Classical Field Theory
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Lecture – 18

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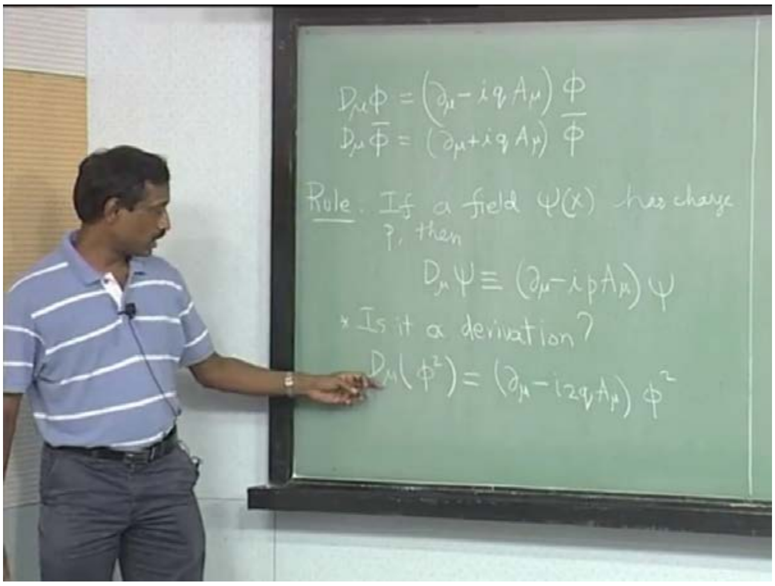

This is how we defined it last time and of course for the complex conjugate field I said it is better to think of phi bar as something which has charge minus q. So, so the rule is that if a field or if a field psi of x by that I could be a product of fields it could be composite it need not be it could be phi square phi cubed has charge p and the covariant derivative D mu of psi which has defined to be… So, we need to check, if two things we need to check. We, we needed to verify is it a derivation? Is it a derivation that is Leibnitz rule, is it valid?

So, I gave you couple of examples hopefully you have checked out. So, the first ones for instance was if you take psi to be phi square, according to this rule it should be equal to D mu minus i and charge of phi, phi square will be 2 q if phi has charge q it is 2 q. Now, let us see if Leibnitz rule works, here this is just using psi as a single object which has charge q. So, now we can Leibnitz rule will tell you.

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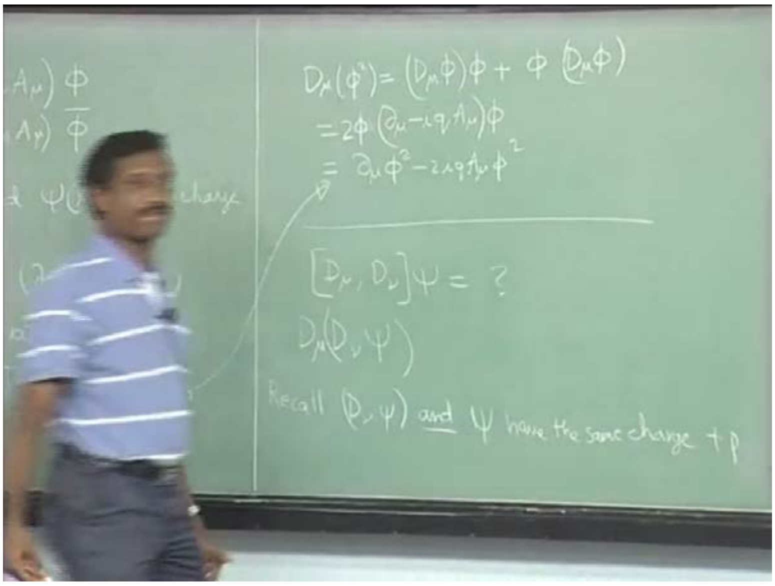

The D mu of phi square should be equal to D mu of phi times phi plus phi times D mu of phi. So, this is just the standard Leibnitz rule which will now, but this is just our initial definition, we can just check that this has charge q. So, we will write D mu minus i q A mu phi and that could be other phi on this side and it is not have to see that this is 2 times that. But this is exactly equal to this, how do we see that? D mu 2 phi D mu phi is nothing but D mu of phi square, and this would be just minus 2 i q A mu of phi square. So, this is what this is right. So, Leibnitz rule is valid, but I like I pointed out the following thing is that the, the covariant derivatives do not commute. So, that is the property which we give up by going to the covariant derivative.

