Classical Field Theory

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Lecture - 2
What is Classical Field Theory

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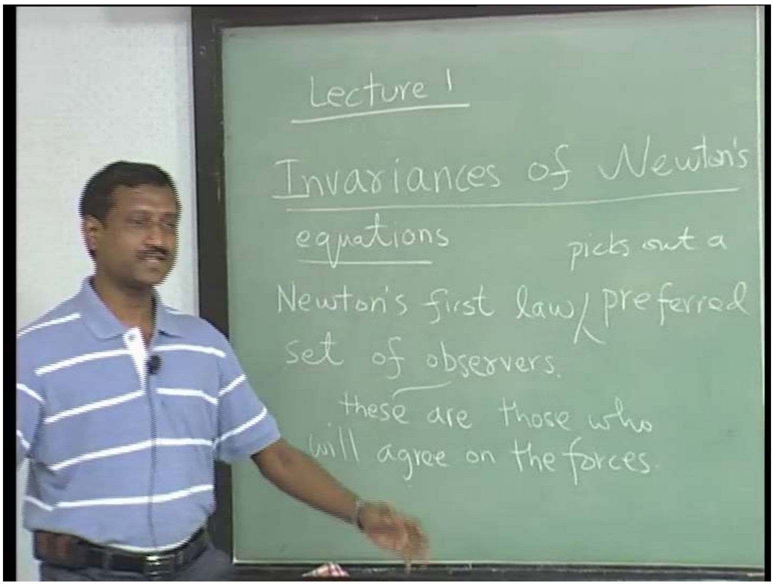

So, last lecture we actually got to see, what the basic definition of classical field theory was we saw that it was just a classical mechanic of fields. And the one might guess that the natural thing to do would be to actually go ahead and start discussing how to write action or Lagrangian for fields. But we will hold that back a little bit, because there is something, which we need to input or understand it, how symmetry in nature occurs in nature, and how best we should formulated? So, the next three to four lectures will be a sort of introduction to symmetry, group theory, etcetera.

Sort of mathematical background prelude, before we actually start doing classical field theory; and it is actually a very important ingredient for what will be discussing in this course, because as you will see that symmetry is something, which sort of an underlying principle; it puts restriction on the kinds of things, we could do and but the weird thing about symmetry is that for instance, we would we would thing that things should be isotropic; that means it should in variant under rotations. But here we are on earth and there is always have preferred direction and that is a direction due to gravity.

So, quite often we are actually even though, the underline force is the one upon rather, the potential is one upon r, we are in a situation, where you know there is a preferred direction, then you may say fine let me not think about the earth, let us take the whole solar system, but even there if you think about it earth is moving in a plan, so again there is a preferred direction if you wish, which is to the normal to the plane. So, it appears that all the time we are in a situation, where there is a symmetry, but it is not obvious to us and this is sometimes called hidden symmetry or more commonly called broken symmetry.

So, one by one that do we need to really worry about symmetry, and the answer is yes because as I mention sometime back, it puts lot of restriction. So, what we will do is, we will start with Newton's equations, which we know very well and as what are the symmetries or the precise statement for an equation is the invariances of the equation. And, we will see that certain nice structure, will emerge and then we can map it into some problem in mathematics called group theory, and then again we can come back to physics. So, what we are going to do first today is to discuss the invariances of Newton's equations.

In fact, Newton's first law already picks out preferred sector of it picks out a preferred set of observers; these are observers who will agree on the forces. In fact, the classic thing is to imagine yourself, in a train moving with uniform velocity and then Newton's first law will tell you, that you will agree with the forces acting on some other particle with a person on the standing on the railway platform, because , that is one example, but the minute, the train either accelerates or decelerates, you start feeling forces and then you have to add to make Newton's law a work, what you do if you add, what a called pseudo forces. Of course more dramatic example is that of rotating system, if you are in a marigo around or something like that, there you will, so the key point here is, that we will look for a observers for whom who agree on the forces.

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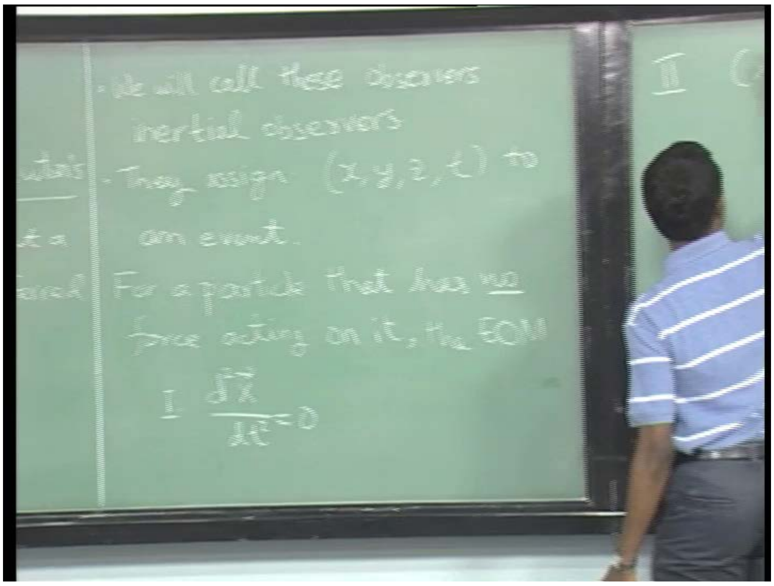

So, we will give them an a hint, we will call these people observers, inertial observers. And the question, we will ask is, what characterizes these observers and every observer? What do they is do they will assign? They will assign a coordinate or whatever x y z and sometime to an even to something. So, if you have two observers, they will assign three, you know four different numbers to something and the question is, how will they how will they be related given the fact that, we have we have said, that they will agree on these forces.

So, what we will start out with is the obvious situation is, to consider thing were particle, which is being observed or whatever as no force acting on it, so the equation so that is so the for a particle that has no force acting on it the equation of motion. So, this will be a standard short form EOM for equation of motion is nothing but we have written a mass out here, which is what you have written, but I can just get rid of it, so this is the equation and this is something. So, let us choose a situation where this is observer one.

