Classical Field Theory
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Lecture – 17

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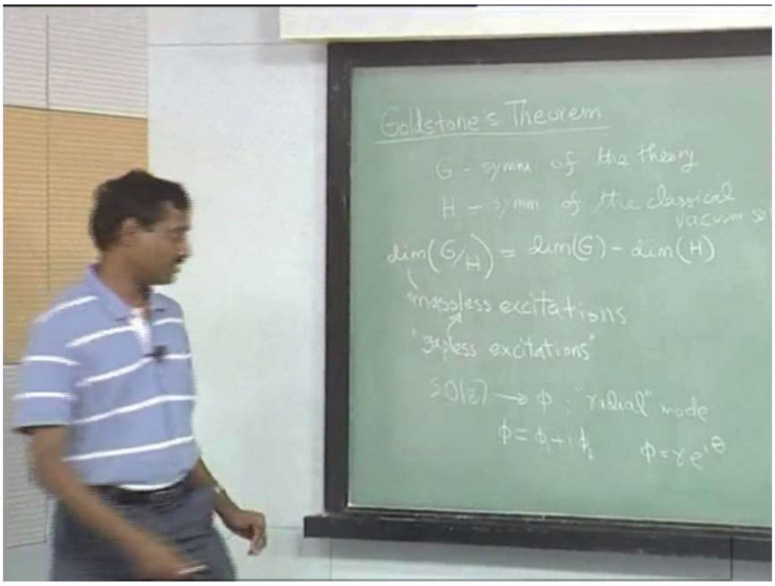

And we had a very nice concise statement to the whole story, if G was the symmetry of the theory and H was the symmetry of a solution. So, when there were continuous symmetries, we saw that we expect, dim the mention of G mod H, a coset space which is defined to be dimension of G minus dimension of H. So, many mass less excitations.

In condensed matter physics these are also called gap less, these are called gap less excitations, these are then are… In other words, if you have particles which you excite and you find that in the energy momentum thing you find that you can that there is, the if you want to excite some particle, you do not need any, there is no gap. As suppose, if there is a gap, then you have to give energy more than the gap. So, suppose so when you have when you are in quantum field theory, you are in a position to create an annulet particle. So, the number of particles is not is not constant. So, if you have a massive particle you need to actually generate energy. So, usually particle and antiparticle are created so at least the minimum energy you would require to create such an excitation would be 2 m c square, where m is the mass of the particle.

But suppose, the particle is mass less, then you can see that there is no gap. So, the scale in the problem is the mass of the excitation. So, in this in the example we did was S O 2 broken to nothing, to nothing, we saw that there was the radial mode which had a mass, so the radial mode. So, what I have in mind when I am calling it radial is that I think of phi the field, there were 2 scalar fields right I can define a complex scalar field phi which is just phi 1 plus i phi 2. And then I can just write phi as mod phi or if you wish in polar coordinates something like r e power i theta. And then this is the direction which should be the massive direction and this.

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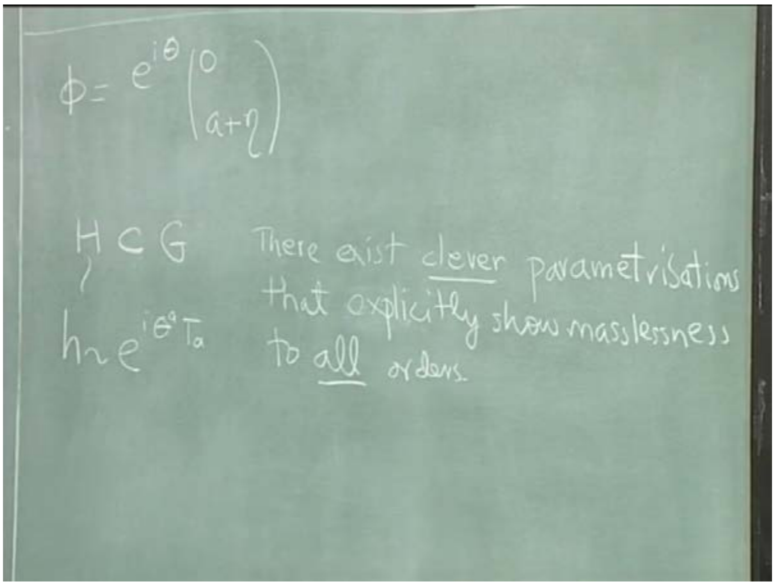

So, if you go back and plug into those equations you would see that the mass square rather of the r field was what we got last time. So we got something like lambda a square upon 4, is this correct or we should look at your notes, or to 4 lambda a square. So I know there was a 4 i do not remember, but 2 4 whatever, some number times lambda a square. So, but I really like to think of this as a second derivative r equal to whatever was the r equal to I guess a scale in this a itself at r equal to a. So, that is what it should be and so for instance, say if you want to see this excitation there in quantum field theory would say that there is a gap and the gap is roughly of the order of magnitude its lambda a square. So, this is the difference between gap less excitations, and so if you are in particular, if lambda a square is very, very large, then those excitations would never compared to the energy scales of your problem. You would not look at those excitations would only look at things like this.

So, this terminology difference between fields, so particle physics people will say mass less excitations and condense matter of people would say gap less, they had really the same thing. And there was a question after class or towards the end of last lecture which said you know I was very clever in choosing a parameterization, I wrote something like this, I wrote let us write we started out with something like 0 a and then I said I will write eta and I wrote e power i theta. So, with worked out as eta and theta where existed, I mean these things and the claim was that. So, this theta will map to exactly this theta you can check, and r will become just a plus eta in this set up.

