Classical Field Theory
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Lecture – 15

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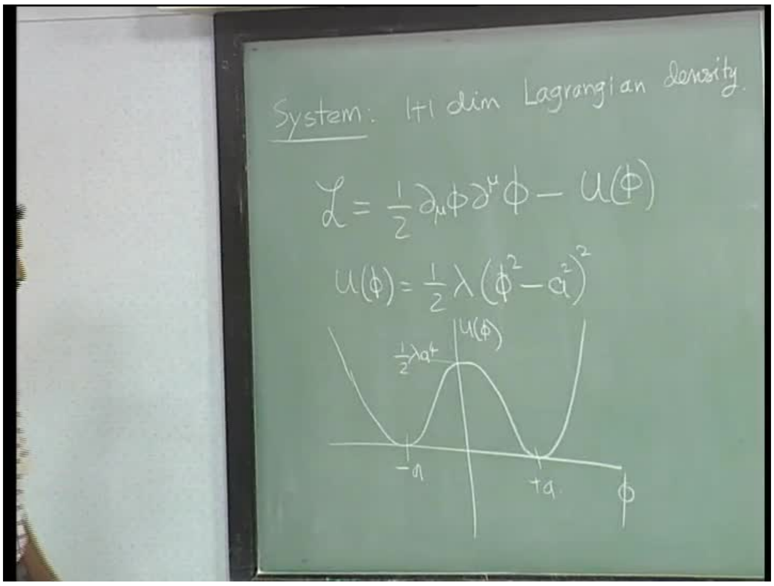

To be eventually discussed as we go though the lecture, because I want to illustrate certain ideas, which seem to play a lot of important role in theoretical physics these days, and so dimensional Lagrangian density. Trace one real scalar field, which takes u of phi is half lambda phi square minus a square whole square. So it has a quadratic potential, but shifted by this things the system is… So, 1 plus 1 dimensional Lagrangian, given by 1 plus 1. So it is a double, it is a classic double well potential, so if I plot u of phi versus phi. It has to minima, so it looks something like this. So, suppose to be symmetric about phi, it is going to minus phi, but I am not sure that my, and the values it takes here at phi equal to zero is half lambda a power 4.

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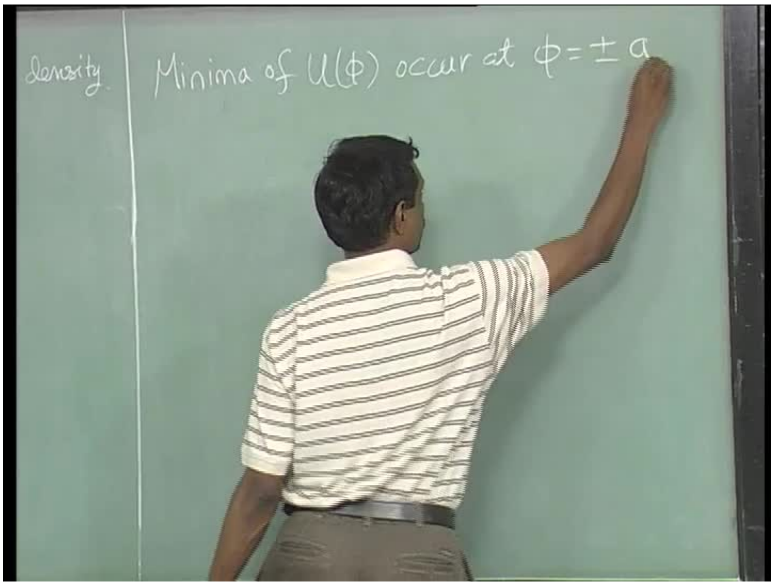

So, the minima of u of phi occur at phi equal to lambda. So this is canonical example, where i you can see I have adjusted, I have shifted things, to make my zero lower energy, values of u of zero.