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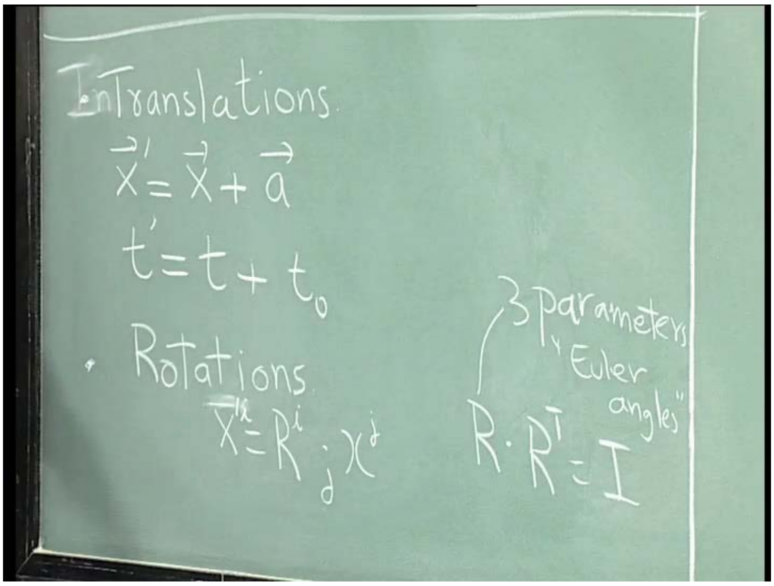

And let us say, observer two assigns coordinates x prime, y prime, z prime, t prime to the particle and this is they do not even have to agree on time, and the equation of no force for this person would be this, so now what we would like to ask is, the what is the most general relationship between x prime, y prime, z prime, and t prime and x, y, z in t, which is the 1 due to…, so they had both assume to be inertial observers and we have decided that there is no force. So, this is a equation they would write. (( )) it has to be t prime. So, let us sort of do it in part. So, this is the invariances or the relationship, so the first thing we could do is translations.

So, x and x prime could differ by some constant. So, more generally we could write something like x prime equal to x plus some constant, but again then may not even agree on some t naught, so this is translation in space and this is translation in time. That can be something more general, we could have rotations as well in other words x prime let us put some index structure to this x prime i if some matrix, r i j x j again Einstein summation, where r is rotation matrix, so to satisfies r dot r transpose is identity.

And if you notice I am sort of preserving something out here, I am saying that look i is an upper index out here it, I keep it as a upper index out here and j is a lower index and you can. So, i this j is an upper index. So, the rule the Einstein summation rule always through this course will be that we some over one upper and one lower index. And. So and it should be. So, this j is a repeated index well i is what is called free index.

So, this is how it is going to be you can ask, can I end up with a situation with two lower indices being the same. The answer is you are doing something wrong for instance, we will get back to, what these things is… So, we know that we could write this rotation matrix you could choose, how many parameters are there in this rotation matrix just one, anyone else. So, it has three parameters one possible choices something called Euler angles. So, far we have got one, two, three, four plus three, seven parameters.

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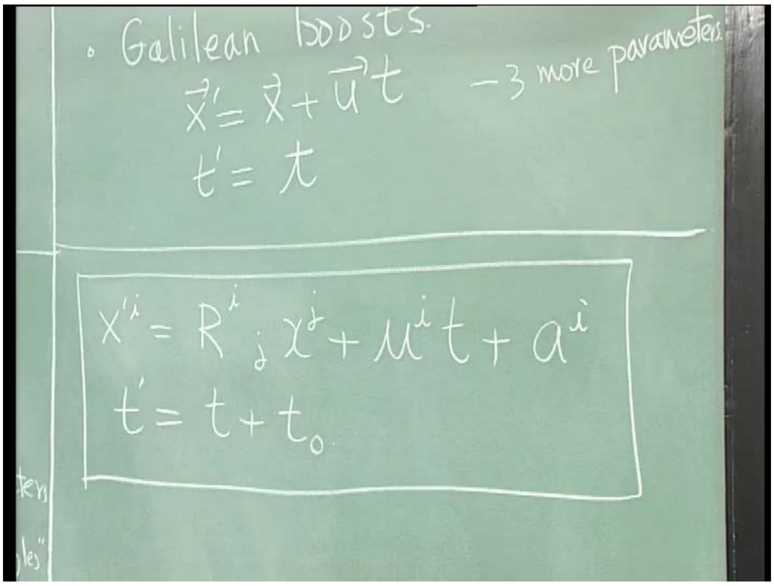

And now comes the interesting part, which we will call Galilean boosts, which is the following, so this is exactly the coordinates, which would u t and t prime equal to t. So, this just tells you that, this observer is moving with some velocity u with respect to observer two is moving with respect to observers one with some constant velocity u. And, it is not hard to check that since the this involves two derivatives it is still satisfy this equation. So, now this is again three more parameters.

In fact, there are quite a few more if the particle, where having no forces for instance there is nothing, which sets the scale of x or nothing, which sets the scale of time. But, we know that once you apply forces, when you go to situation with forces, you get a scale for x you get a scale for time. So, those kind of scale invariances, which is actually the symmetries or the invariances of Newton’s equation without a forces is much larger than, what I have written, but I am going to keep in mind, that they will be forces, which introduce scale in these things.

So, for instance m which we wrote here when there is the force acting on it you cannot. So, what we see. So, I will restrict myself to only these three things. And So, now you can see there are 10 parameters worth and of course, if two arbitrary observers are there, they could be it may not be just one of these things, it can be the most general thing, which could relate these two things.

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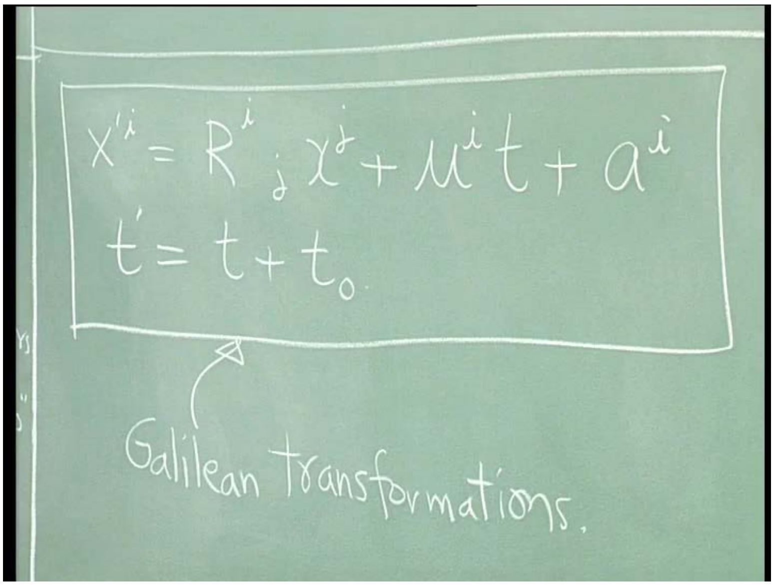

So in other words, we could write something like x prime, so this would be the most general sort of thing i equal to r i j upper. So, this is the most general thing and these set of transformation are called Galilean transformations. So, i started out here with a with a particle with no force acting on it.

But, you could do the following thing, you could ask, you can take shading us equation take the time dependent shading a equation and ask yourself the same question. But, just bear in mind that in quantum mechanics, the way function may not go back to itself there is phase freedom. So, you can use that phase degree of freedom. So, that is a nice exercise, which is the following, so I will keep giving you exercises and I expect you to go back and play with it, because that is the only way you can sort of learn and understand these things.