And so the thing is there a clever way of parameterizing? The answer is yes, and in general you could so H is the symmetry of the solution. So, what you would do is, so you given H which is a sub group of G and we will assume that they are all e groups, so in the fact that everything can be written as exponentially or something. So, what you would do really is all elements of H so typical group element you would write H as some e power. We will see something it can be return something like this theta's a's and T a's. So, a's would be as many as, as many as the dimension of H that that many parameters. So, if it were S O 3 it would be 3 parameters and some sort of generators we will see that we can write it in such a form. And the remaining stuff, you could write in again in a this thing but the key point here is that, you can see that I am not working lowest order in theta or anything is true to all orders in theta.

If you write it in this clever fashion, you can see it that it is true to all orders in theta or you just look at in a global way. Again you would see that this, this is true to all orders it is not just saying that the mass, the quadratic term is 0, saying that all terms has no potential. And so there is so in other words I just want you to remember that there exists clever or let us leave it at their parameterizations that show, explicitly show masslessness to all orders. So, but so we have been discussing classical field theories, but you can ask what happens in quantum field theory and there is something important which comes.

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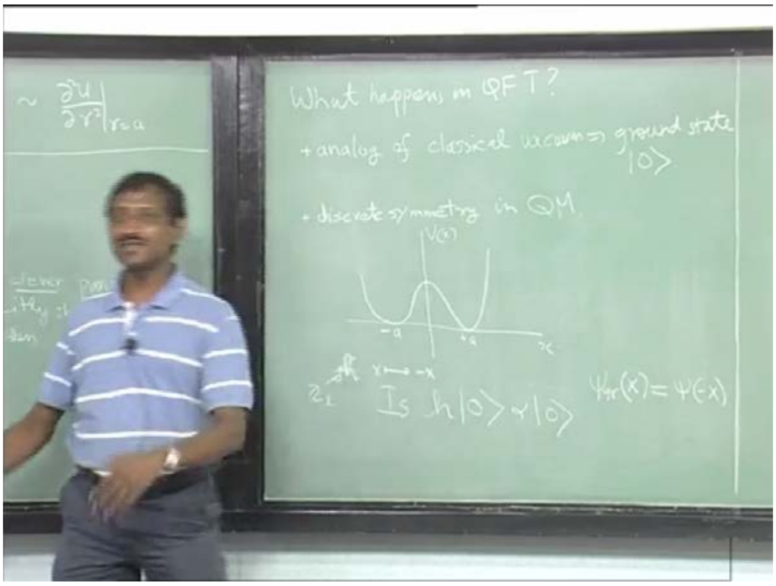

So, what happens I know this is not a QFT course, but never the less the result is something which can remember and so. First thing is we so by that I mean what happens to Goldstone's theorem and we will see that there is a dimensionality dependence in the story. And so first, so the first point remember, so is I keep using classical vacuum solutions, but the analog of that in quantum mechanics or quantum field theory is a classical vacuum, is that a question? So, vacuum is the ground state. What we will call it ground state and usually use the symbol something like this. So, that is the first point, the second thing to ask can that be what happens to spontaneous symmetry breaking or Goldstone's theorem.

So, first let us consider discrete symmetries in quantum mechanics this is something which you, if you have done a quantum mechanics, you will know about. So, for instance suppose you have simple potential in quantum mechanics, again a double well kind of thing, and say minus a, plus a and minus a. And let say this is a 2 symmetry which is takes X goes to minus X. So, the potential has symmetry in the theory etcetera, but the thing is so let say let called this generator h. So, the question is h equal to 0, not equal to 0. So, this would be then analogues state. Now I am just getting to the point of stating what I mean so by the. Only then we would say the symmetry is broken. If h acting on this thing is 0 that means its symmetry is it is symmetry of the theory.

So, this is just the statement that it is, so lively you would think here, I mean the first guess is yes I mean equals to 0. So, in terms of way function is a way function symmetric, is if you if you want to sink of that you say psi ground state of x is it equal to up to some phrases it may I mean will permit phrases that is not so is this, so X go to minus X this is what I mean, so h is a generator. So, this is not correct, I should write something like this. So, h now it is I think it makes sense. Why should it be 0? Tell me, so it should map. So, all I am saying is h maps the 0 vector to 0, it is not 0. 0 is different from the vacuum. So, what it could have done is it could have it could have done the following thing.

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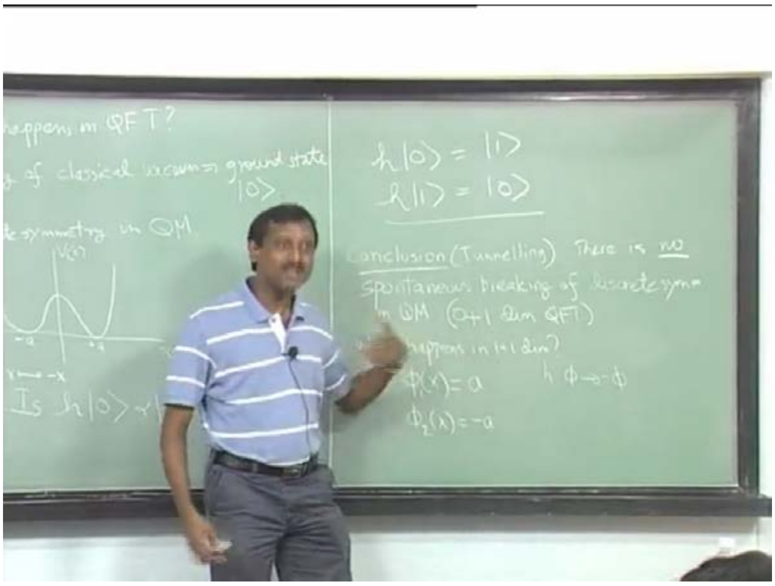

