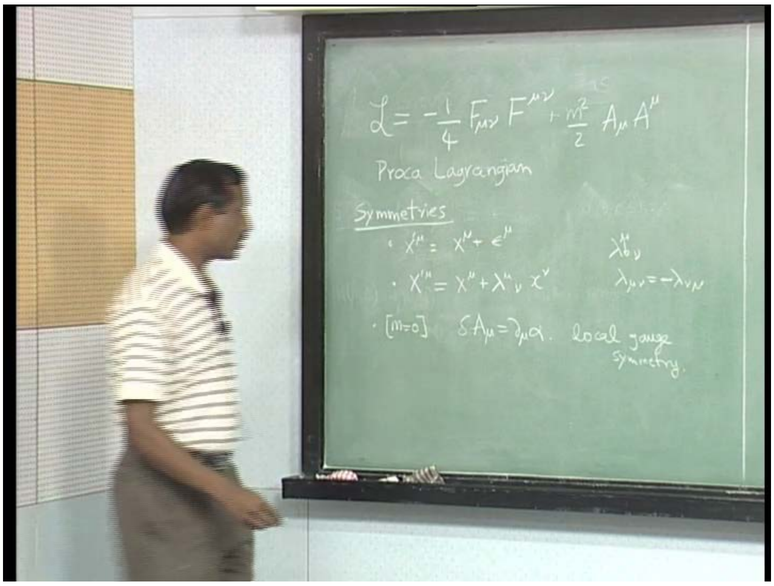

So, the Lagrangian is very simple. Just to add some structure to it. So, this is what would be the Lagrangian density for pure electromagnetism or u 1 but I have also added mass term. If you add such a term this Lagrangian goes back to Proca, so it is called the Proca Lagrangian. As I mentioned just as a quantum field theory this thing is sick without low rates like I mean this breaks gauge it is not invariant under local transformations because of this particular term there is a potential for the gauge field. So, you know that this is bad but from a view point of just taking a classical Lagrangian, getting it working out what you know the currents are, etcetera under various symmetries. This is perfectly good thing to start with and also you can go ahead; at the end of the day there is one part where you when you set m equal to zero something nice happens and that means something.

So, what are the symmetries? So, the first symmetry of course is translations in space and time and I will use epsilon because I will use a for the arbitrary parameter, so epsilon is some constant parameter in this thing and so this is one symmetry. The other symmetry which we look at is x prime mu; we can let us just write it this way. So, there we always work to first order in these parameters. So, epsilon and lambda are suppose to be first order parameters, and so for here there might be no higher order terms but in this case there will be more terms and we also saw that this lambda mu nu was if you lower the index you get lambda mu nu it is anti symmetric; you have already seen this. So, and the last symmetry occurs only when m is equal to zero and that comes from and that is just the usual get symmetry which is I think I use alpha or something, that does not matter, so this local gauge symmetry. So, the first point here is to work out what the equations of motion are but I do not think I will work it out; it is rather simple but yeah, so what I will do first is to work out the current for translations.

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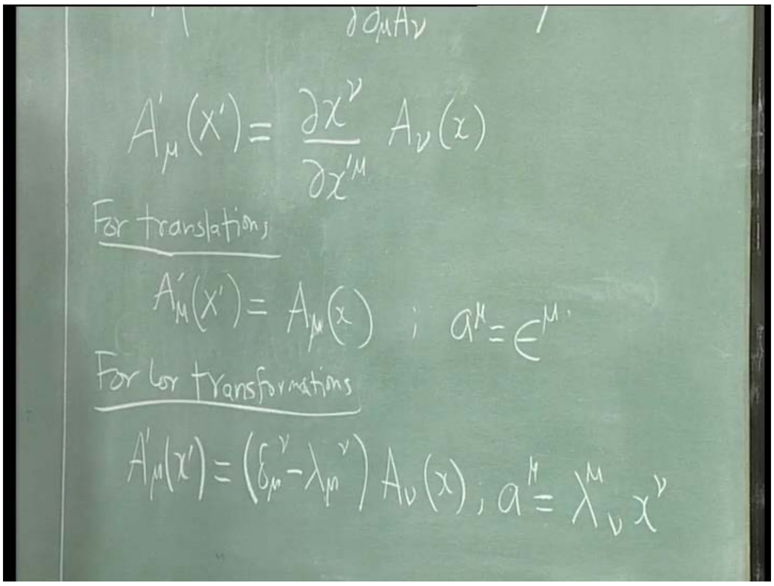