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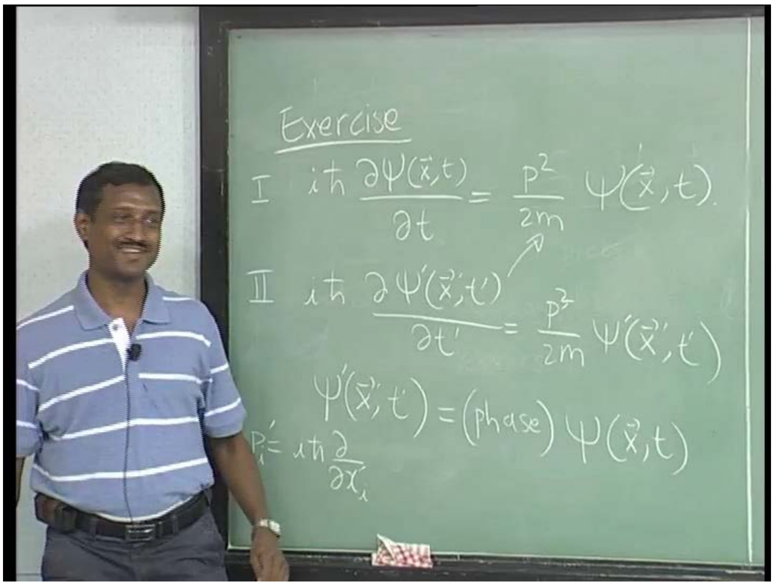

So, the exercises the following, so take a different equation in this case, i h bar d psi and x and t by d t equal to say a free particles, so let us just write it p square upon 2 m acting on. And ask the same question ask, when two different person to observe us will get the same will agree on this equation.

So, the point here is that, so this would be observer, one observer two would write out something like this, so you can replace p with del square the laplacian minus of the laplacian with some h bar put in there. So, now you need to work out, what is the map and really all you need to do is to realize, that psi of x prime let us put even a prime out here, because you may not be same function should be equal to after some phase, which you have to fix by the way. Now, comes the amazing thing the amazing thing is it looks like, we are doing a completely different equation right, you go ahead you will discover the same thing.

So, the invariances of the (( )) equation is the same as Newton's laws. Now, you can ask, let us look at there are equations let us say, how paper burns for instance. There are some beautiful exponents called the Kardar Parisi Zhang exponent, which come from that how paper burns. Go ahead and work out it is invariances and low and behold again, you will discover Galilean transformations taking various to equation, which tells you, how a fluid flows all of them give these things. But, intuitively at the end of day, it should agree why is it? Why should it agree? Suppose, it did not agree, suppose let us assume that fluid flow was such that it did not respect these thing, then we can think of, we can use the fluid flow to say for instance distinguish, but we might be able to measure velocity might be some absolute velocity or something, because there are two different setups, which differ in some way and then these two observers can actually differentiate something. So, here we took a particle there, you could I mean instance, we can take a fluid and see what is happening?

But of course, we also know that fluid flow, it comes back from taking many particles at the underline equations is still the same. So, there is something special about the Galilean transformations, there are 10 parameters. And the surprise here is, so we know one more setup equations, which is Maxwell equations and you can ask, what are the in variances of Maxwell equations?

And, there is something, we will work out may be later today or may be in the next lecture, because I just want to focus on this setup transformations, and ask certain questions about this (( )) this p, ya p is an operator now in quantum mechanics. So, in these thing here it would be the operator would be d by the, so p would be lets write it, this way p i will be I, so maybe I should put a prime here, if you want if you are comfortable with that, and one without this.

So, each person will write the equation in there frame, they do not care about somewhat, someone else is doing the question, we are asking is how would this two things be related, they should be related, so this an exercise for you to do. So, lot of fun fixing the phase and actually after you finish doing this things and you get the phase what you find is that the phase is a very nice interpretation said I could have written it before when I started you know. So, that is why I will not do it. So, right now what we did was, we took two observers and we worked out, how these things get related, but the key is we do not have to restrict our self to two observers.

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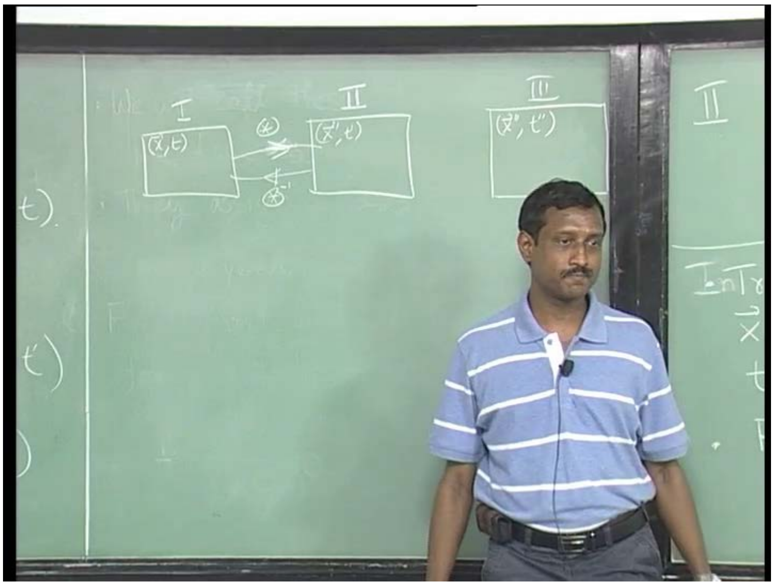

They can be millions of observers. But, we will simplify things we take only three different observers, we call them 1, 2 and 3 observer 1, observer 2, we already have two of them and we have a 3rd observer and let us say, that before x and t x prime t prime and the numbers of prime are increasing. So, we will put a double prime out here. So, these are the coordinates chosen by observer 1, 2 and 3.

And, what have, we said that x prime is, so let us draw this arrow this way. So, this you get x prime as a function rather, react this is the way it goes; so the way, this equation that we have out here star tells you, how to relate get x prime and t prime in terms of x and t. Obviously there is a opposite one, which we should call star inverse in some ways. But, they need thing is there is nothing, special out here this, the inverse transformation also should have form with the except the 10 parameters, that we see out here are different. So, you can work out, so exercise again for you to work out, what is the inverse. Now, what about 2 and 3 there has to be a similar transformation.

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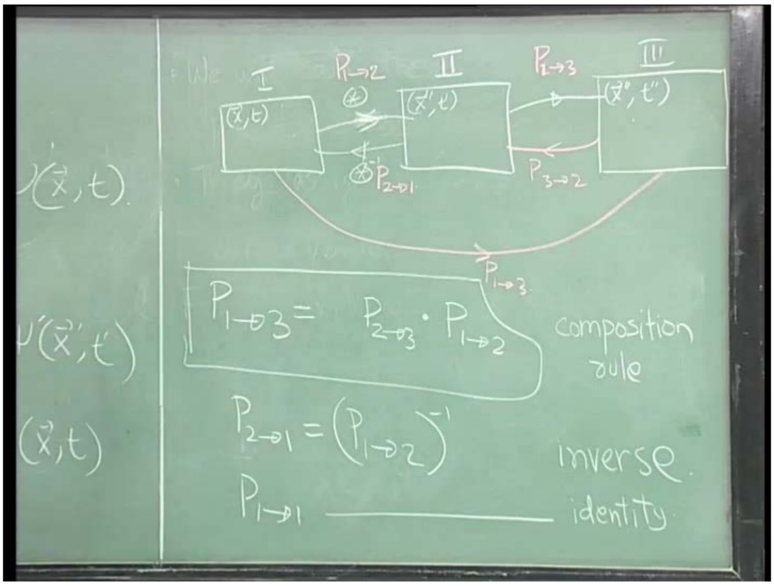

So, let us sort of may be, there is another way of doing this, let us say that this takes let us put all the parameters and say, and this takes observer 1 to 2, I will replace roman this thing with the normal numerals. So, out here r 1 goes to 2 for the parameter, this is 1 goes to 2 same thing out here, so this is the (( )) this what I would do. So, the parameter here, so let us just put all the parameters and just call it p for parameters 1 goes to 2. Similarly, here that should be p 2 goes to 3 and this will p 2 goes to 1. Similarly we will have p 3 goes to 2. But, now comes a need thing 1 and 3 are also in observers, they should agree with each other.