It has taken h of 0, we go to say some other state, and then h of 1 could be equal to 0. Here is, so if it is symmetry of the thing, it would it will take it to the term that is what it is proportional is what I wrote, that is what I am saying? It should give the vacuum. You do the following thing operate on a h is in the state symmetry, h square is identity. So, trivial way to check I mean to sort out my error for instance is to act on with h square. So, if I did this way, so the proportionality I put here is because that it could be minus of it does not matter. But so now the key point here is what the ground state is? So, this is the kind of thing, I want so these 2 are same the statements. So, we know that so that the 2 ground state of this theory is indeed is actually has no such this thing cannot happen, if this happens we would say it is thing, we would say that symmetry is broken.

So, roughly what one has in mind is, some wave function with supported this well that is the nice guess, first guess and the second guess is one more here. So, there are 2 degenerate vacuum, this is what you would think. But quantum mechanics you know this does not happen, because there is something called tunneling. So, because of tunneling, you find that the actually true ground state is a coherent super position of the 2 states which you have written in some place. So, the true ground state actually reserves the symmetric. So, the conclusion is that there no, so just based on tunneling arguments breaking of a discrete symmetry in quantum mechanics. By the way we can also, it is useful to think of quantum mechanics as 0 plus one-dimensional field theory. It has time, but it has a discrete number of degrees of freedom countable or whatever so you just call it 0, because if you say 1 dimensional it does 1 field worth of there are no fields in the theory. So, is best to think of it as 0 plus one-dimensional CFT. This is the trivial way of stating it, but because it will connect up beautifully with everything else.

So, similarly, since we are saying this would be 0 plus 1 dimensional quantum mechanics is nothing but 0 plus 1 dimension quantum field theory. Now, the question is let us go to, raise the thing to one extra dimension. So, let us go to let us go to 1 plus 1, what happens in 1 plus 1. Now, we have infinite degrees of freedom phi of x, and let us say that we choose phi of x to be equal to whatever is that which breaks it is again exactly like this. So, instead of the so I just set the field I takes values with plus a or minus a, I again have a discrete set to symmetry and the symmetry of course, h will takes phi to minus phi. So, this is one ground state and phi 2 of x equal to minus a is the other ground state. This as an example we have looked at in the last lecture. So, what the question here is now suppose I prepare the system in this state, I could have done the same thing in quantum mechanics I can I can start with an initial configuration which where the particle you see localized this well choose a way packet which is localized there.

And then but you can show that after sometime there is a finite non zero probability, because of tunneling, and that shows you that the I mean that the 2 ground state is indeed something else. So I am doing the analog of that I am saying let me go ahead and prepare the system in this first ground state. So, put all the field everywhere which set a in all of space. Now, the question is can it tunnel? Now the point here is, if it has to go to the other, if it has to go to the other ground state, what you required is sort of spontaneously everywhere it has to tunnel and go in one fells hoop to the other thing.

If it went only in some region we know that it will like the kink or whatever. So, this sort of I am not giving you a proof of anything, because we are not doing quantum field theory, is what you give you a entry to field of things what this tells you is that that cannot happen, because one more thing you have to remember is that, the tunneling probability is actually expansion is surprise. Now given that it has depends height of the potential, but it also now you have to do everywhere in space. So, more or less it becomes more and more likely and improbable. So, the point is that in any field theory, other than quantum mechanics, actually discrete can be spontaneously broken.

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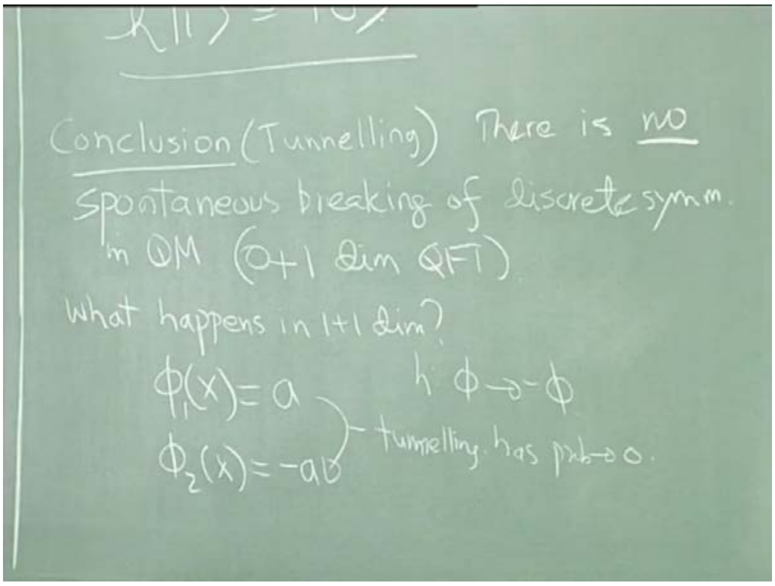

So, really tunneling has probably making 0, if discrete is thing you see that require some bunches of thing, if take the continues limit you gone you get 0. So, this theorem was actually originally done in context of statistical mechanic system where Mermin Wagner theorem.