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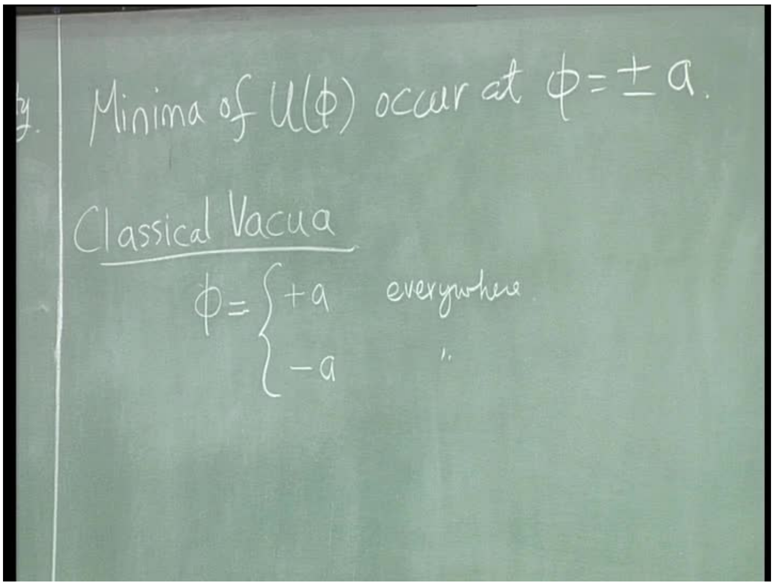

And so the classical vacuum of this, are of course, phi a equal to plus a and minus a everywhere of course, so these are, where two different solutions and. So one thing I should point out, is that you know, quantum mechanically we know that, if you have a system with this kind of potential. It is not this example, but, you know you know that the ground state of the, true ground state of the system is actually, a symmetric one. But out here, this. So let us call this solution one and two or plus or minus probably.

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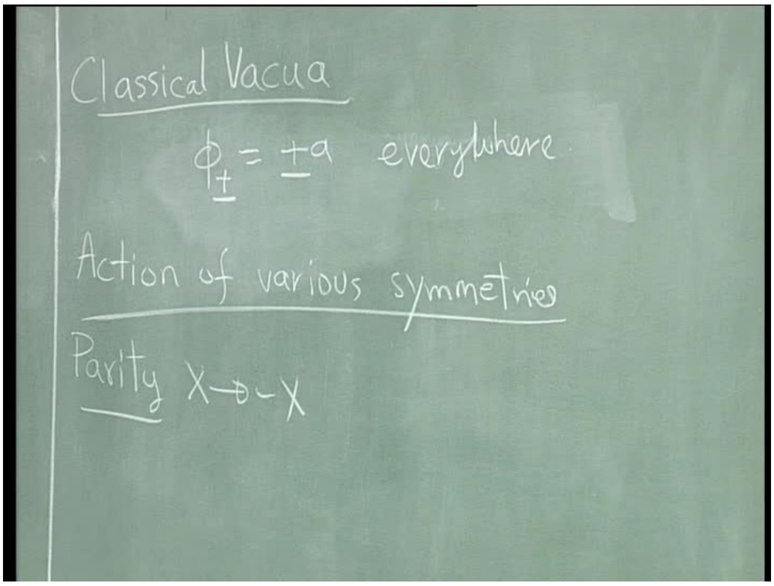

So let me erase this, and just it phi plus, it is a solution where it is equal to plus a and minus a. so, let us look at the action of various symmetries. So for instance if you look at parity, which is x goes to minus x, we are in one plus one dimension, so if you look at parity; of course, parity is the symmetry of the Lagrangian density, it is very easy to see.

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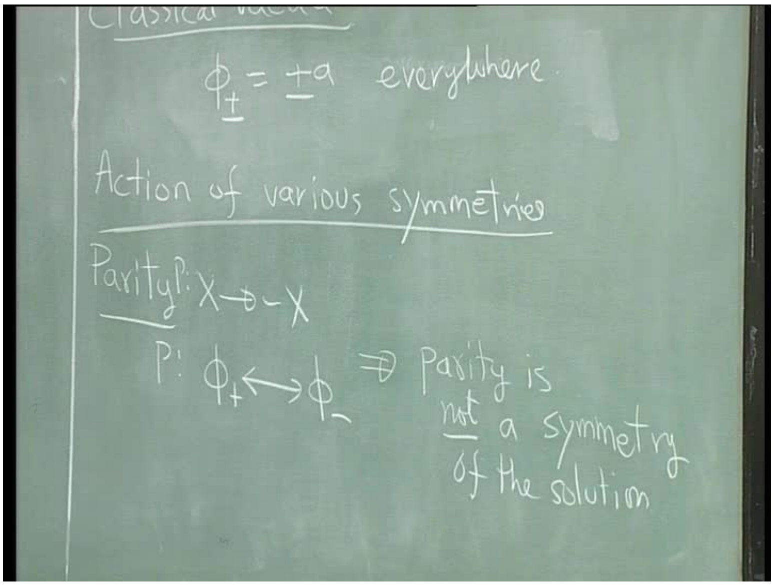

So but, if you look at the solution, under parity, so let us call it p, under p you can see that phi plus and phi minus get exchanged. So if you are in a world, where phi plus equal to plus a everywhere, then it is one would say that parity, the action of parity takes you to another solution. So parity is not the symmetry of the solution.