So, we have to check so let us instead of picking you know one of these things I will just choose an arbitrary fields i with charge p an act on it with D mu and d psi and ask what you can So, what, what I want to compute is the following object. So, the question is what should this be of course, there is one situation where it, it would be 0 that is when psi has charge 0, then the it becomes the ordinary derivative and we expect things to commute. And the standard practice in this is not to go ahead and work this way what you do is to just compute D mu D nu of psi. And we can do it in steps; first point is that by definition D mu of psi has the same charge as psi, because it, it does not mess up the property. So, we need to you remember in this case, this psi has charged p, so they have charge plus p. So, let us write out what we would write here.

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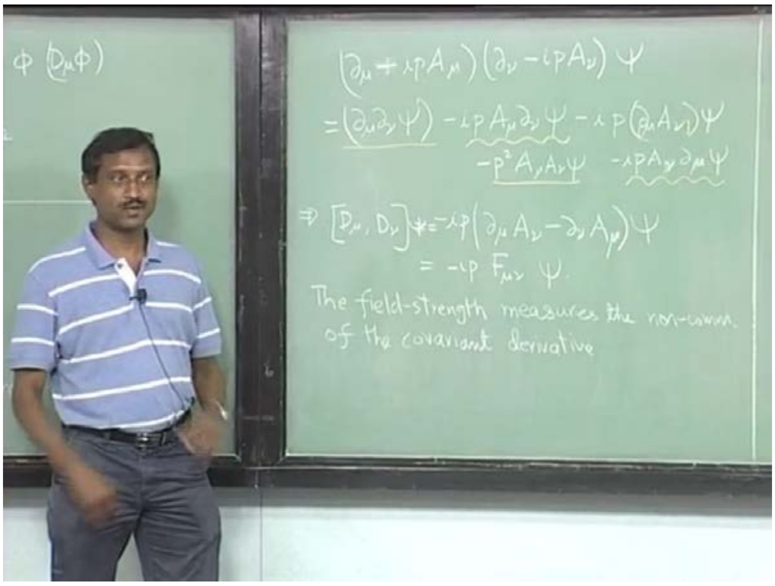

So, we write D mu plus rather minus i p A mu acting on D mu of psi which D mu of psi which again may know what its acting on something which has charge p. This way of writing it is little bit dangerous in the following senses some ambiguity, because this D mu could I mean, what I mean by this is this think of this as an operator. So, it acts on everything to the right of it. So, these derivatives here will act on not just psi, but they will also act on the end A nu A nu.

So, let us write out the various terms, this is one term, and then there is another term which goes with this. So, we are done with one set of terms. Now, let us look at the next term. So, here when D mu acts on it you actually get several terms you get 2 terms; one of which is minus i p D mu A nu of psi. Then let me write it below that minus i p A nu D nu of psi that is this term when this goes and acts on this. The next one would be when this acts on this, and that is just minus minus will give you a minus p square A mu D nu psi.

So, I have written off all the terms there are 1 2 3 4 5 terms, but now the key point is we just wants to takes the commutator. So, it is just simple to anti symmetries by, by seeing things and just look at terms which has symmetric and just through them out. So; obvious, symmetric terms, if we anti symmetries this would go to 0 so would this. So, I do not need to worry about these two terms it is just the other two terms I need to anti symmetries. So i anti symmetries this guy and then I also, somewhere I have missed something here, this is so now even if you look at this; this is a this is a symmetric combination of these two guys a this is just mu nu exchange them with the same sign both of them come with minus i p.

So, let me so they all these things which I have underlined are 0 here these this by itself is 0; this by itself is 0 this combination is 0 the sum of these guys So, you can see that so this implies as only one term which is this guy, and let me now explicitly expand it out. But what is this combination? We have seen this before is F mu nu it is a field strength electromagnetic field strength. Now, we have something very nice we understand the field strength as, so field strength actually measures the non commutativity. This is actually a very deep result when we go to more complicated situations which we will see in this course when we go to the situations where the group now right now we started of the S O 2. But we could have started off with any other group, but we will see that there is exactly this sort of a setup.

And we will define a covariant derivative, we will find out the, we will see something which comes out to be the object with measures the, I mean how, how non commutative the covariant derivative is. And we will call that the field strength and that will be the definition. And one nice thing is you can see it depends on the charge. So, you have to pull out the charge here. Similarly, in those cases we will find that they depend on what it acts on. So, here it remembers this p is something intrinsic not to the commutator I forgot a psi out here. So, intrinsic to this, but it is actually a measure of this thing. So, this is independent so you have to separate these two things out. And so the object that you get here is what you would call I mean there are i is in minus signs etcetera that is not so important.

And that you can always say that is the definition, but we will get a proper definition later. But this is the most important point you get the field strength in a natural fashion, is this clear? So, yes, but that just.