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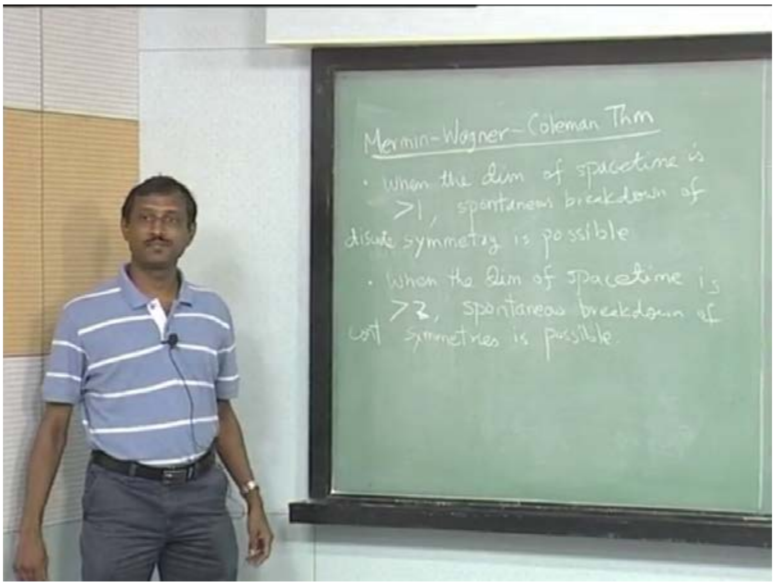

So it is called the Mermin Wagner theorem and it was adopted in the context of one field theory by Coleman let us call the Mermin Wagner Coleman theorem, most express is the Mermin Wagner theorem. So, what it says is that in, in dimension in dimension when the dimension of space time it is greater than 1 spontaneous break down of symmetry. So, just 1 in the context of stat make systems the analog of dimension, because great space time becoming 2 dimensionalizing model, thus are few critical temperature definite value. But for the one, the one dimensional the icing chain the critical temperature is 0 which basically tells you that there is no there is no accuracy, no face where actually symmetry is broken we can go up to 0. So, this is for of discrete symmetries, but they also have something to say about continuous symmetries.

Now, for me it is, I mean we cannot use arguments like tunneling, you can that there are certainties, because if you if you have a continuous symmetry, I can smoothly change things. So, so that is a little bit more freedom. So, the question is not I mean I one should not use these kind of hand waving arguments even this I think is hand waving I mean the setting is completely off, but because we are doing classical field theory I just wanted to give you an extrusion, for continuous symmetries the statement is that you require dimensionality of space time to be greater than or greater than 2 for continuous symmetries.

So, in 1 plus 1 dimensions for instance you cannot have a continuous symmetry. Now, you can see the change here for continuous symmetries, you can go from 1 to 2 spontaneous of symmetry of continuous symmetries. It is usually written in a negative way it says that for, so one say's that the critical dimension for this is 1 or 2. This is another way of saying is this clear? So, many of the considerations that we are do, we are going to look at, you should remember that these are the issues and this is what memorizing. So what I am going to do now is to this is what I have to say about Goldstone's theorem. The next step is to actually understand how to convert. So, we have so far we have been discussing global symmetries, what will happen if you want your symmetry to become local. So, in other words, so when we let us take let us be more specific.

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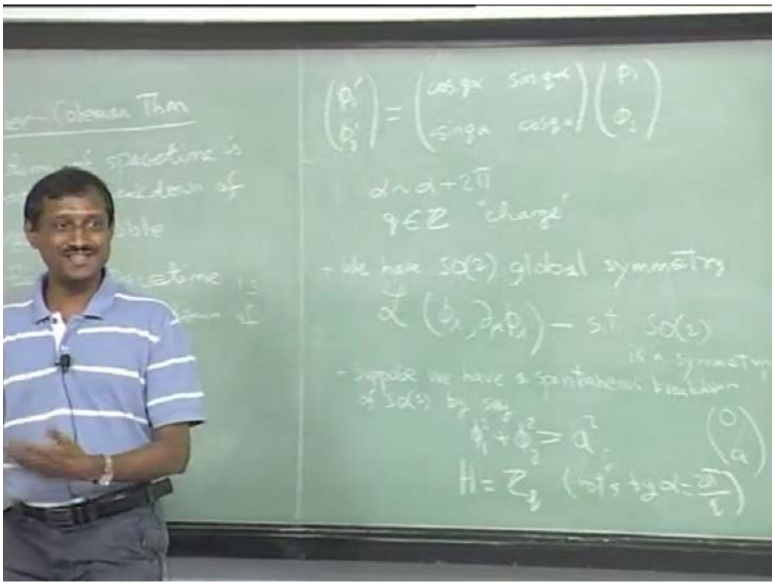

Let us consider a simple example of S O 2 and so I think, I should write from left to right, not from right to left spacing's get may stuffed I am going to introduce some parameter which I call q become pretty soon clear what I mean. So, this is just some S O 2 global symmetry with the parameter theta I have written as q times alpha, and alpha is supposed to be an angle. And alpha is identified with alpha plus 2 pi. Now, you can and q is some integer which I will call charge, if you have just one scalar in your theory I do not think it is a big deal, what this q means, you can always say fine I mean I leave with I could have redefined it and define q alpha as theta. But suppose you have the same S O 2 acting on 2 different fields. So I could have one field which has q equal to plus 1, I can have some other field which has equal to 2, then there is a distinction it means that in field space when one of them is rotated by 2 pi. The other is rotated by 4 pi.

So, we will call q the charge we will see that this relates to charge in a in a physical setting. We know the charge is quantized etcetera. So, this will map to something like that, so right now alpha is global, so rather so we have a, what we have is we have S O 2 global symmetry. And let us assume that we have some density which is a function of phi i and d mu of phi I, such that S O 2 is symmetry. By global I mean its alpha is just a constant it does not depend on space time.