So we need the master formula which we had worked out several lectures ago. So, the only change I did in the master formula is said that we are done it for scalar phi a but it does not make any difference; I replace phi A by A nu, I mean it is the same thing, think of nu as a index like A. So, the same formula goes through; there is nothing deep in the whole thing whole business and delta bar was the change in the functional form and yeah, so one more thing one way to we need to know how the gauge field transforms but I prefer to think of it in more general settings like in general theory of relativity but it holds even these are also these two are also examples of coordinate transformations. So, I will write something more general. So, all you have to do here if you want to work hard what it is under translation is to look at d x prime by d x this thing d x nu and what do we get? We get delta out here.

So, for translations we see that a mu but for Lorentz transformation. So, you may think from here it is easy to gets d x prime by d x nu but for small thing you can just invert it and just get put a minus sign and you will be fine. So, since I want to write it in this form what I will get is a prime mu of x prime will be equal to. Now you can see that the first time we are actually seeing something more than normal. So, this looks like exactly like you had for a scalar field but out here the fact that this index transforms under Lorentz transformation shows up here. So, what is it we need to do; really the only things we need to work out are and to write out what this a will be, what is delta bar and these things. So, I will just do these two things and so first thing to work out is what is A mu. A mu is nothing but epsilon mu out here and out here A mu. So, by the way is this was that my notation, I mean am I okay or is there a sign issue? I am okay, right; well if there is a sign issue I will fix it later but right now it is not so important, I mean it is important but I do not think the world is going to.

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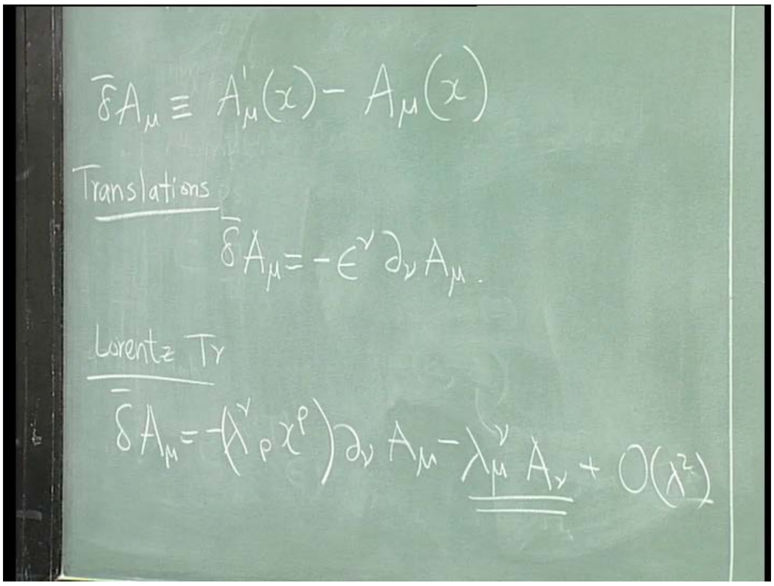

So, let us first do this part. Now this is something lots of students have trouble with you know what do I do with this; you know the first thing is so I have to go ahead and do the variations in here and this has d nu, A mu, etcetera, your dummy indices all these things. So, first thing you should do if you are totally confused is to not do what I have. Here I am interested in d nu, A mu and these are all dummy indices. So, first thing to do is to write this as rho and sigma something else then you expand this. Each of this is two terms, you get four terms and each one of them will give a contribution and they will all add up and it will eat up this four and that is about it. So, you will end up getting minus. So, you need to fix you have to look only at one of the terms the minus comes from here this minus; sorry F mu nu, thank you. Sorry, I was intent on getting the sign corrected got something wrong.

So but the thing is I never like to explain in more detail how this four goes away because everybody has their own way of doing communitorics. I know that I cannot follow someone else's communitorics; what I do is to do my own and then check. So, but this is something which you should do once in your life, actually expand it out in gory detail, write out every one of the four terms, write the delta functions everything and see that it works and then after sometime I mean you can just do it visually. But the first time around it is not something this step to this step I mean it is to spend 5 to 10 minutes. It is not, do not feel ashamed that ho, it took me so long but after a while it will be like ten seconds. So, this is this is one bit. The next part is to work out delta bar of A mu and that is defined to be. So, it is a change in the functional form, this is the definition. So, for translations it is similar to what we got for a scalar field. I will just jump some steps and do it from memory it was minus A mu d mu. So, it was so delta bar of A mu was minus a.