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So, you can ask, how does F mu nu transform under local S O 2, so which is the same as the u 1, if you write it in a complex basis it is called u 1, u 1 transformations. And again you know this if A mu delta A mu is D mu lambda then. So, it is easy to see that delta F mu nu is 0, but this you know already from electro magnetism, but this is very special through the abelian which is S O 2 all group elements commute has to do with this only for the abelian case this is true. But in the non abelian case we will see that delta F is not 0, but it is the next best thing to not be 0 in the sense that it will transform nicely; which one? But that how does that helps? Yes, you are saying by yes so yes I agree.

So, the statement here is we know that the left hand side transforms nicely if psi has charge p left hand side transforms like something which has charge p. Now look at the right hand side, psi already transforms like charge. So, whatever it is multiplying it should not transform, if it transform, then it will change to the charge or something. So, it does not transform, but absolutely right this can be; this can be. I just have one common in fact, I claim that all of you have seen the covariant derivative before, the context is the coupling of the electron or whatever through a magnetic field what was the Hamiltonian? We have it was in the one of the problem sets also the Hamiltonian was p minus e or let write it e A by c whole square up on 2 m. And in quantum mechanics you know that you, you replace.

So, when its acting on this thing this combination p minus e A over c p would become the operator i h bar del minus forget this c let us put c equals to 1 and even h bar equal to 1. So, that we do not have any of these things minus i h bar right this minus should be there minus i del minus e A, if I pull out i e s etcetera it will look almost like that up to some signs it is not I mean I think there is a sign difference, but you can see that it depends on the charge So, in some sense this is the origin if you wish. So, in fact, there was a question outside of class or people asking me is this, what is this, how things were discovered? I do not think so I mean already in quantum mechanics we have seen things like this. And then when you go to first thing is you make the next step to relativistic quantum mechanics there you replace this with the four vector kind of thing and the A is already a four vector.

So, you can ask what p mu minus e A mu would look and the derivative form, and then you would see naturally something like this. But there is something slightly I mean we are not here there, there this e out here is the electric charge it is a coupling while this q out here is just a number it is an integer, because we said that it had to e power i q alpha. So, q was necessarily an integer so there is a little bit of redefinition I need to do to bring e back into the story that I will do a little later. But right now what we should do is go back so now we have we know we have checked that covariant derivative that we have defined has the properties the promise properties.

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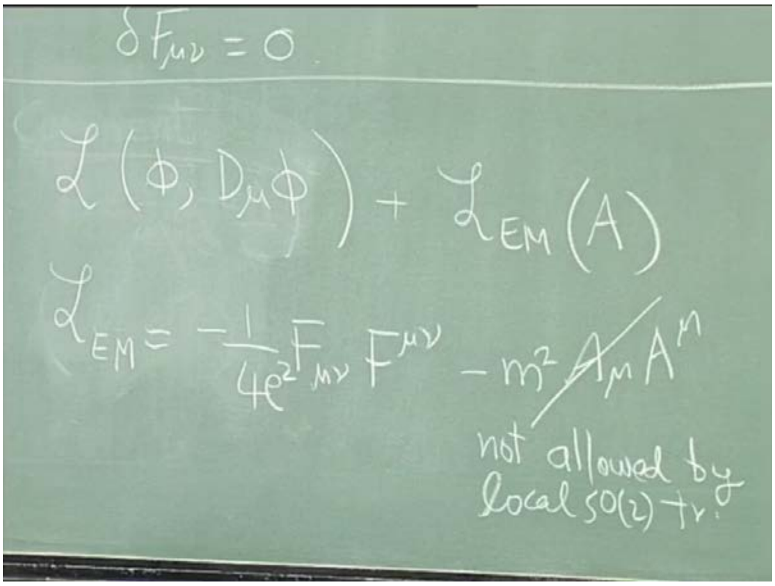

So, now we have a nice Lagrangian density which is a function of phi and D mu of phi and it is complex conjugate, but I will not I mean just write those things which is invariant. But actually this is little bit of a fraud it is fraudulent, because this has a D mu as a field in the thing in the story. And in principle you would like to say why not have a kinetic energy for that for that field the gauge field the vector field? Why not add kinetic energy or even a potential energy for that why not? And there is no reason not to do that. So, you can write the analog of L E M of A I am using E M you have to remind you and so what would L E M be it would be and in my convention I will put an e out here I am do I need an e here no yes, yes I need an e this is my e. This e is supposed to be the electric charge and q just measures everything in terms of the electric charge. In field theory electric charge is a coupling constant it is like m, m square lambda in the potential these are like terms like that similar to that you get an e square.

So, this is what you write and you you the key point is you, if you try to write a mass term suppose we try to write a mass term the natural mass term, you may think there is a sign error out here there is not it has to do with our sign convention. And you would write a mass term you could you might think I could write something like this which is a mass term, but this is not invariant at the gauge transformations or local S O 2. So, this is not allowed, but local S O 2 which you also called gauge transformation.