So, now remember I also told you some time back that there is a way of breaking S O 2 to some Z q z and some discrete group. Suppose we have such a field phi 1 phi 2, such that they pick up a wave. And you have this breaking of, breaking of symmetry S O 2, but now you can see that there are rotations. So, let us assume, let us for, let us assume that suppose we will come back to the local part I just have a comment to make about the worth we have a spontaneous break down of symmetry of S O 2 by the fields by say something like this phi 1 square plus phi 2 square we are getting a wave, we call it a square. So, this breaks so you, you could start by like, we did last time, we could say phi 2 equal to a. And now we can ask what is the, what is the symmetry of? So, let us choose phi so let us choose this like we did last time, and ask what the symmetry of this is the unbroken symmetry?

So; obviously, the S O 1 is nothing but you can see that I can make rotations of phi if q is 2. Then I can make a rotation by 2 pi over 2 which is phi, such that this will still be a identity, because there is a 2 in there in this transformation. More generally if it is q, you can see the I mean it S O 2 is broken down to the sub group z q which is just rotations with angle 2 pi over q. Now H is actually z q which is a rotation by. So, this I would say that this situation where g is S O 2 and H is z q, but of course, Goldstone’s theorem is still say that there is one goldstone boson, because in terms of continuous counting this is still 0, the dimension of this continuous dimension is 0. It of course, there are certain elements so this is now the, the idea is we can…

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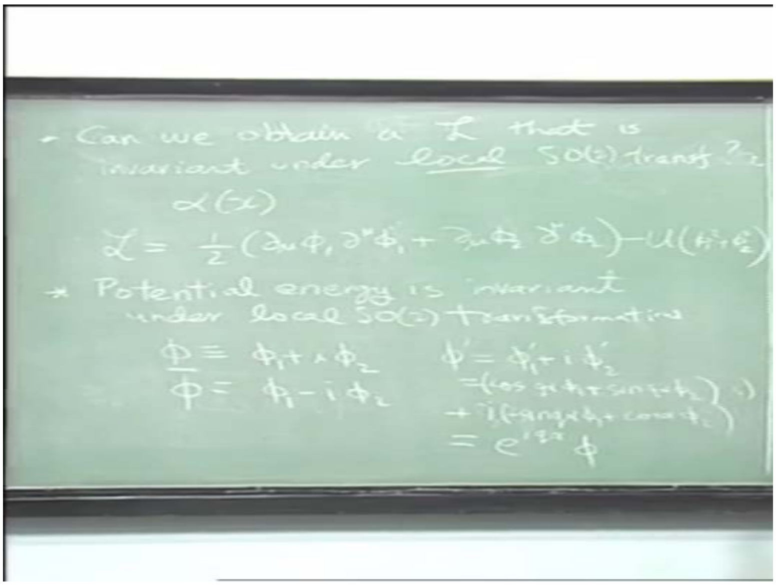

So, now we will can, we me, can we obtain a Lagrangian density that is invariant under local S O 2 transformation. So what I mean by that is just go back and think of the alpha parameter as a function of space time. And let us go back to the Lagrangian density and look at it. So, you are given that this looks like something like this, u which is the function of phi 1 square plus phi 2 square.

So, first step is to so what we will do is we look at first. So, what have now, we just go back to the same transformation and ask what happens if he look at the, the 1 with phi prime, but alpha now dependent on x is it you see that the potential energy this is easy to see, because it does not matter it did not really matter that it just depends on the combination phi 1 square, but what about this guy? Now, it is no longer we made and what I am going to do now is to make a slight change in notation which is to go to a complex, I will go work on a work with complex fields which rather than real just for convenience and you will see.

So, let us define phi to be phi 1 plus and define phi star or phi bar as minus i phi 2, and I need to write out how this transforms, and so you can see that phi prime would be phi 1 prime plus i phi 2 prime and we put things together. So, that this will be phi 1 prime it is cosine phi 1 plus sin phi 2 plus i times. This may look very complicated, but actually this is nothing but so phi prime just, so it is like going to the Eigen basis of this I am diagonalizing this matrix if you wish and.

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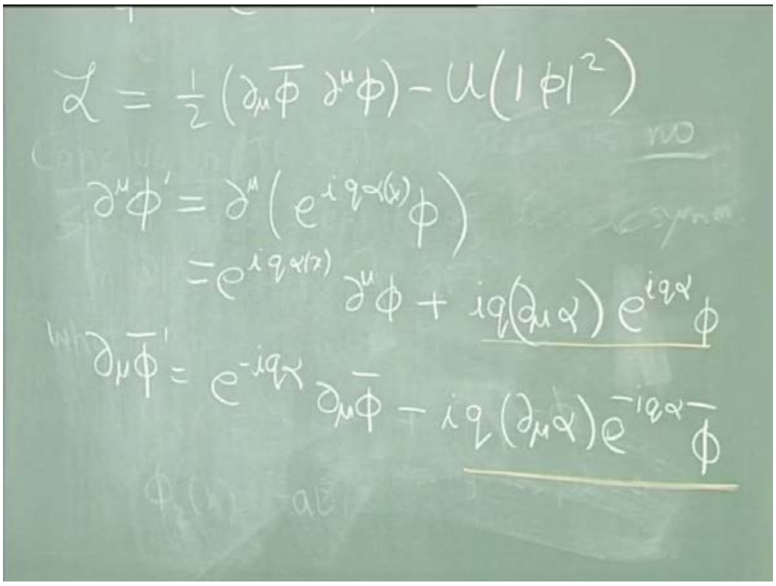

So, phi prime is e power i q alpha again, and this looking at this thing. We will say that phi, the field phi as charge plus q and field phi bar as charge minus q. So, in quantum field theory we will see that these are like particle, anti particle problem, anti particles. So, electron and positron if these are bosonic that does not matter I mean what I am what I mean by that is that they will have opposite charges and I can rewrite this Lagrangian in the following fashion. So, for I have done nothing I have just rewritten in terms of a complex basis. Now the thing we will see is that see we will address the question whether the kinetic energy is independent is invariant under this. So, for that what I will do is, I will look at only one of these pieces rather I could have done it in real thing using phi 1 and phi 2 it is a little bit messier, but you do not have to feel bad about it you can go home and work it out.