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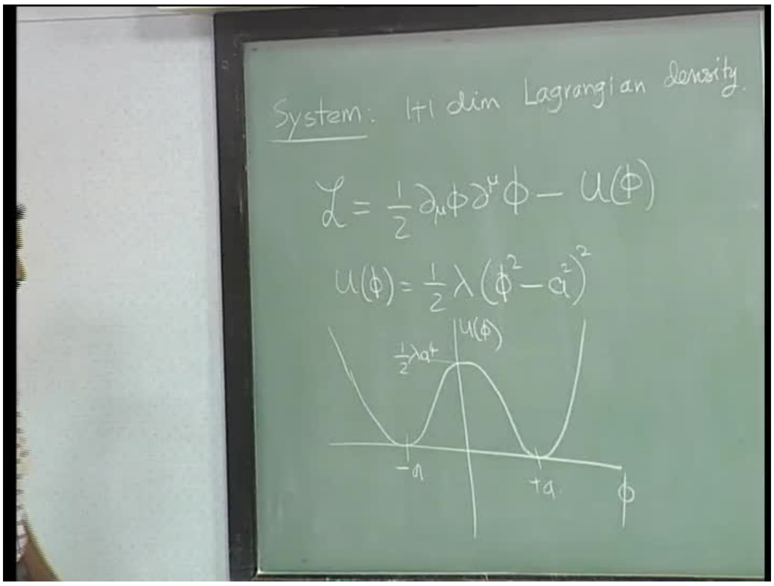

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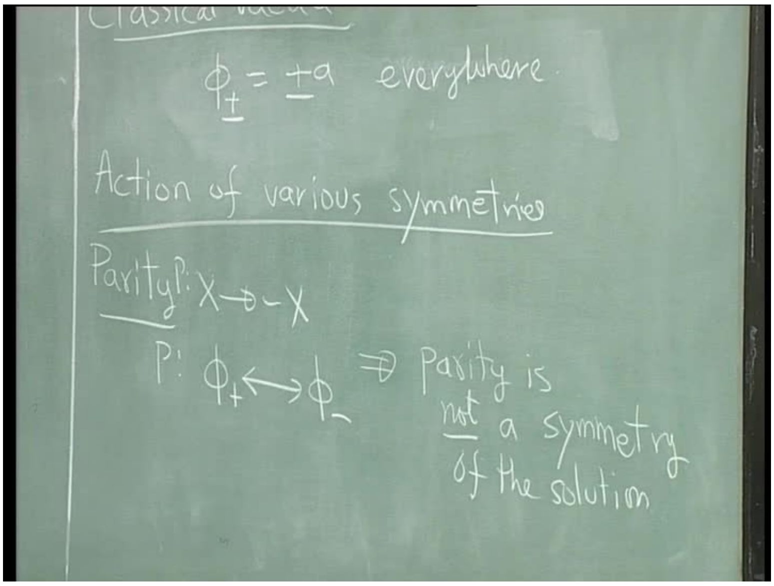

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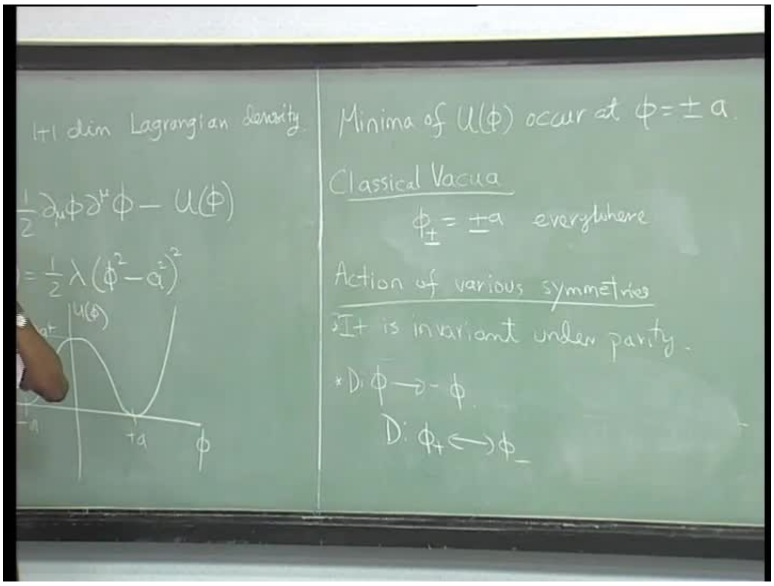