So, now there is a free index mu, so I should use some other index. So, let me call it d nu; it is a little bit more involved for Lorentz transformations. So, for Lorentz transformations so we have to make so one term, so let us first forget so this one term which will look exactly like this which I will write out and A mu A nu is just epsilon. So, we go back to the constant parameter, we find delta bar. A mu will have one piece which should be minus A mu, so it is minus A mu lambda nu rho x rho times d nu A mu. So, I have just written A mu d nu but A mu is this but I get one more piece which comes from this. We are ignoring all odd lambda square pieces. So, this is the biggest change that happens because you have this has an index which transforms. So, until this point everything went like it went for a scalar field. So, first step is to get the stress tensor for this that is just the conserved current for under translation. So, let us get the stress tensor.

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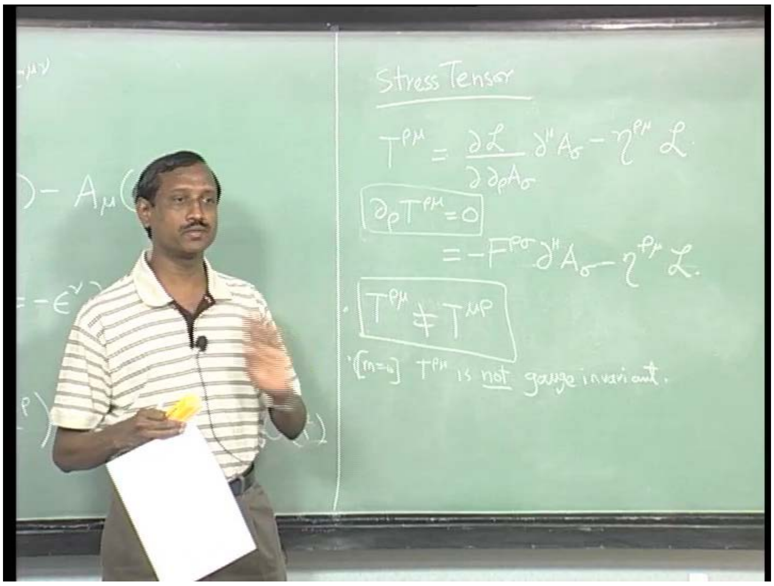

So, now putting all these things into this thing and there is only one. So, there is a parameter epsilon which is the constant. Remember A mu in general is not a constant because here is an example where A mu depends on x. So, you cannot pull out A mu outside the derivative. You can pull out only constant factors like lambdas. So, what you get is T rho mu. So, this is what you get and the conserved current here is d rho. So, the rest of this talk lecture rho will be the conserved current this thing. So, this is the point. So, we can look at this and we can just write out we get minus F rho sigma d mu A sigma minus eta rho mu L. So, first thing you observe is that it is not symmetric anymore; that is one issue. So, which was true for the scalar fields, so just one concrete example but there is one more issue.

Especially we know that you know that this is a toy model; with the model we are really interested in this when m is zero. When m is zero you look at the energy momentum tends around here; this is not gauge invariant. This is F which is invariant but this is not invariant. It is also not gauge invariant. So set when m equal to zero which is not a very nice thing; I mean so the question is that somewhere we can actually make that you know F and along the way also make it symmetric and so the answer is actually more or less staring at you in your face; you can actually figure out these things but there is some logic which goes behind what you can add. But at the end of the day this is the conserved current which was given to you by no other prescription, I mean the formula. So, now the question is can we do something to it such that the charges are not affected. The charges should not be affected but some nice properties are recovered such as we recover symmetry and may even get invariance. In this case you will find both of them come in one phase.

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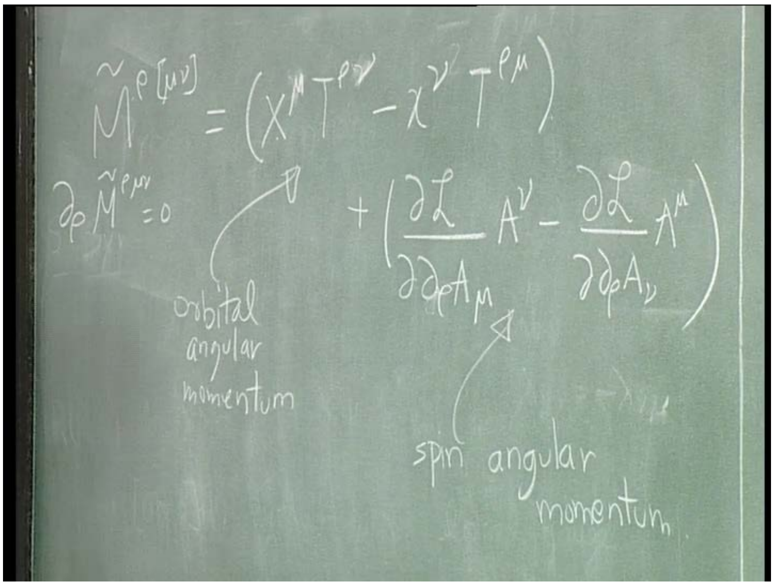