So, there will be another set, similar transformation with just the parameter changing and we will just call this p 1 goes to 3, there will be an inverse. So, let us leave this, but the most important thing is, there are two ways I could have gone, I can go from here to here, but I can also go directly.

So, we get a very very important property; it says that p 1, 2, 3 is equal to p 1 to 2 compose with p 2 to 3. It just needs follow the arrows and please remember that, because of the way mathematical operation said then the first one comes here and the other one comes here this is very simple. So, physics tells you that this has to be true. We will just call this composition rule. And there is also the other thing, which I just mentioned p 2 to 1 should be equal to the inverse of p 1 to 2.

So, now comes the… So, it is you can go back to the equation, start and keep putting these 1 to 2 labels and 2 to 3 and you can work out, what is the, how these setup parameters get labeled. It is a again an another exercise, which will come in your assignment, where you actually should work out, how the parameters precisely change?. It is not like they, it is a product of the parameters, it is a complicated one, because I am not doing if we just did two translations of course, they add up or if we did two boost together of course, they add up with the vector addition rule. But what I am saying here is, take the most general thing do not assume anything about this things.

And of course, there is a trivial one one goes to one which is doing nothing which will call the identity. Actually these three composition rules these three rules under like something in mathematics called a group, there is just one more property, which you need, which you require of fourth one which I have not. So, again the last, but it is there is fourth one and let us just call it, so p fourth one.

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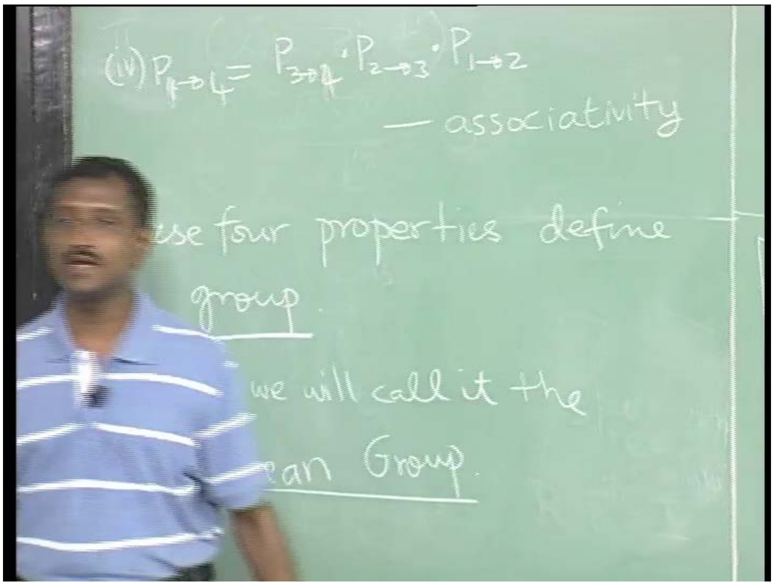

So, I am not drawing the other box, I am just writing it out p 4 1 should be p 1 goes to 2, say composed with p 2 goes to 3 composed with p 3 goes to 2. Now, the thing is the answer, should not matter whether I in first do this composition or if I do this composition, and what is such a property call associativity 3 goes to 4 thank you. So, 1 goes to 2. So, the order should not matter. (( )) thank you. It is 1 to 4 thanks.

So, this should not depend on the bracket. So, there is the obvious property, but we should. So, the amazing thing is that these four. So, let us call them 1 2 3 and 4 these four properties define a group, this the mathematical, the thing will give a formal definition later. And in this instance with this rule, we get here because it is obtain from Galilean transformations.

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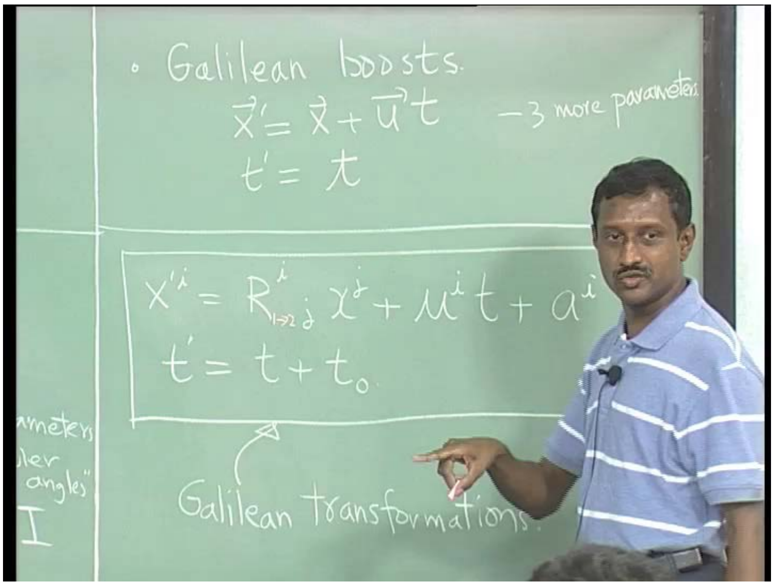

Just one more point is that, here for instance I could have put even, if scales are fixed, I could I mean I did not put a minus sign here, this would remain for instance this would correspond to reversing sign. Similarly, I could put a minus sign out here also. I thought of being very loose, but usually in terms of time, we always agree in terms of how time would increase. So, you want to observe us to all agree as to, what is the positive direction of time? No one does not usually consider that.