In fact, no potential is possible and we always restrict ourselves to two derivatives there is not much this is it, this is the only term you can write if you need. If you allow permit high derivatives there are there are there are many more possibilities etcetera. So, you can see that in fact it is a very tight fit if we are given we started out with a system which was just a complex scalar field which had local S O 2 invariant, a global S O 2 invariance what we did is we, we want reasonless. Ask the following question can we make it locally invariant we found that yes modulo adding this extra gauge field through the covariant derivative, but nothing tells you that, that cannot have dynamics So, you can put this whole thing.

So, you end up with a nice system which has a where you find that the interactions or the coupling between the phi and the gauge fields are fixed for you, you do not have freedom in that. This is that is why it is called minimal, you could have done more exotic things you could have you can you can do but this is kind of the minimal thing that you can do to get a locally invariant starting with a global, the amazing thing is it this, this kind of structure goes through for any group, any global symmetry you want if you want to make it local you can just go through this process. So, now let us sort of play with this and go back and lo at this particular Lagrangian let us expand and see what it contains? It is kind of a fun.

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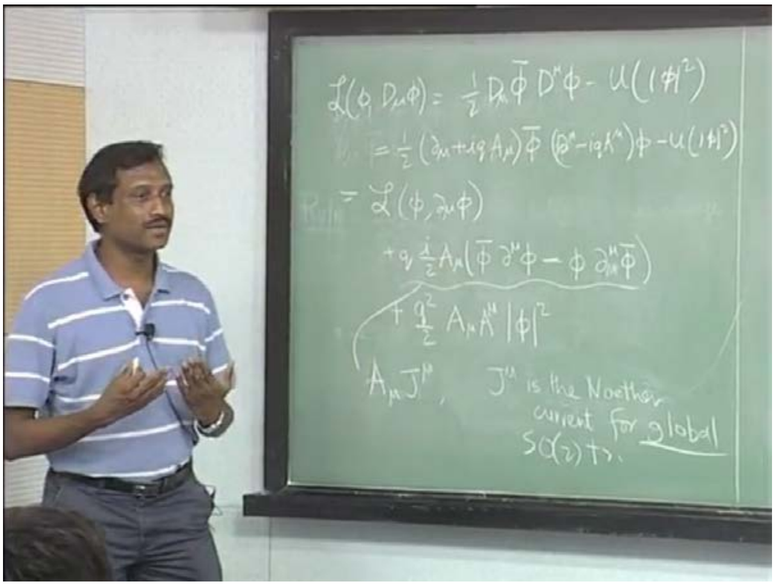

So, the matter part so the Lagrangian for this is what you have now it is under this. So, what I want to do here is to write everything out and expand it as a power series and m u So, there is 1 m u buried here there is another m u here. So, there will be a piece which is independent of that, then there will be a piece linear in that and there will be a piece quadratic in that and the linear piece is the interesting one. So, we just, we write this out let me, so first step is at 0 this thing so that you have L naught phi, the linear piece there will be two pieces; one from this guy and the other from here. So, let us write that out. So, let me start picking out this guy plus so this is the linear piece and then that is a quadratic piece which is just this with this. Now I have a question for you have you seen this linear piece any where? Probability, but that we got also as another current fall so you can see that this piece if I write this thing as J mu I write it as this where J mu.

This is what we got i by 2 phi this thing or if you write I wrote it in this fashion also, you get all these factors everything will work out, probably at that time I just said q would be 1 it should to match that thing, but that is not, so that is not a big deal. But this is something related to a comment I had made when we discussed on the currents is that the you could always sort of act like the parameter was local and then you would find it as the co efficient of what was going with the. So, how does this transform? D mu of alpha, so I said that that was the term which should measure and this is true. So, you can take it as a general statement that always the linear coupling actually would be this way. In fact, there are there are I know of instances where this kind of method if you have some symmetry you could actually start out which you want to make local.

So, you start out with the global part, you write the linear piece and then of course, you would need the next order piece you want to fix. So, now you go I do the variation of that thing and then you will find that you need to add one more term in this case it would terminate here. But there are examples for which it does not terminate ever it goes on to infinity and you keep doing these things. And people have used I mean sort of very clever tricks I have seen clever tricks in the literature to push things beyond the when you have non-linear completely this is at least quadratic at it ends. But you can have situations where it goes on and on then I mean if you want to get the full closure you can you can prove the existence of closure may be. But but here it is you used to sort of find this other piece among in fact, lot of things about the S O 2 case is really very simple we will see later that there is not in some, some instances there has not much of a distinction between infinitesimal and finite we will see that.

So, that is that is one part of the story and another thing which you can see observe out here is that this piece. This piece has A mu A mu kind of piece and this of course, its multiplied by this phi square and we know that this this is completely invariant under local gauge transformations. But just bear in mind that this is similar to this except that is a phi square so but this was not permitted, because it violates, it violates what do you call gauge invariance but we are able to generate something like this now do we.