And we can go ahead and also work out what is the No ether currents for Lorentz transformations; call them this thing. So, again rho is the, so this is such that d rho and tilde rho mu nu is zero and this is antisymmetric in mu and rho nu because that lambda is so the parameters which come with this. So, this turns out to be, what am I writing, x mu t rho nu minus x nu T rho mu. So again when you do this, this term comes from exactly this term similar to the other guy but this piece will give you an extra term and it is better to write that out. So, the thing is you do get something but the point here is it actually consists of two parts. This piece looks pretty much like r cross p in physical’s way this thing.

So, this has morally speaking role of orbital angular momentum while this guy, so whenever I say angular momentum I mean a generalization; my angular momentum has six components because we are in three plus one dimensions, so rotations are examples of them, rotations only have three parameters. So, the usual angular momentum which we discussed has only three components corresponding to rotations but we have also three more currents which come due to boos. So, there are six of them and so what one does is to use in the relative stick field theories whenever we use the word angular momentum we mean a generalized angular momentum; in fact this is the angular momentum in that setting.

And so now this has do, so this is what we will call but you can see orbital angular momentum is not conserved and this came because this object has this extra transformation, where does this come from? This comes from the fact that this transforms under Lorentz transformation; it is an index which transforms. When we go to fermions, fermions carry spinner indices which again transform exactly like this, I mean when I say exactly like this it is not a vector index; it is a similar property you expand it, you will get lambda in except here this indices would be Spinner indices as a post to vector indices but if the structure will be similar. So for fermions also you get the same story. You get one more piece for the same reason and so this is what we should call the spin angular momentum. So, now what we will do is to recover symmetry of the stress tenser or the energy momentum tenser we will modify to get something.

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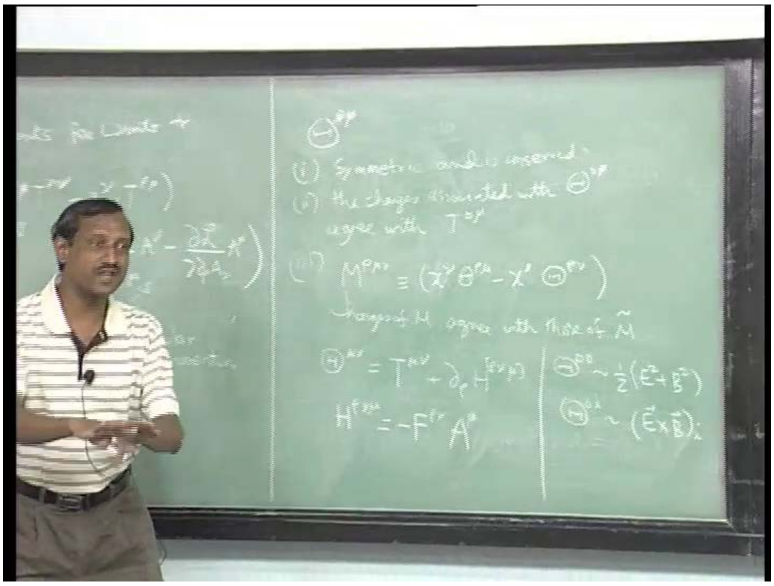

So, we will construct something which we will call theta mu nu which has the following properties. First thing is it is symmetric and it is conserved. Let us say rho mu. The four charges the charges associated with the theta zero mu integrated over agree with T zero mu and some more properties we will put in. M rho mu nu, so this other term is just antisymmetrising with mu nu, I mean there is nothing to remember. So, this M is not quite M tilde that is why there is a tilde, but the key point is that the charge is again and the charges of M agree with those of M tilde. So, the property two actually gives you a hint as to what you have to do. You want the charges to be conserved.