So, what I want to know is what d mu of phi prime look is like? So, I mean so let us do this so d mu phi prime will be e power i q alpha phi and alpha is also a function of x. Now the derivate can act on 2 things. So, you can see if alpha were independent about constant this term will drop out. And this is transforming nicely it is just transforming like exactly like phi. And it is not hard to see how this is similar to this except there will be a minus sign coming out here. So, again now you can see that this product will when alpha is independent, it is a constant only these 2 terms and there is a nice cancellation.

So, the offending pieces are out here these 2 guys and we see that it is not invariant. It should be excepted once you make things local things which involves special derivatives should not work out the cancellations which happened should not work out. And but you could take 2 lines of that one would be to say fine y I do not care about local gauge transformations I will only work with the global things. And that would be the end of the discussion, but there is something very interesting which comes by saying that no I want to make, make things invariant I want to make it locally invariant. And so again there are many ways of doing it, but there is something called a minimal way.

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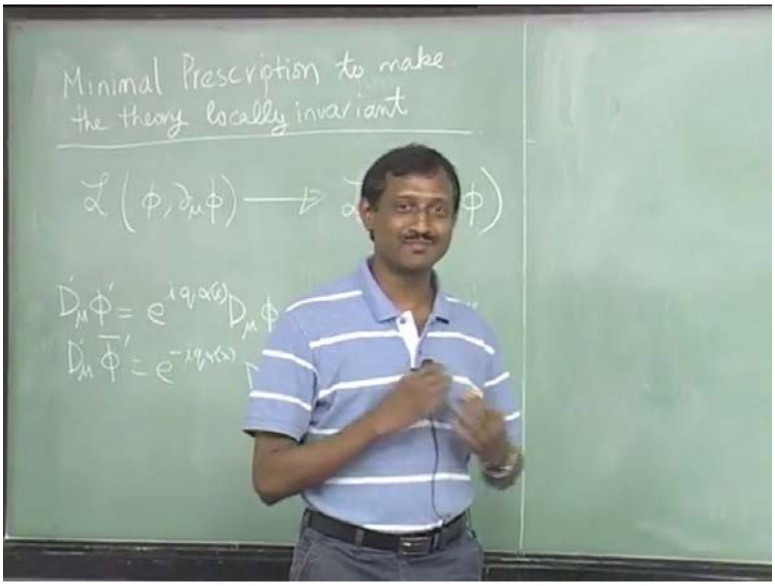

So, what we will say, prescription so this is means that it is just a way of doing it. So, minimal prescription to make so again I do not have to prove anything I just have to tell you the prescription and you can go and check that in its I mean it is invariant under local and the prescription is very simple. So, if you have a whole bunch of fields. So, phi and d mu of phi, the prescription is very simple you change what you mean by derivative, you introduce some new derivative. So, this is some new derivative and we will call it the covariant derivative. And the, the condition the requirement is that. So, here the guy the problem was, was with these derivatives I will just postulate this will work out what properties we need.

And then I will construct that object for you. So, so this is what will happen, D mu of phi prime I will put a prime even on top of this. You will see why, but if you do not put the prime right now, I will forgive you for that, that means it transforms nicely I mean that is what that would, I mean our wish. But alpha here remains local, I am not making it and of course, the same covariant derivative acting on phi bar should do the something. Now, you can see that if I if my covariant derivative satisfied this condition trivially it is invariant unlike becomes good. Now, we can see how this how this is done? But before that we should it is useful to go back to the days of lenience we have 2 people invented calculus; one of them was Newton who was other Leibnitz, and most of the notations that we use today is due to limits except in physics course.

So, derivatives in I use this in this course, all the time which is dot for derivatives right dot, double dot, triple dot so on so forth. So, that was the notation which was used by Newton which was very useful, because he was trying to solve problems in mechanics using by inventing calculus. So, most of the notation that we use today is modern notion is due to Leibnitz. So, that is an important rule called the Leibnitz rule.

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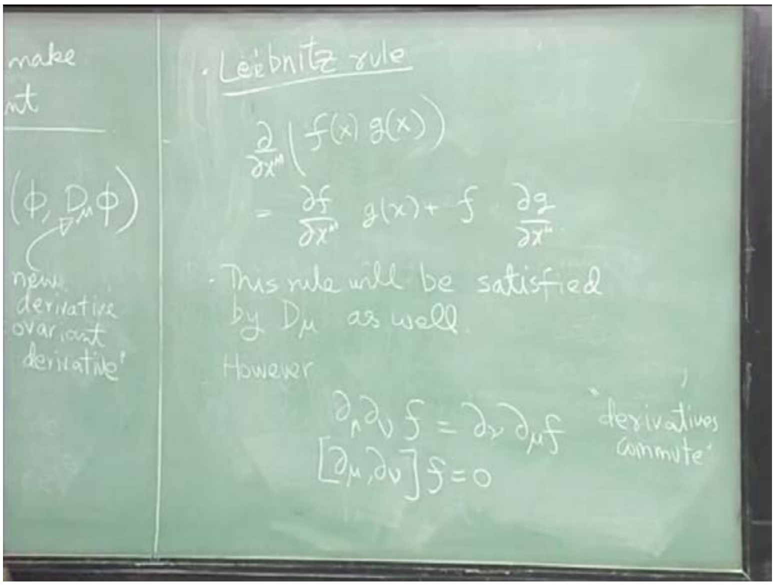

And this is the correct spelling e i right Leibnitz again there is no universality I see a t put here. So, I am so Leibnitz rule which is, which is an important rule for derivatives and which says the following; if we have 2 functions, so let us just do simple. So, this is this is the important rule and whatever derivative we, we create here that better satisfy this rule this is true for one variable; this is also true in multi variant calculus. So, here these derivatives are multi are multivariable d plus 1 variables.