So this is standard pattern we will observe, if you have a symmetry in your theory. It be a symmetry of your Lagrangian, or a of the equation of motion, but, that need not be a symmetry of your solution. So action of the solution could be to be an invariance, or it could map it to another solution, but, what it will do, if it is not invariance of your solution, it will map on solution to another solution. (( )) you are right, so parity. (( )) its invariant under parity, sorry its invariant under parity, but, not under this other discreet, sorry sorry yes thank you. So it is invariant correction, so it is invariant under parity thank you. But, that is another discrete symmetry, which I would say phi goes to, so let us call it d for discrete symmetry phi goes to minus phi, which is also a symmetry, and it breaks that. So under d phi plus goes to phi minus and phi minus goes to phi plus, thank you.

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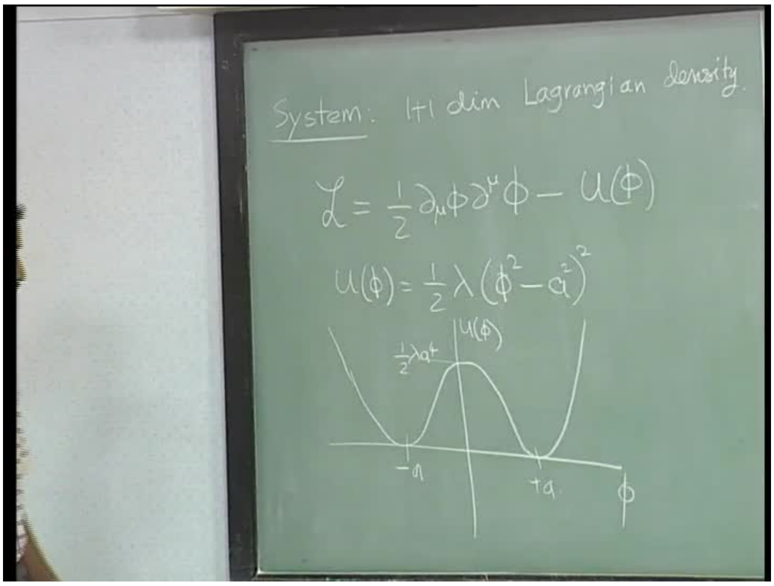

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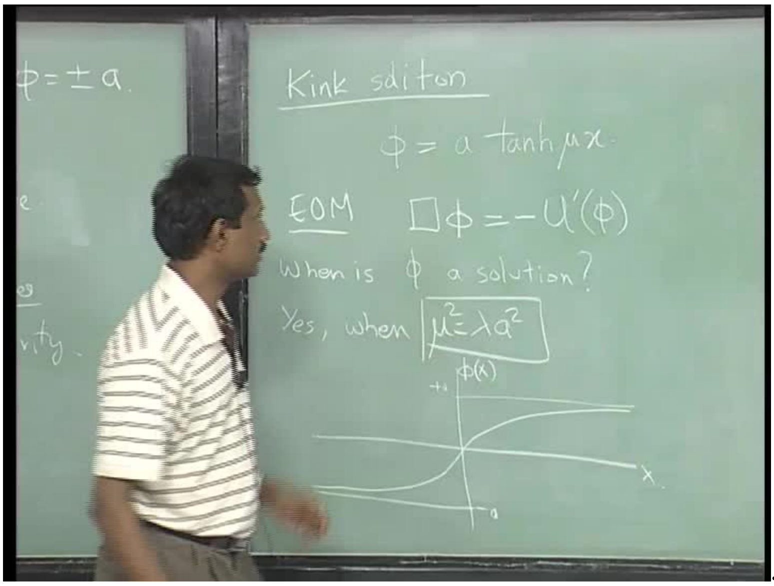