Now, you can see that suppose if are in situation where u of phi was such that was such that the ground state had a spontaneous symmetry breaking so phi equal to a was some finite number was this thing. Then you can see that I can get something which lo like a square. So, the mass so there it would start looking like the, this object if this is a gauge field this, this; this cannot have a mass we as we discuss that but you can get something which los like a mass. And it appears I mean so this leads to what is call the Higgs mechanism which you will discuss in other probably 10 minutes, but you can already see that structure out here.

So, it just very, very simple example I, I resisted the temptation to call, call it by what people refer to it is in news papers. So, I will not repeat it but I am sure you know all heard about it so what it is called but this we called. Now, let us just sort of before, so this is the structure of Lagrangian let us ask what the, what the equations of motion look like which in go back and ask what are the ground state etcetera? We can ask these things.

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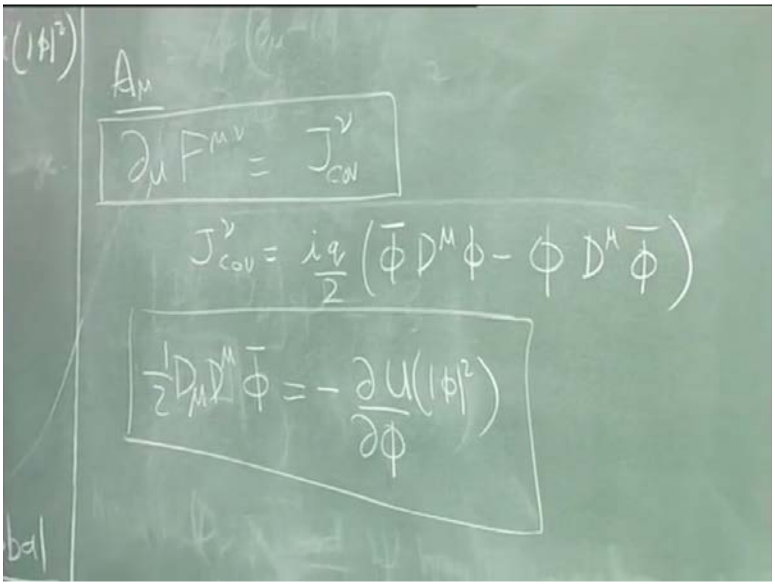

So, equations of motion by the way when we when we wrote Maxwell's equations we wrote the we wrote the equations of motion as something like this right D mu F mu nu equals to some I mean factors are not so important equal to j nu. In fact, in this is if you write out the analogous Maxwell's equations for the gauge field you will get exactly that because it is a first variation it is coupled to J mu. So, you can think of this as J mu and you can see that we actually get the source part. So, from the view point of the of the electromagnetic field or the gauge field this the this coupling is just this is like this is really like the current which you will get an extra you will get one more piece out here and even that is interesting.

So, I will just write out what I mean by that I will write something called and what is that? So that would be it should not matter, but where will, will slightly differ from this so let me just write that out. So, this if you or if you look at this, this object does not transform nicely under, under local gauge transformations, but this does. And this transforms like something which has charge 0 in which so I do not need to do any deep thing to see how this will come and fix that. This a power of a, what I what you called covariance. So, what the, what you do is you write things in terms of objects which transform nicely and then proof become much easier. So, the hard work goes in defining those things and then you are done. So, here you can get this so this is the equation of motion for A mu, now I have do the equation of motion for phi.

And gain I should tell you there is a nice method of doing this, you might think that how do you do this have should I go to real's and do the equation of motion for phi 1 and do the equation for motion for phi 2. The answer is no what you do is you think of phi and phi bar as independent fields. So, when we do the variation here, the delta if I am doing a phi variation I only varied this but not this I think a phi bar is independent of that, then all factors everything will work out.

Now, now let me just go ahead and write the equation. So, let us write out the equation of and just a little bit. So, the equation of motion would be box of phi equal to minus u prime of phi now I was worrying about a factor of 2, because when I do this variation I will get. This is what it should have been in the absence of the gauge field, but now again I think now I am. So, the reason I was thinking a little bit is; first thing is that if you see if I do a variation of phi I get the equation of motion for phi bar, because that is what rebates number one. And of course, here this you just do the derivative with respect to phi so that is what I have done. And an easy way to understand that this has to be corrected or phi in the denominator is like charge minus q.