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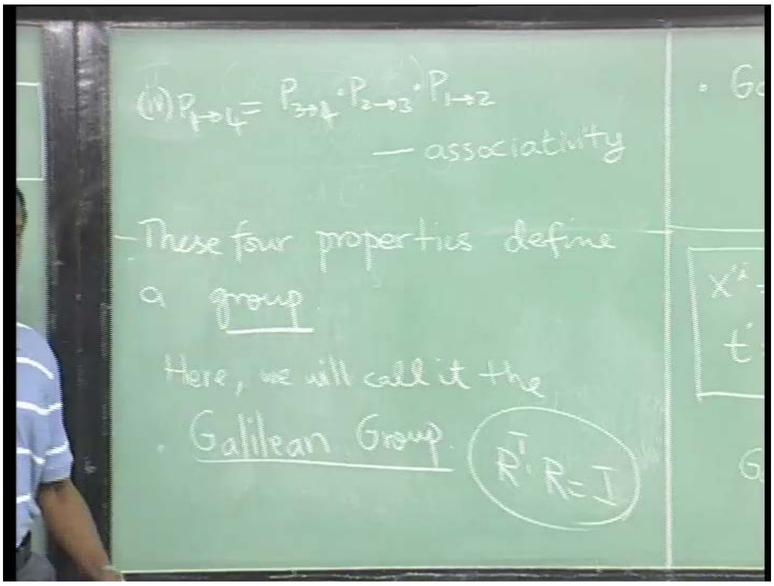

But, we do consider situations, where we you would put a minus sign out here, but that is actually more or less; this was the condition, that we had put on these things. So, that will allow for these things. So, this is the example of the Galilean group. So, you can see that, the simple idea that started out with that, we expect inertial observers to agree on their forces. Let us do something, which is far more general than the equation, we started operate. So, are there any questions? Ok. So, what we will do now, is to gear ourselves towards asking, the same question for Maxwell equation and what we will find is that you get something different.

So, that is the first instance, where there is a conflict and this is the theoretical conflict, which can be only resolved by experiment. And like I said there are enough examples, where this holds in we have only one sort of sort of equation, for which is does not hold. So, you may things hard, this is the worst guy I may have forget about it. But, there as I mentioned to you light is somewhat different, max i mean electromagnetic radiation is also different. The only example, where we have of a velocity dependent force, which is also a conservative, which already sort of tells you that there is something special about it, but at this point, we are just going to be very nice, we are going to say fine, let us ask what are the invariances of Maxwell's equations.

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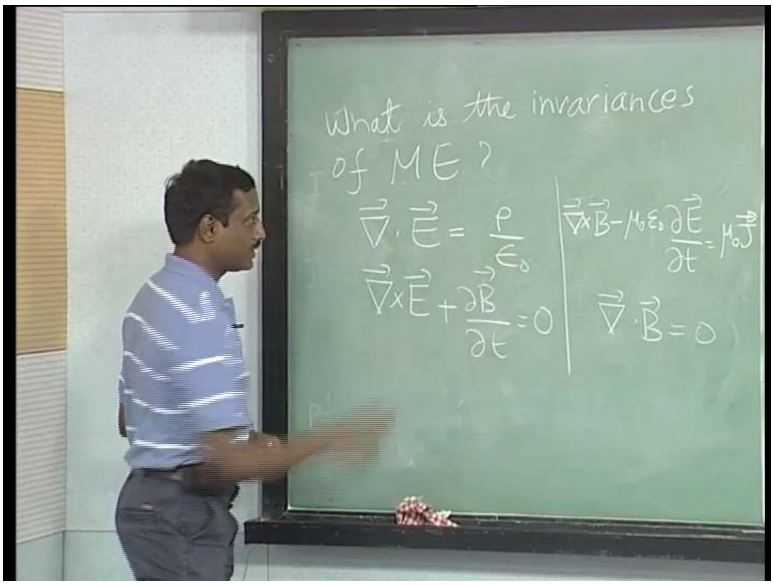

So, in order to do that, it turns out that Maxwell's equations in s i units are not the best suited for getting its in variances. And so we will get to writing it in Maxwell rewrite Maxwell's equations in a slightly different max ME stands for Maxwell's equations. So, let me write it out for once, in s i units, so I am writing it in free space. Hopefully there are no sign errors. So, you can see that, I mean most of you would be used to, writing these things on the right hand side, but there is a sign, there is a sign error here and this should be minus. So, there is a difference between the top line and the bottom line.

The top line have sources, so this is the charge density and this is the current density, while these are just this hold independent of whether, there are sources or not these are the source free Maxwell's equation and these are the once with sources. Yes. (( )) Where (( )) Here Yeah one by (( )) Here good thank you.

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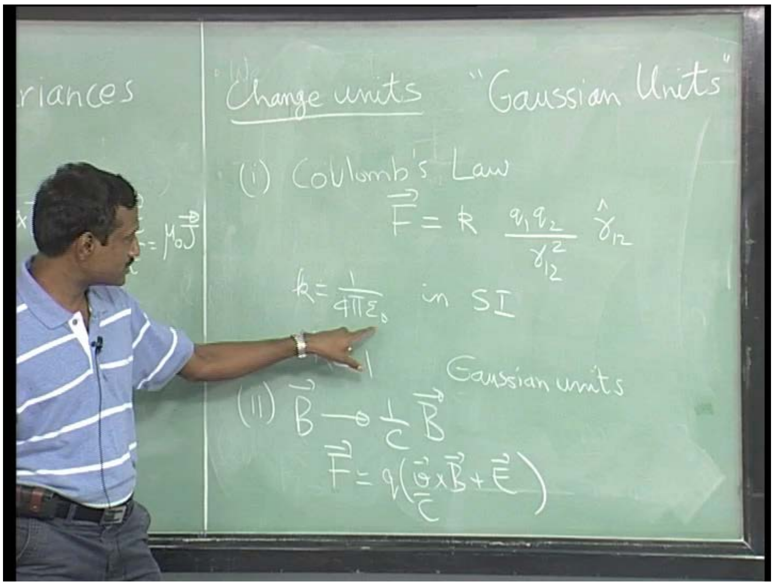

So, what we will do now is to change units, we go to what I call Gaussian units. So, this is call the Gaussian units standard in any e n m book, will have it at least in the appendix. But I will tell you, how it works? So, the first there are only two steps, we need to do really; the first step is to write coulombs law for a, so this is the force between two charges, so 4 pi. So, r 12 here is the is the vector, which connects 1 and 2, so this k is the constant; so k is equal to 1 by 4 pi epsilon naught in s i units. In Gaussian units k is equal to 1 in Gaussian units. In s i units charge was measured in coulombs, but in Gaussian units it measured it stat coulombs. So, this is the first step.

The second with is that, what we will see later is that both electric fields and magnetic fields can makes in some ways, and so you need both of them to be measured in the same units and that is still not true. So, you what you do is you make the replacement, I am just making it as a replacement, but you can make it formal, but wherever you see b you put. and the way to see that, this is correct is to yes use something very elementary, which is the Lawrence force law, which in s i units was just q into v cross b plus e, where c here is the speed of light. So, if you make this change, there is to put wherever you see b you put b upon c, now v upon c is dimension less. So, you can see that e and b are being measured in the same thing. So, now we can go back to this equation and start modifying things to make it look like the same things.

So, what I am going to do, now is to use the same equations, but I will just input these things. Now, you can see that, there is a simple trick of going from this to this by just say noticing that, wherever I see a epsilon naught, I just put 1 by 4 pi. So, I see an epsilon naught out here. So, what did it, I do, I go ahead and put one by 4 pi. So, I get a 4 pi this equation, I just put a 1 upon c. Yeah.