So this is the kind of structure which we would see, but if you look at so, but its if you look at the solution it is invariant under time translation itself, independent of time any case, so also in independent space translation, special translation as well, and its trivially invariant under boost. Now comes the interesting thing, which we notices, which we had discussed in class earlier, was that we can have interpolating solutions, and this is an example, the reason to look at this one, is this gone the few example, where we have actually explicit solution, analytical solution. So we will look at a solution, which we will call it the kink solution, and we will put on (( )) of the following form, and we need to check if it is a solution. So the equation of the motion, or a Langranian equation of motion, which we derived is just box phi equal to minus u prime. So we need to put this into this equation, and ask when is phi solution. Obviously because this is the non-linear function, but we will find, is that for arbitrary values of mu, we will not get the solution, but we will find that, the answer is yes, when as many you would have checked, when mu is lambda a square, where mu square is lambda a square. good So the thing is that, you do not have much freedom, but this, I mean this solution of course, takes cares of the following interpolation, at x goes to minus infinity, it goes to minus a, so this is the kind of solution, that we have this is phi of x versus x.

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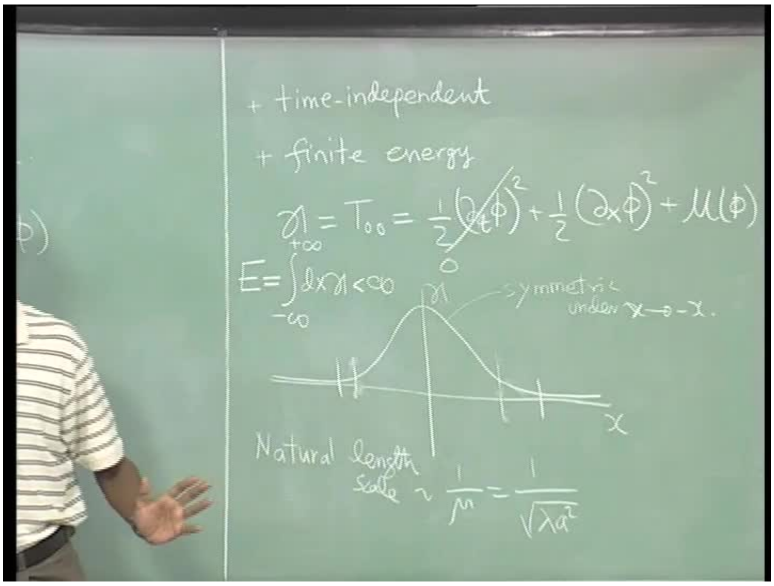

So now, the need thing about this, is that this is at, this is of course, solution is time independent, but another feature of this solution, is that it has finite energy. So the Hamiltonian in density for the system would be t 0 0, and it would have been d 0 of phi whole square plus half d x of phi let us put t, and for this particular u of phi look this is also a plus definite quantity, it is a square. So, we have, it is a sum of three independent quantity, and this is zero, because it is time independent. So it is a sum of two such quantities, and so you end up. So if you plot a Hamiltonian in density versus x, for this particular value, what you will observe, or you would have observed, is that first thing is agintonically it has to go to zero, because it agintodes the classical vacuums, so it has to be this way. And it is also a symmetric function of x. So the thing out here is that, is you can work how; that is the natural length scale associated with the problem, what is the length scale. (( )) so, the natural length scales here is one upon mu; so, that is a natural length scale, which is one upon mu, which is equal to, in this thing it is square root of one by square root of lambda a square.