So, what you do is you suppose we did this, suppose we look for something like this theta mu nu actually there is a fourth one which I could add which it says that it should become gauge invariant; in fact it is a bit of a cheat in the assignment I did not write that and so theta mu nu plus t mu nu and h rho nu is totally antisymmetric. So if you have this is like a total derivative, so if you add things to it and you do an integration, you will find that the charges this gives surface terms and if your fields drop off which is what you assume at infinity, then there is no contribution coming from this and so we need to figure out n h rho of mu but actually it is very easy to ask what that would be because you use my trick which is to say let us look for the gauge invariant guy and up to some sign we see that H, if you plug this in, if you if you plug what is it this value of H into this you get this theta and theta is really if my memory is right theta, what did I erase I forgot, F mu sigma; that is what it is?

And you can also then it is kind of nice you get something which is gauge invariant and you have this thing and you can actually go back and look at what you get for theta zero zero and think of it in terms of electric and magnetic field. So, in your assignment I am sure you put M not equal to zero but for my class for this lecture I will just set M equal to 0, because I can write the answers; otherwise I have to look up some notes or something, I do not remember the answers. But if this is like half and what was again this tilde here implies I do not know the factor. It is probably half; was it half, but the key point here is that you end up getting something which is positive definite. What about theta zero I; it is a pointing vector and another thing which is very interesting is that theta i zero is equal to this, but it has a different meaning you see. In some sense it also is the energy flux.

So it has a dual role, it is a conserved charge. So, you can think of this as momentum carried by the electromagnetic wave. We also think of this as the energy flux where you think of it a low this thing, theta i j is something which you should write out in terms of electric and magnetic field; just see it for yourself, it is a little bit of a mess but it is okay. I mean it is so it is a momentum flux. So, it has two directions; one for the momentum and one for the physical directions of the plane. So, this is actually important; this tells you that the field carries some energy and also can carry some momentum. In fact you can come up with examples if you look at books like may be Jackson electrodynamics I am sure it or even Landau ellipse, I am sure they discuss examples where you actually if you just looked at particles charge particles moving, you will think that energy is not conserved or momentum is not conserved.

But if you actually go ahead and look at the combined combination of the energy of the particles plus the field energy and the field momentum, you will find that energy and momentum are conserved. There is a problem with saying what is the field due to suppose you have a point charge, there is an electromagnetic field all over. If you try to ask what is it is the total energy it could be infinite, so you need to regulate a bit. So, one way to regulate is you put it in a big box and box is big enough that size should not matter and that is not relevant for your this thing, and so you can actually come across situations where the total energy which is conserved is energy of the particles and this thing. So, at a fundamental level you can see that this is proof if you wish to say that this is the object you should call energy density of course here and this is the momentum density.

So, there are actually in classical electrodynamics enough examples of this kind and the last bit of thing which is actually also. Before I forget, there is actually a completely different definition of the energy momentum tenser and so the thing is that in last lecture we discussed local internal symmetries, if you make them local what will happen but there was nothing to say that you cannot make translations local but that will naturally take you in some will take you to the elm of general coordinate transformations. So, that is the correct thing; so in some sense general coordinate transformations subsume both translations and Lorentz, there are special cases of that. So, you will ask how things transform under general coordinate transformations and then if you want to make things so there will be analog you have to introduce a field the corresponding field that will turn out to be something called the metric.

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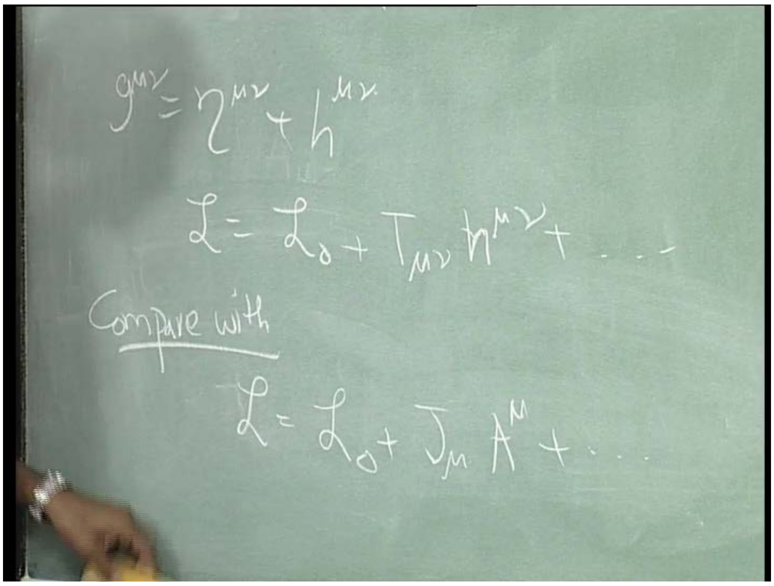