So, this is the kind of thing that you except, but so this is a property, so this will be satisfied by this rule. So, actually so let us right it this way and the rule will be just replace these by the covariant derivatives, this going to be satisfied by d mu as well. So, that is why it has a right to be called a derivative, this is this property is called derivation. So, the derivation property is something satisfied by this thing, but it gives up one interesting property which is however, we know that if we have smooth functions, the order of taking derivative does not matter for smooth functions. But so I can rewrite this in a slightly different fashion I can write it as a commutator of D mu and d nu acting on a is 0 it is just the same statement by I just took this to the left hand side. So, in other words we say that so we say that derivatives commute, and the key point is covariant derivatives do not commute, do not necessarily commute they may commute.

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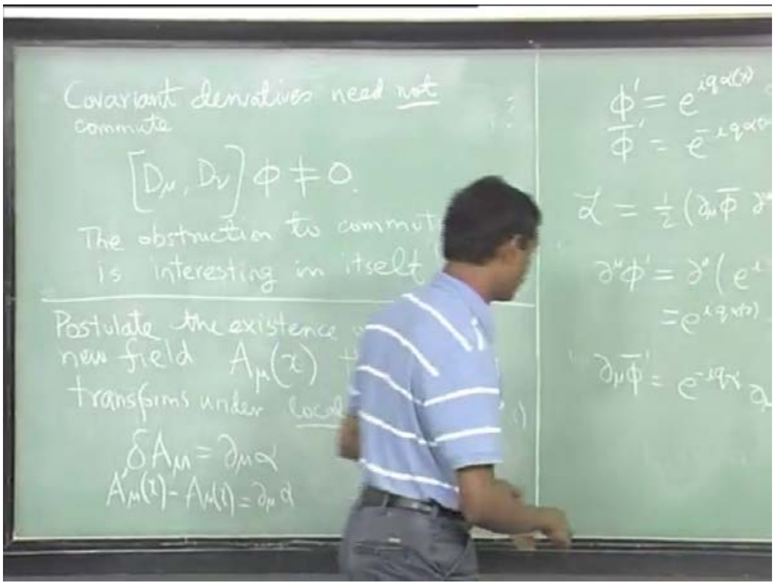

Need not commute, so by that we mean something like this D mu d nu acting no some field phi need not be 0 in other words there is an obstruction to commutativity. So, the obstruction, this is actually interesting, so I have not, we will see what this obstruction? Is it will turn out be something which we know and we will see that this way of defining things will be a will generalizes beautifully. Some of you have done general theory of relativity; you would have seen something like this out there as well. So, what is happening is that the, the covariant derivative does one thing for you it makes your fields transform nicely it removes that horrible derivative piece out there. And next bit is that it satisfies derivation rule, but does not satisfy the commutative these are the things.

So, let us now implement the prescription. So, what we will do is postulate the existence of a new field and this field is a vector field. So, it is a vector valued object that transforms under local S O 2 as delta a mu, this is how it changes. So, by this I mean and there since, since this coordinate this thing does not act the S O 2 does not act on space time fields the argument is still the same. So, there is no worry about delta\/, deltas bar all of them are same. So, this is just a postulate now you can see that this guy has just about the right the thing to remove something like this so I will.

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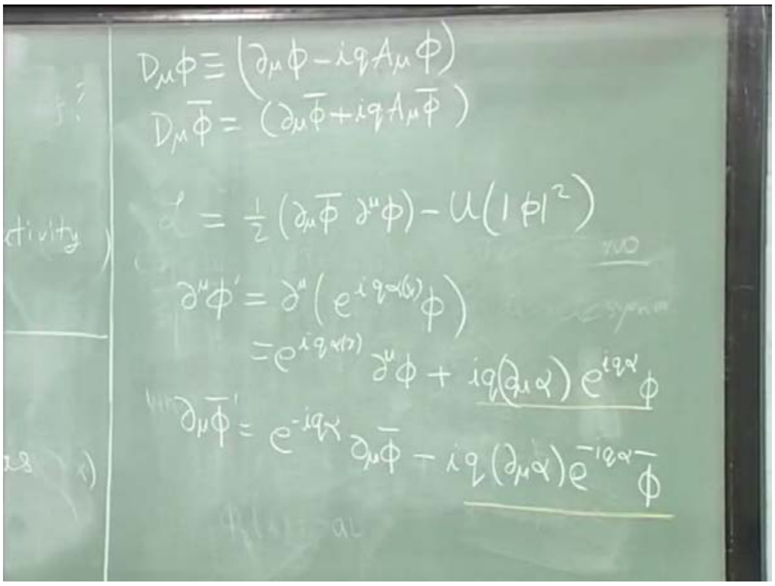

So, let me define so this is just definition. So, there is a way to read this it says that if phi was a field which had charge q then you put in the charge q comes this way now you can ask phi bar what was the charge of phi bar minus q? So, it acts differently suppose there was a field was which did not transform under this there was charge q equal to 0, then there was no need to it does not transform under this thing. So, there is nothing to be done. So, when charge is 0 the covariant derivative reduces to your normal derivative.