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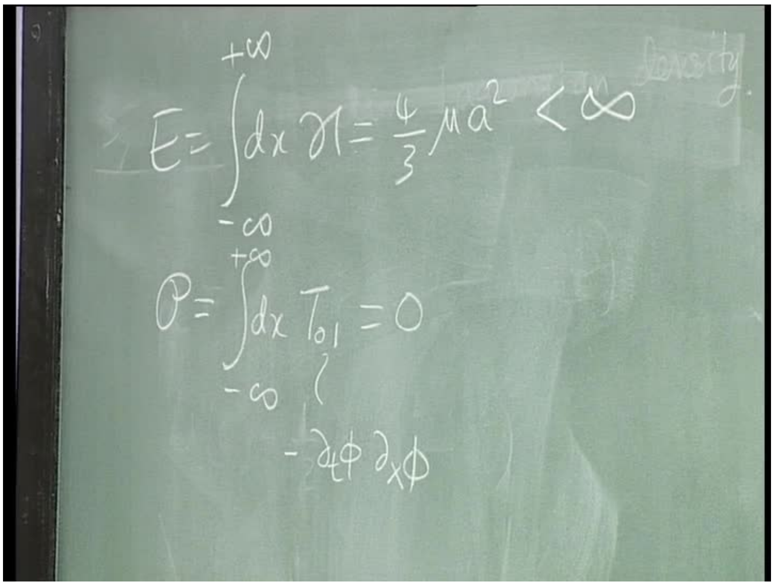

So what this will tell you, is this roughly of that order of this length scale, the energy actually goes to zero. So, it could be up to some factor, numerical factors of order one, but you can ask the following question, where is, and yeah the most important thing is at the total energy, which is integral of d x from minus infinity to plus infinity h, is less than infinity, it is actually finite, I have the number here somewhere, so lets me write it out here. So, the energy, which is for that solution turns out to be 4 by 3; obviously it's less than infinity it is finite. You could also compute, what the momentum of this thing is, of t zero one, and this turn out to be zero, because there is time derivative, and this is, if you look at the expression for it, it is just minus d t of phi d x of phi, but d t of phi is zero.

So what we have out here is a solution, which has finite so most of it is energy, is actually in a finite region is space. If you look in, it is exponentially suppress, if you look at how it behaves out here, in this region, it is exponentially suppress. So 99.99 is actually cover, most of the energy out here, it has finite site, it has finite total energy. And what you will see is that this is behaves, exactly like a particle, which has some size and some mass, this is. So it is exactly like this table, this table has some finite extent, going from here to here, it acts a certain mass, and times c square is an energy, and it does not carry momentum, exactly like this. And the key point, is this, this we think we are doing a classical field theory, but we have a classical solution, to the equation of motion, which behaves exactly like, it is no different from this, and to show evidence for this, what I had asked you to do us to boost the solution.

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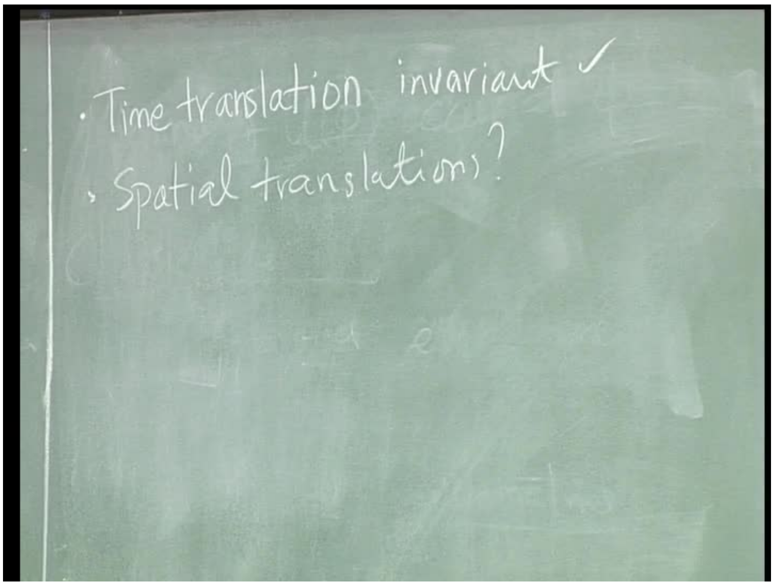