So, this is now again the charges are fine. So, this has a nice transformation property, how do we make this correct just? It is just the covariant so covariant fixes it, but you do not have to believe me and you should not believe me at least once in your life you just go ahead and take these things and check that you get that you will I mean there is no doubt about that. So, so this is the equation of motion. So, now we want to know again, what is the ground state etcetera? And of course, we need the Hamiltonian as we will see in the next lecture. And some of you have you already proved it in the assignment you know that the Hamiltonian for just this part is just is some positive definite thing, it is e square plus b square with some halves or whatever half e square plus b square in this set up you tend e square in the denominator it is anyway it is some positive definite thing.

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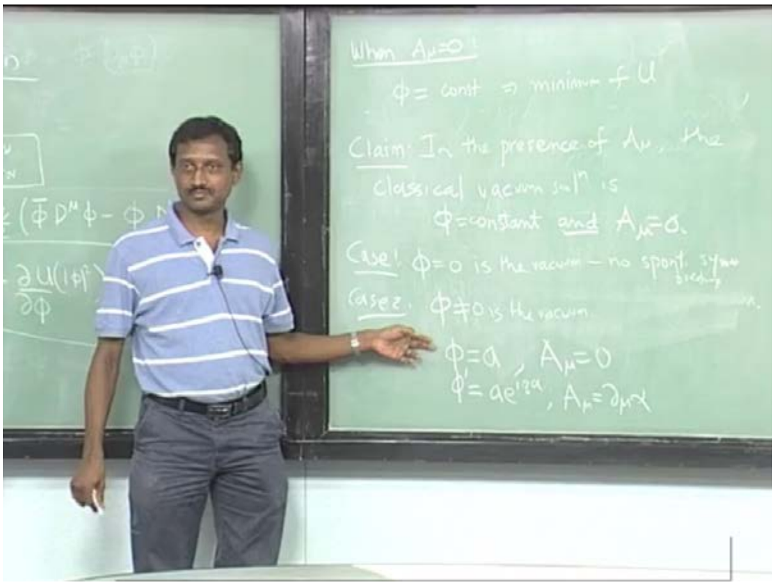

So, let us say that when A mu was 0 when A mu is 0 when, when we are back to the old classical vacuum and what was it? You look for phi was equal to some constant and it was the minimum of u. This is what we would have chosen and so, depending on these things so claim in the presence of A, the classical vacuum solution phi equal to exactly this constant and A mu equal to 0, because what is that doing? Setting A mu equal to 0 tells you that the energy density from the higher from the gauge field sector is 0.

And this is this is the minimum that is so it is really you got the 0 thing, because the Hamiltonian density we will see it will be a sum of positive terms and I have achieved that I cannot go anymore, any lower this has zero energy, is this clear? So, whatever we the hard work we did earlier again minimal prescription or whatever is nice. It, it tells you that the classical vacuum that you had was the same, but the actually that there is something different now from the earlier thing which is that so. And let us, let us let us now look at the case when, when the constant is 0, when phi is equal to 0 is a vacuum this is the case when you do not have spontaneous breaking of S O 2, phi not equal to 0. But suppose phi is not equal to 0 is is vacuum configuration when you then there is something very interesting where happens I can I, I could right I do not need to just write this I could choose a new solution.

So, let us say that phi equal a, A mu is equal to 0, A is that constant, but I could write gauge equivalent solution which is like this, because I have local gauge symmetry. This configuration is not different from this they are physically equivalent configurations in particular suppose a is a complex constant.

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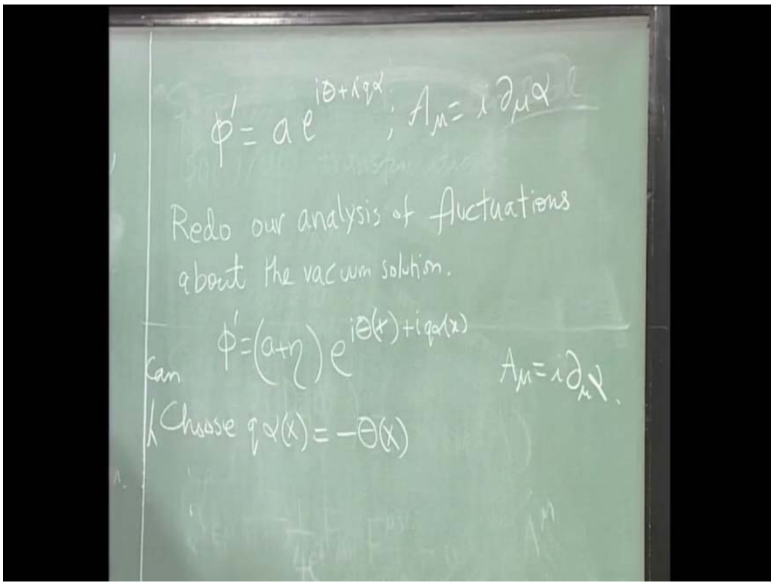

Suppose a is complex I mean, because the ground state would be mod phi square equal to a square, a is I mean let us write it no, no let us write it as this as a, let me write it this way so that when I do not need to so that a is real. I will re write this I need to just re write this so that phi prime I could write that as some a e power i theta plus i q alpha I just written a gauge equivalent configuration. Now, we can go back in redo, we can redo our earlier analysis of fluctuations about this thing. Now I could what the way I would do that is by shifting the a so then you would write phi prime equal to a plus eta and e power i theta. So, first step is that you can see that if theta were some constant I can always the phase of it I can eat it up by shifting alpha and A mu will continue to remain 0.