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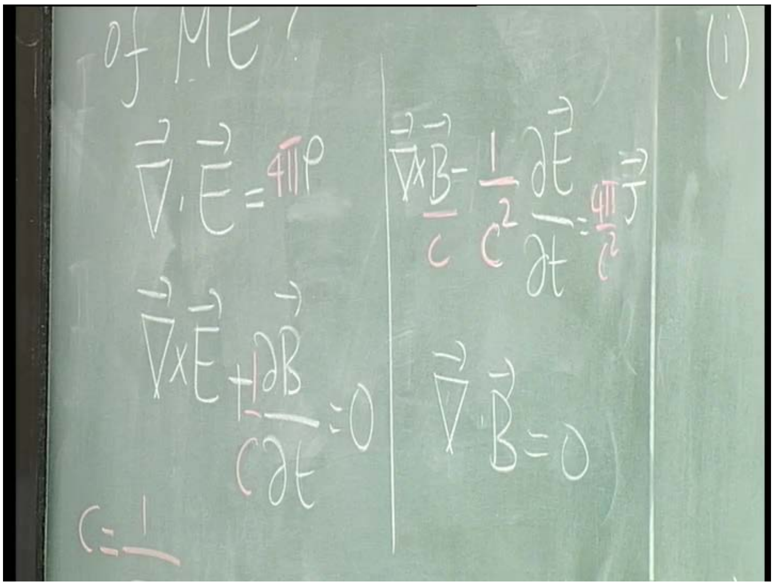

So, just one more things, which we need is c naught; c is nothing but 1 by square root of mu naught epsilon naught. So, instead of doing various things all, I can do here is to replace this with 1 by c square. But, we already know that epsilon naught should be 4 pi. So, this would just tell you that, mu naught square mu naught epsilon naught equal to 1 by c square, but epsilon naught is 1 by 4 pi. So, this implies that mu naught in is 4 pi upon c square. Now, let me use the other color and nothing is to be done, here 1 by c, but here I need to a one more 1 by c.

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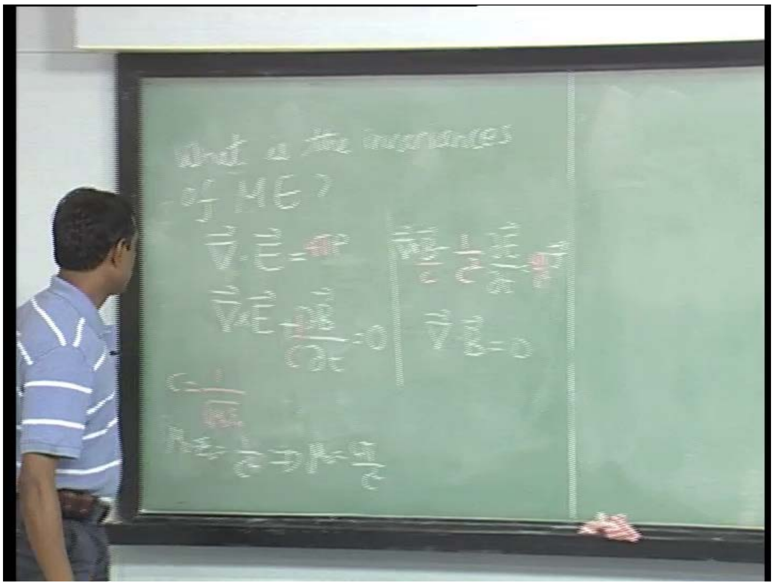

So, you can see that, what I have done is it just it looks like the changes are sort of very simple, but let us see what we get, so I am going to now rewrite the whole set of equations.

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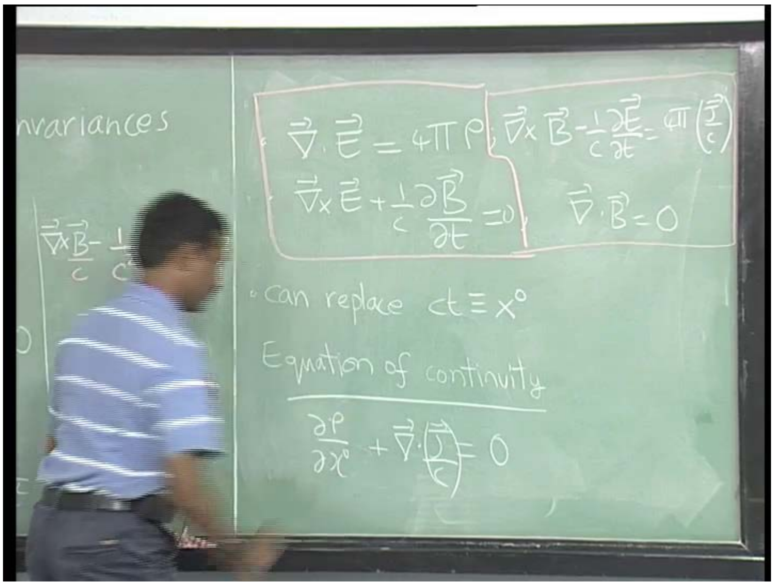

So, now you can see out here, I can just cancel one factor of c out here, and then I get. So, this is Maxwell’s equations. So, let me use this color to separate these things, so this is Maxwell’s equations in Gaussian units. There are already nice things that, you see out here. For instance, we look out here, we see that wherever t is coming it is coming in the combination c t. So, we can replace c t with something, which I will just write as a x naught, I will define this combination x naught, then c disappears from this equation and out here, there is only one factor of c, which remains out here, but actually both rho and j satisfy an equation, that is a consistency condition, if you wish of Maxwell's equation.

And what is that called the continuity equation. So, let us look at the, so let us try to understand that factor of c. So, you have the equation of continuity, which says that d rho by d t plus. But, if you wanted to write in terms of x naught, you have to you have to put a 1 upon c out here. So, what you see nice in a nice manner is that, either I could write j upon c or I could multiply this guy with ah. I could either divide j by c or you multiply rho with c, I mean you have that freedom, but in some sense there is, so you can see that this becomes now. So, now we are sort of almost ready to work out, what are the invariance of the Maxwell's equations.