So, it only sees fields which transform under this symmetry, now it is a trivial exercise to go back may be will locate it next lecture is to go back and see that this is precisely this minus sign which I have put here I think it is correct. So, you can see that when I do this the delta of this object while get one extra piece which would be a d mu alpha with a minus i q d mu alpha phi which is exactly what I have here it will cancel it. So, I have, I have just chosen it in such a manner, such that it transforms like this, maybe I will do it since I have a couple of minutes. Let me just do that now these primes will come into play I have put a prime, because a also changes it is not like a does not change.

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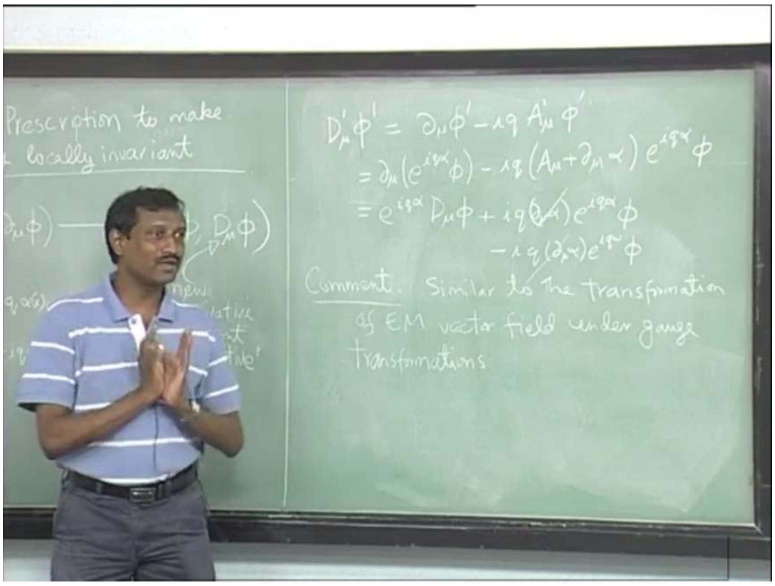

So, D nu prime of phi prime now you just look at the definition there d mu of phi prime minus i q a mu prime phi prime. Now I have to use all these things, now I can go ahead and collect terms together so d mu acting on phi will give me e power i q alpha d mu of phi, I can combine it with this piece. Then I can rewrite this as, now what remains I have to write out. One part is when the D mu acts on this, and this piece which was left over. So, these 2 cancel out. So, you can see that I have achieved what I wanted and before I finish I just want a point out if you see if you would see so familiar something similar to what you would have seen for the electromagnetic field? This is how it transforms under what we called gauge transformations.

So, just a comment is that similar to the transformation of the electromagnetic vector field 4 vector field. And there are instances in which you can I mean and in fact, you can think of electro magnetism as related to S O 2 or u 1 in tunnel symmetry which is local. So, I am not saying this particular thing has to be that could be many many symmetries in nature not everything has to be electromagnetism. So, but it could be thought of as electromagnetism. So, next time we will go back and look at the structure of all the terms. Now, we have already achieved what we set out to do which is we know what to do, and may be before we meet next time I recommend that you check what you get on this side and there and Leibnitz rule is very interesting.

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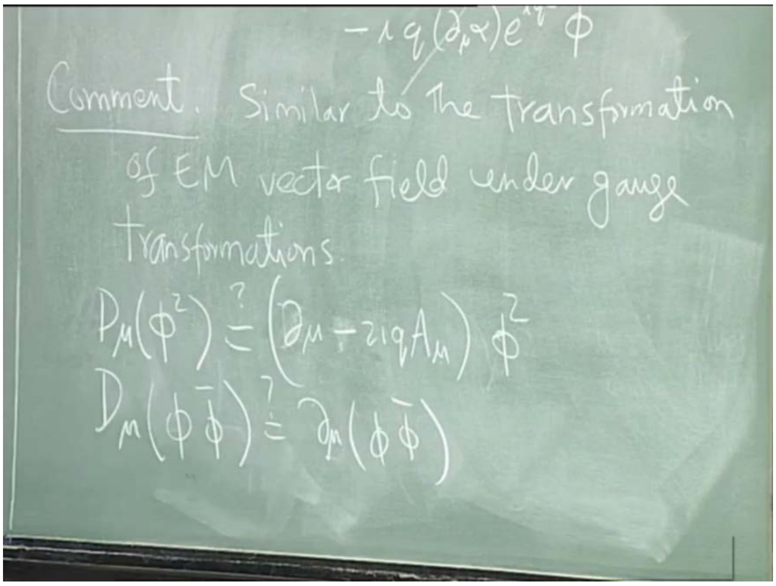

So, let us consider an object like you know what should be D mu of phi square. You work out these 2 things. So, one let us look on this phi square, what would be the charge of something which is like phi square, 2 q? What about an object like this phi phi bar? It is 0. So, according to this I am just looking at the charges and saying that how did it go with a minus sign 2 i q A mu of phi square? And this should be just ordinary derivative of phi phi prime.

So, it is fun exercise to check that I can make Leibnitz rule worked and it will be compatible with these 2 statements. That is because that covariant derivative does depend on the charge on of the object on which it acts very important. So, use Leibnitz rule and check that this is indeed true; that is one thing and check that it does not commute and work out what this is this is fun I mean I will work it out next class. But it is going to be boring if you have not worked it out you should see something familiar. So, we will continue there in next lecture, we will we will work out what, what these things are the obstruction to commutativity? And then we will produce from there.