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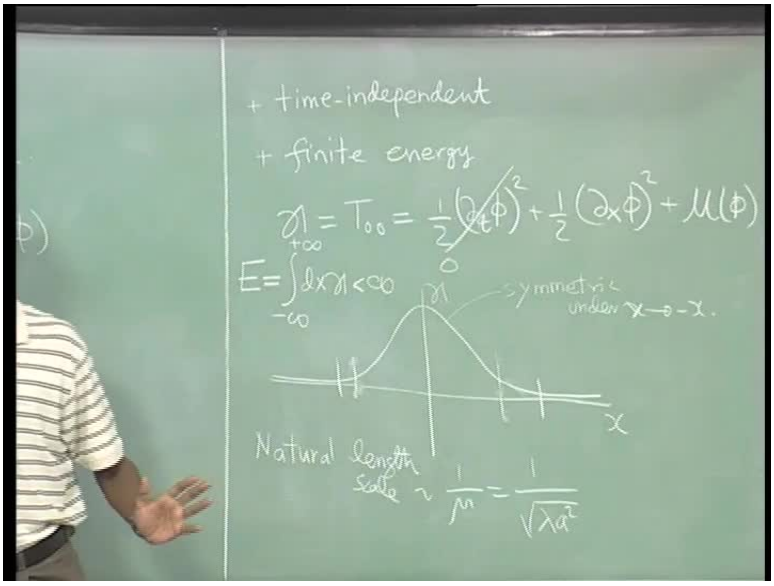

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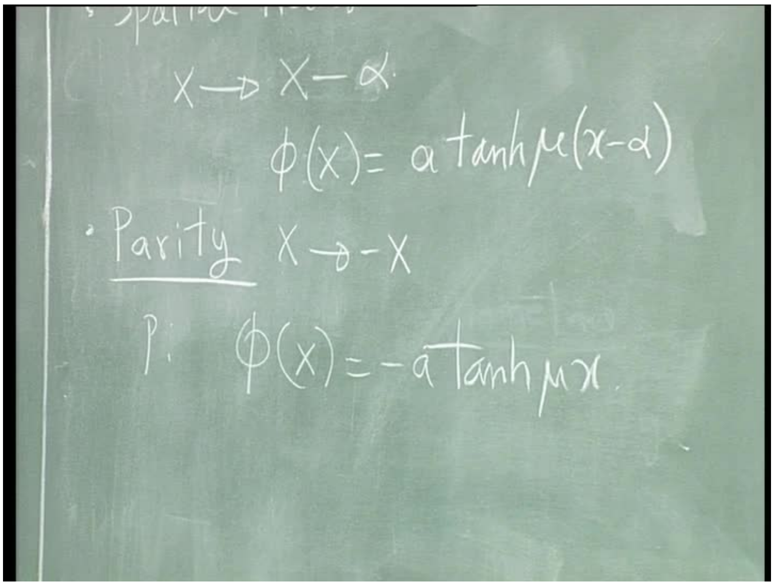

So before doing that, we will go back and look at the action of various symmetries in the theory, and we will see that, solution may not be invariance under everything. So first look at discrete symmetries first, or translation or time. It is, the solution is actually time translation invariant, because it no time dependence, but what about special translation. If you shift the o origin, or you move things, so this thing is an lump of energy which is centered at only origin, the way we have chosen it, if you do a shift, then it moves, so you get another solution, so answer is no. So what does that do, its suppose we took x, it goes to x minus a we get an, not a, because at you stop.

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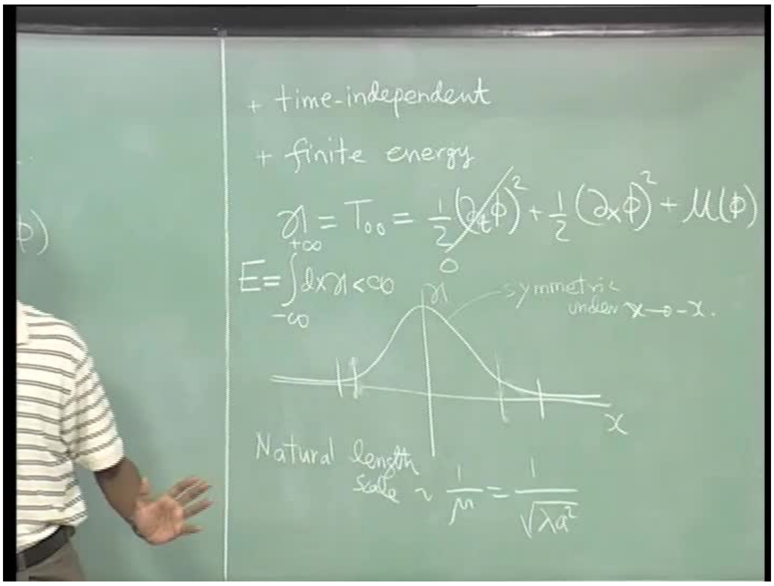