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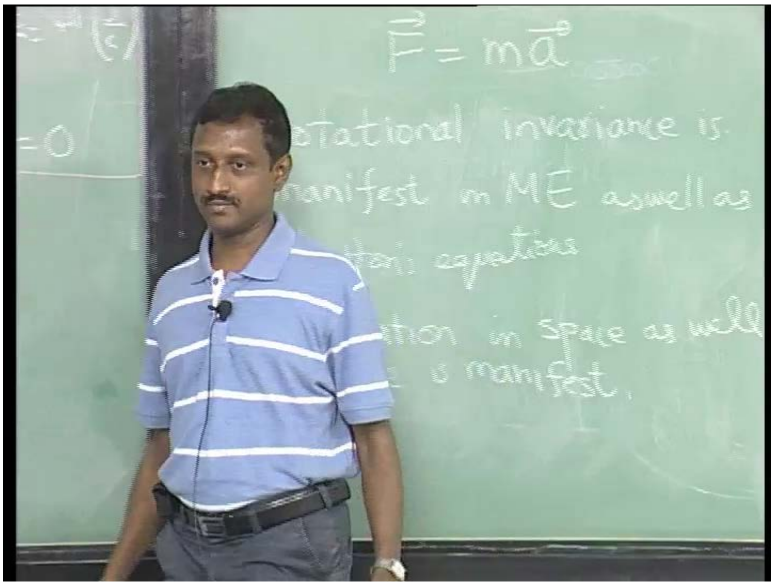

And in principle, I could do it at this point, but there is something we could do, which is analog is to what we would do, when we looked at an equation like this, when you look at the equation like this, we say that the left hand side is a vector, but what about the right hand side m is taken to be vec scalar and a is a vector; so product of a scalar and a vector is a another vector. So, this equation is a vector equation. In fact, this is true about all these equations. So, this the first rather this equation and these two are vector equations, while these are scalar equations, while these are scalar equations. So, if you want to count, these things you have 10 such equations.

And so, the advantage if writing it in this form is the fact, that you rotational invariance is manifest in ME as well as Newton's equations. So, one obvious invariance of Maxwell's equations is obvious, is there even before we did anything even before going to Gaussian units, then has to do with the fact that it is are all these are all nice equations under rotations. So, we know how they transform? But, we also what about translations, you can see the there is no explicit dependents on time or special coordinates in any of these equation. So, translation invariance space as well as time.

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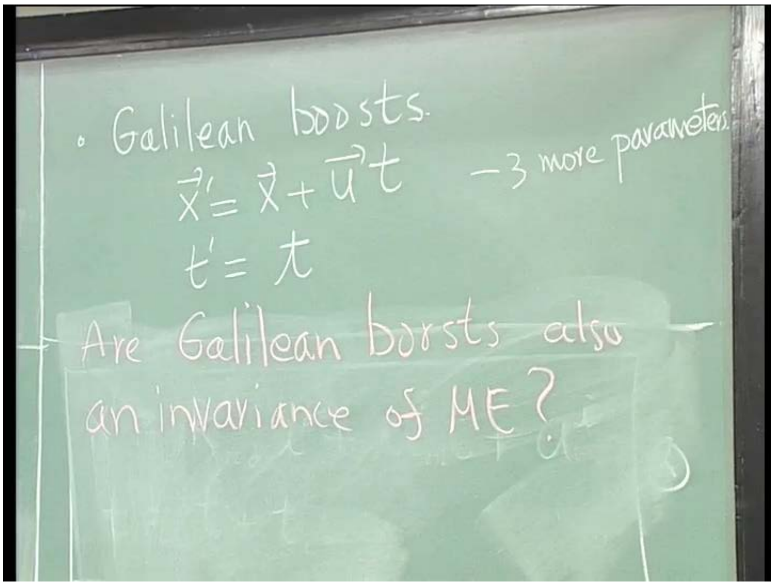

So, it only leaves only thing, we need to check is it in variant under Galilean boosts. So, an invariance. So, we will see that it is not, but we will work it out explicitly in next lecture. But, what we will do, now is to use a third experiment to show, that it cannot be so . And I will use make, I will make use of the fact, that the Lawrence force law is velocity dependent. So, consider observer one, who sees let us say that, this is a charge particle and I am observer one, I am standing out here and let us say that, we are in a region of a constant magnetic field. So, observer one is sees the charged particle address says, there is no force active on it.

What about observer two, Let us take observer two to be moving with uniform velocity; that means, with a that a Galilean boosts in a has a Galilean boosts. So, it is moving with uniform velocity with respect to this. So, suppose he is moving in this direction, he would perceive this stationary object for me, as if it were moving with a constant velocity in the other direction. So, now he would perceive the following thing, that there is a force acting on it, and what is the force? That is just the it is a charge, that is a magnetic field and there are there will be a force, which is u cross b acting on it. But now, that really conflicts with what we saw earlier, that the these two inertial observers should agree on the fact that, there is no force acting on these thing.

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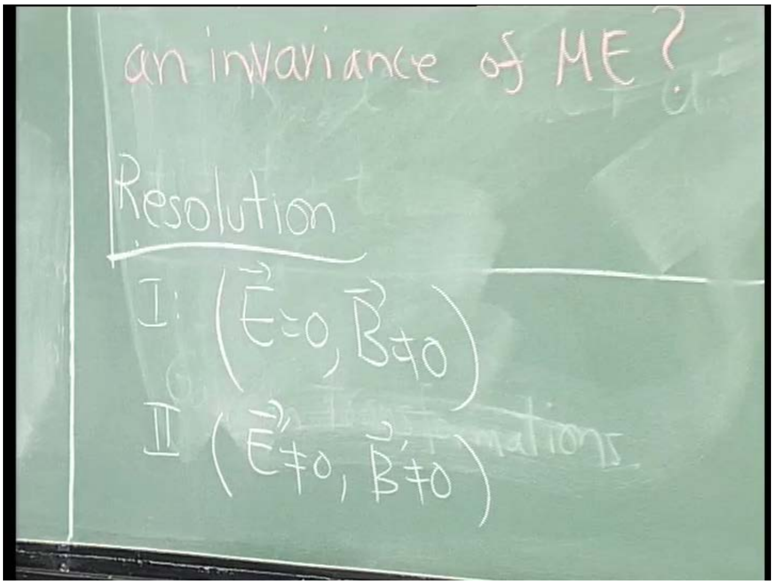

And there is resolution to this is that, the there is a resolution to the third experiment is very simple is that e, so we are in a situation, where e is 0 and b is non zero in frame one. In frame two, you would see that e prime is not 0 and d prime is not zero. In other words, what happens is that what was the situation with only E B magnetic field in here, becomes map situation, where b and e field makes. So, in some sense well you may think that it does not conflict with Galilean boosts.

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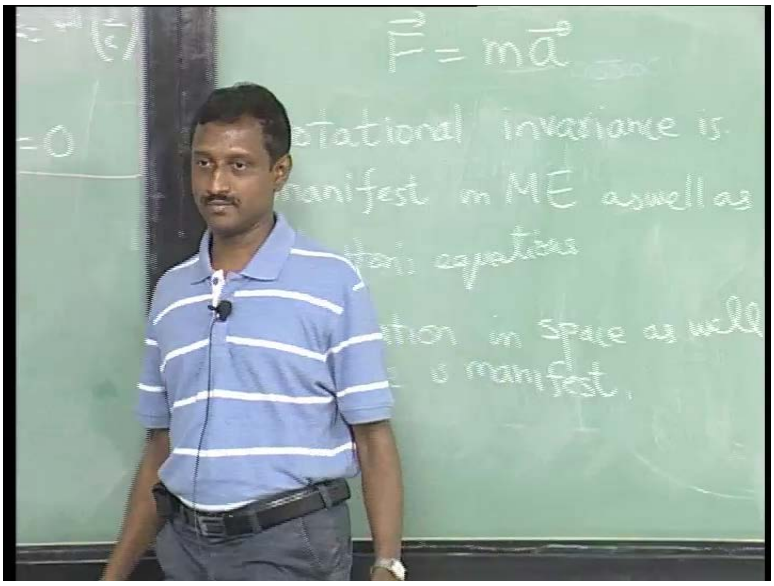

But, the answer is that it does, because we have to prove that, these equations are invariant. So, what we will do in the next lecture is, to rewrite this equations in a manner, where it would be invariant under something larger something different. So, we will find the Galilean boosts are replaced by something called Lawrence boosts. So, our goal will be to work out what Lawrence boost are, but I will do it, I will do it in a easy manner, just as the way we define rotation group as r transpose r equal to identity, we will see that there is a similar representation of Lawrence boost.