So I can by I mean I can keep making global transformations such that I reach phi prime equal to 0, I mean the phase becomes 0. But now when I am doing fluctuations I will write i theta will be a function of x but nevertheless I can still have this category of freedom, but of course, now alpha can be x dependent. Now I can choose rather can choose alpha of x q times alpha of x to be minus theta of x.

So, I have much more degree of freedom not just taking away a global phase I can get rid of the phase permanently, but what happens is that, because of that you will see that this, but the gate field will remain I mean you its it is just a picture of. So, in some sense you can see that the, the pure gauge part it does not change the field strength is tend 0 the pure. So, this kind of thing in terms are called a pure gauge it is just a gauge transformation, but that can be used to eat up the phase. So, they are equivalent to each other. So, the, the idea here is that I just get rid of it and always work with so when I look at the phi prime I just have the eta fluctuation which is one real scalar that was the radial fluctuation. Now, coming back to this kind of term now this just becomes a plus eta square and now you can expand it out and you get an a square A mu this thing.

So, you find that you get a mass, mass term in the presence of spontaneous symmetry breaking. Now, it is more precise than what I just said in words you get a mass and you know what the masses is… And what happens to this term you can convince yourself this is only the phase part it will disappear. So, there is only this piece to look at, because see if you write phi as some if you write out phi as some radial part times e power i theta of x in general you will find that this term is just D mu of theta. And but you have just set theta to 0 if you wish by our choice of gauge choice. So, this term vanishes, but this will remain and we can expand this out and we get crucial thing we get. So A mu is not by the way I should write out I, I would like have to add fluctuations even for this even this can so this part will be there.

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So, you will get terms like this plus interactions what I will call. So, there will be a term which would lo like eta A mu A nu with some co efficient plus another co efficient eta square. So, these are if you wish this is cubic in the fields this is quartic in the fields. So, if you lo at the quadratic piece which is this guy you can see that there is a mass term. So, it tells you that it in this back ground the lower, lower energy fluctuations of a photon is as if it has some mass.

So, in other words so if you has spontaneous symmetry breaking in the presence of a local gauge symmetry, the result is somewhat different. Originally what did we have? We had a mass less fluctuation associated with this field theta, theta of x there was no this thing, but now what has happened theta has disappeared. And it shows up as a mass, mass for a for a the gauge field which was not allowed which was not possible. And the the; the reason this has to happen is because if you have a massive photon that should have in other words it should have a longitudinal mode. Longitudinal mode is roughly equal to one scalar particle, one scalar filed kind of thing while a real photon mass less photon which we know in real life has no longitudinal mode plus only 2 transverse modes if you pick a direction it has 2 transverse modes. Similar to that there as so the massive guy actually has 3 modes; 2 transverse and one longitudinal and the statement people say is that the photon by this now I am using terminology if you quantize the thing the guage fields expectations would be what we will call a photon.

Photon picks up a mass or equivalently longitudinal mode by eating the goldstone boson this is what it is colloquially this is what I said and. But it is a nice I mean it is a graphic picture, but it captures exactly what and the process of eating we have already done we did it by doing this. And we end up getting a mass but remember as coalman emphasis symmetry is not broken it is just hidden not just global even in this case the local symmetry exists, So, in the presence of local symmetries the Higgs I mean you, you get a you had the you get a mass for the gauge field and this is called the Higgs mechanism. And by the way this is kind of exactly what happens in what are called type 2 super conductors, in super conductors you have you find that you have an expelling of, of; of flux, flux cannot go through super conducting regions. So, what happens is that if you have a flux trying to come out through the thing, what happens is that you will find regions which are normal.

But then the, that the that the part here regions which where it is not allowed photons I mean magnetic field if not allowed that is major to this fact. The region the super conducting phase actually corresponds to something of this kind, so there you have a complex scale field whose which if you wish is the this is related to phi, in that example it is related to the density of cooper pads. So, you know that super conducting state occurs when the density of cooper pads picks up a non trivial wave or non trivial expectation value, this is a kind of thing in this a would be the density. So, once you have something like that then then you then you end up in the super conducting phase. In the normal phase the, what you will find is analogous of this analogous to this. And you may wonder how can I do this in a single model, the answer is very, very simple.