So you replace, so the dynamical field will be eta mu nu replaced by something called g mu nu; it is little bit confusing. It is not a vector field, it is a spin two guy, it is symmetric, etcetera. But even if you when this is sort of a particle physics way, antigeometric in some ways, what we are doing is to just say that what will happen if you convert something symmetry to a global symmetry that is not the way Einstein taught about the general theory of relativity but even this sort of approach leads you to a similar thing and so what you do is now you can ask give me some Lagrangian, can I make it locally invariant under these things and you couple it to a metric and so you can write something in again in a minimal fashion. So, you define t mu nu as follows. I think there is some square root of minus g factor g is determinant of this thing and upper lower math form on my part but this by definition is symmetric. Now it is a big deal to show why is this equal to that.

That is a different thing but there is a straight ahead way of getting a symmetric stress tenser by thinking of it as the current which couples to gravity; just as we got the No ether current which couples to I mean so in that sense it is similar. So, the coupling another way of saying it is that you take g mu nu to be eta mu nu plus some small parameter h mu nu and you want the linear coupling such that it is invariant under general coordinate transformations to first order and there is a minimal way of doing it. And so what you will find is you will get s will be s naught plus rather L where L naught is where you use eta mu nu whatever you wrote plus up to factors t mu nu h mu nu.

Similarly compare with for gauge symmetry again it would be similar. There are infinity order terms but only the linear pieces again gave you j mu a mu. So, this will naturally give you something symmetric. So, the last bit was you are asked to find the trace of this theta mu nu and you should remember to trace with eta not with.

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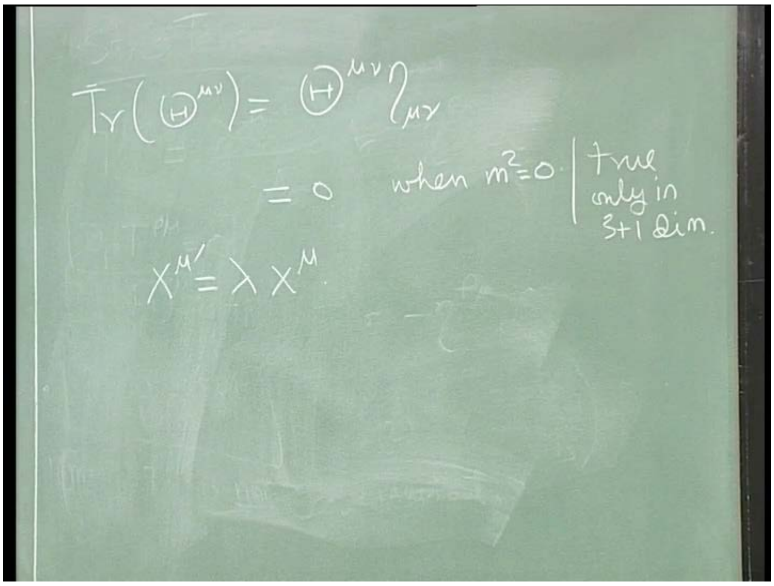

So trace of theta mu nu is not the sum of the diagonal terms or whatever; we have discussed this before. It turns out that this is equal to zero when m square is zero; does anybody know why? I will give you the answer. There is a new there is an extra symmetry at least at a classical level which is scaling exercise show that the conserved current due to this scaling symmetry is actually the trace of this conserved current or charge or whatever, so you will that that is traceless; you can prove that. So, this is actually indicates that you have this thing. So, you will find that we will see more general things when you take the non-abelian gauge theories; that means you have a non-abelian group Lie group which you gauge you will make symmetry local, you will find even in such cases classically.

So, if you take SU 3 for instance you will find that classically the trace is zero, but quantum mechanically that need not be zero. So, a classic example of that is q c d where the group is SU 3 and it is known that there is a natural scale. So, one more way of understanding is when m square is 0, is there is no length scale in the problem. This is true only in three plus one dimensions, but if you go to other dimension that is not true.

So, this is actually dimension true only; in all other dimensions gauge couplings are dimension full; we will do some analysis at some later stage. So, I will stop here.