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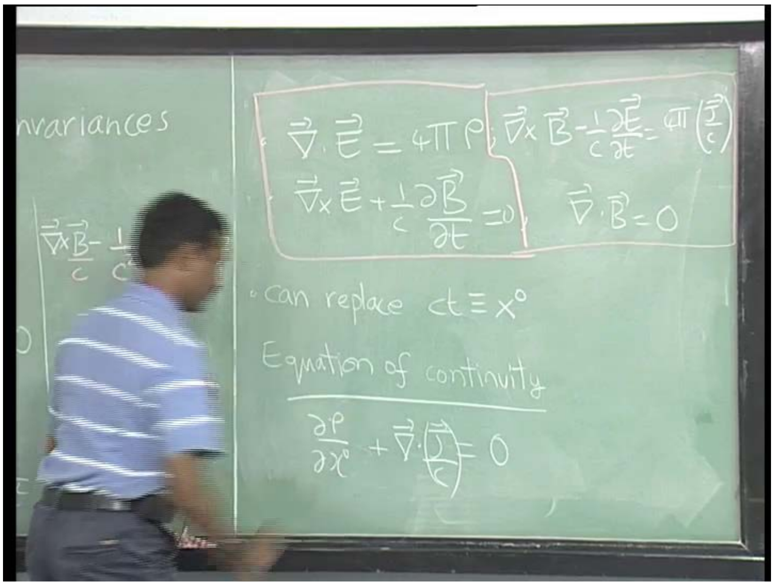

But, the key point here is that, already you start seeing here that a 0, which we define out here is now, time is being measured in units of length. So, it exactly, what we did for making E and B to measured by the same things. So, it is kind of tells you, that there can be a situation. So, where both at 0 mixes or time mixes with the coordinate. In particular this particular thing, we always thought that there was some universe something universal about time.

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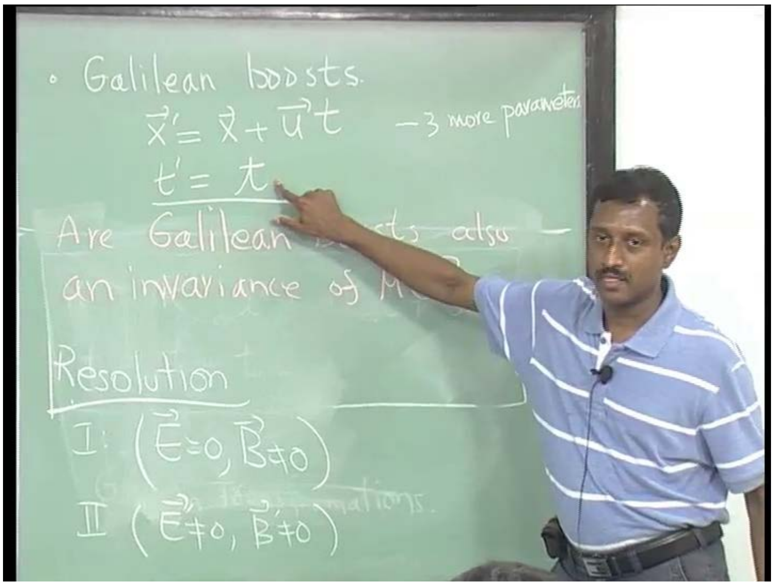

Time just played a spectator role in these whole thing and we will find something much more complicated then this, but there will be exactly three Lawrence boost, just as we had three Galilean boost, again because it is the invariance of Maxwell's equations; it will satisfy a group property and we will call that group the Lawrence group. But, now the thing is it looks strange, now that there is a problem there are some sets of some sets of objects, which have a symmetry, which is a Galilean group and another set of object, which have a Lawrence group. Obviously one of them should get corrected, and what experiments tells you that it is a Galilean boost, which get corrected.

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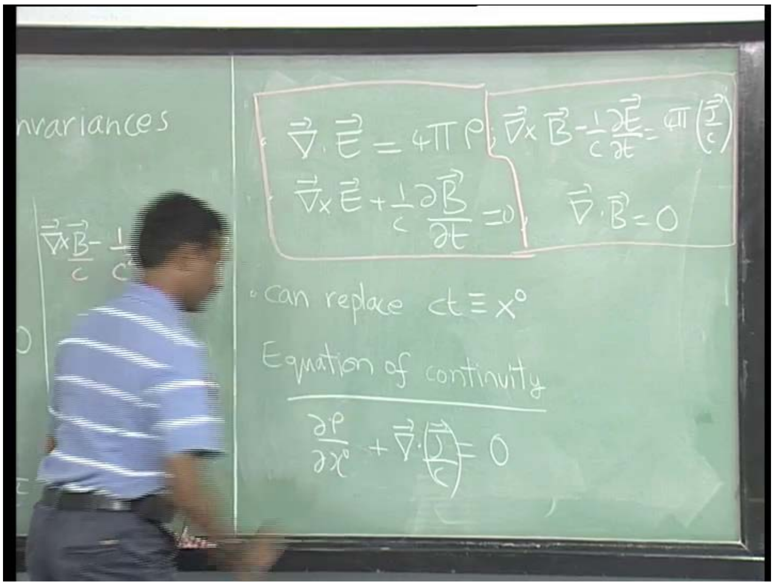

But, there is also a beautiful mathematical structure, which relates the Galilean group to the Lawrence group, and it is a it is called contraction. So, the contraction of the Lawrence group actually gives you the Galilean group. That is the mathematical statement, but the physical statement is that, it is not as if special theory proves Newton's equations are wrong, it just says that it extends the validity of Newton's equations to a much larger regime to velocities, where velocities become very very large.

So, it is not that Newton was wrong, it is just that the validity of his equation are much more limited. So, science always progresses in this fashion, you never go there and there is nothing radical, where you go why put everything that was known to mankind before, that is total nonsense, what happens is that you add something to what was existing there. So, when you become future in the future scientist, and doing something be sure to see that, you know you your results are compatible with, what people have done.

People even the people in the past are not completely idiots, they are but the, but the point is that, there is some method. This is called the scientific method and. So, this is. So, we will see this captured in this simple example, for which we know the revolution. But, we will in the future go to situation, where things are much, where we do not know the answers. And there you there is this condition, which you are imposed that there should be some limiting things, which you by which you by which you require earlier results. Thank you.