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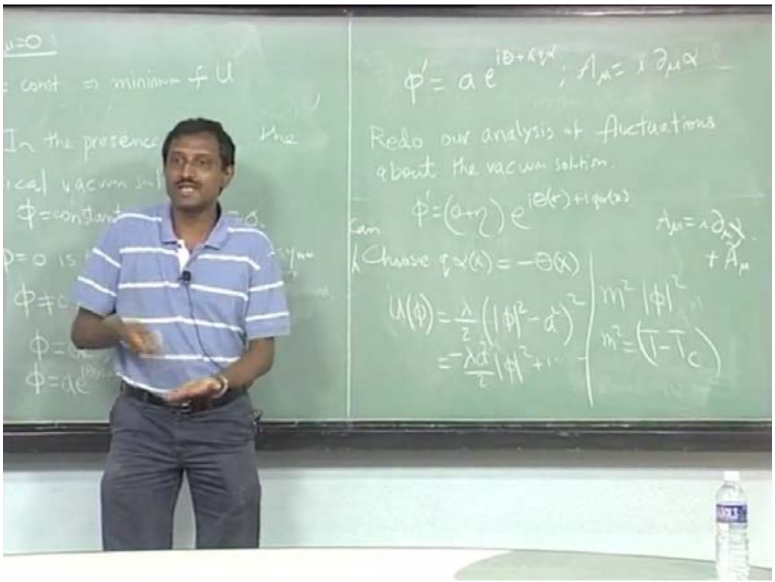

Suppose we did this so we or let us back to this one example it appears that that is the only one I know but let us look at back in a more detail. So, that was if you expand this out let us lo at the quadratic piece that would be the cross term between these so that would be minus 2, so it would be lambda minus 2 lambda a square mod phi square. So, it appears that this thing has a negative mass term, but this is related to if you draw the picture that the, the curvature is the wrong curvature so this has a this thing. Suppose, we have situation where mass square for some, some field has the following property where T is some parameter. Now, you can see that it is when the parameter T is greater than T c what is happening? This is this has regular this thing so you, you are in a situation where the lowest energy configuration is when T equal to 0 the rather than when phi equal to 0. But if it goes the other way when T is below the T c, then you got a negative mass till the five fourth comes into play and you get this double hummed kind of thing.

So, you can see that the if the mass square or the quadratic coupling varies from positive to negative sign has we change a parameter T. Then you can go from a situation so for T greater than T c, it is this for t less the T c it is this and now obviously, what is T in the super conducting example? T would be the temperature. So, you, you; you can see that there is a physical example where indeed this sort of thing happens. So I mean in in principle one, one does not need evidence for the Higgs mechanism it already exists, it is another part a thing that we need to find a Higgs particle, because that has to do with something else it has do with fundamental particle physics.

But there is enough evidence of Higgs mechanism in condensed matrix systems. So, so I do not think the LHC runs are going to actually be proof of this trivial I mean this thing there is enough examples for that. But the question in that context is, what is the precise mechanism for giving masses to various particles not just where in not just the bosons there is much more detail in that in the thing. We will see those examples also at least in in this course, but this is really the world's simplest example. And and this picture that I have drawn also is the way we one gets around derricks theorem, what did derricks theorem tell you? You cannot find you cannot find finite energy time independent solutions in say 2 plus one dimensions. And but in the whole argument which we had there it involved only scalar fields, but now so for that is part one I mean it is not the whole thing.

So, in this thing what we have done is we have already changed the setting, we added gauge fields into the story. And then there was say if you if you remember I told you there was a problem I mean what did we do? We said that when we when we have a kink kind of thing we had 2 I mean the boundary of space which has 2 points. But if you have a plane the boundary of space is a circle large infinite circle, now the thing is that if the phase is what we have seen here in the presence of local symmetry, gauge symmetry it does not matter the phase is not relevant phase is not a gauge invariant thing. So, I can go around the circle I can make it phase to wearied things. So, somehow you find wriggling space and you can find time independent solutions and so. And that again is a, what we will find is that every time we add 1 dimension, you will get some new feature.

So, we will go in this course one step at time we will start, we have we have we did 1 plus one dimension finite energy solutions. The next thing we will do may in a lecture or 2 is to look at finite energy solutions in 2 plus one dimensions. And then we will go ahead and sees things in 3 plus 1 and then may be in 4 it is a rather in 3. And then we will go to 4 and we will the course will kind of end from that there. So, I am done for now. We will we will spend the next couple of lectures, we will spend some time looking at just the pure gauge field, you will understand certain aspects about the vector field I mean you have done the problem sets. So, from a view from my view point it is just solving the problem set, but I will go into more details.