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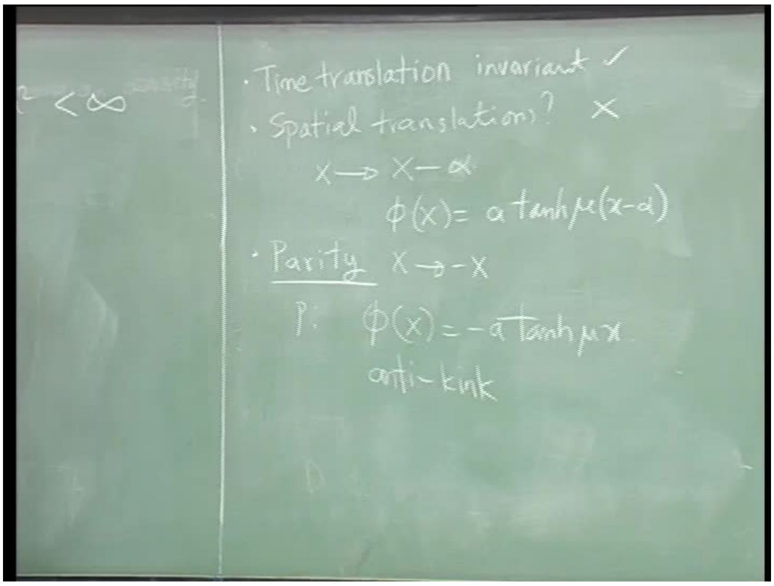

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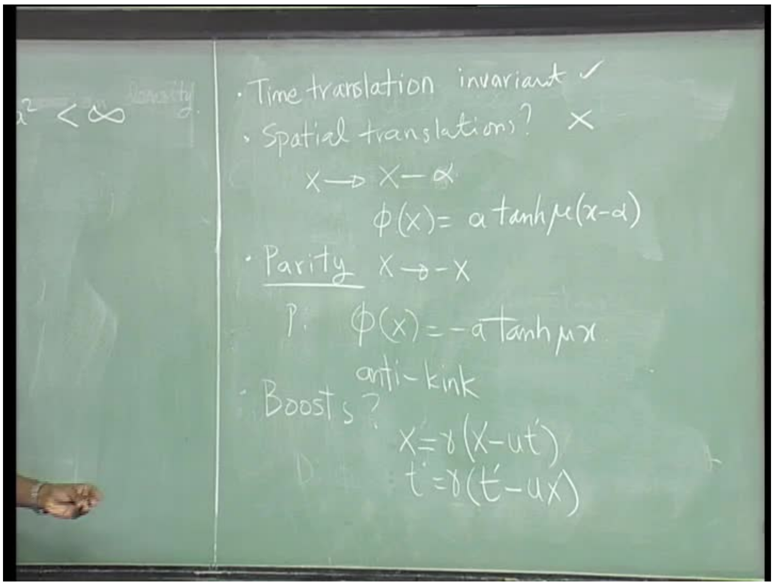

So let us say alpha, and then what we get is a new solution, which is. So that analog is to saying that I can move this table, from her to here, I can shift it by a bit, and that is exactly, and we will not say that they are the same solution, they are different solution. So, here is the case, whereas symmetry of your theory is, not a symmetry of the solution, but it will always map a solution to another solution so if, but sometime of course, it will map a solution it itself, like it did here, what about again parity. Parity would be x goes to minus x, so the solution under parity goes to phi of x. Again it gives you a new solution, it is a new solution because, the asymptotic at plus and minus infinity get exchanged. So if you call the first solution a kink, this is sometime called anti kink.

The reason it is called kink is, I guess, because of the structure of. It is a kink in space, having some finite energy, and this is called anti kink. It is just symmetric what I call kink and what I call anti kink, it is just this thing. So now comes what about boost, so under boost x goes to comma into x minus some mu t, and t prime goes to, and what is t prime. By c is one in our notation, so it is just, it is a same thing. (( )) it has to the other way, does not make sense. So gamma, it should be exactly like this; t minus u x, there had been factors of c square etcetera, these, what we have would get. So the point here now is, this, if you go ahead and plug it into the new solution, or maybe I should do it other way, so that passive active business, let us do it this way. So I plug this into the same solution, I get in new solution; x z now it is not a time independent solution, now it depends on time, and it is a nice fun exercise, which you have hopefully enjoyed doing, which is to work out.

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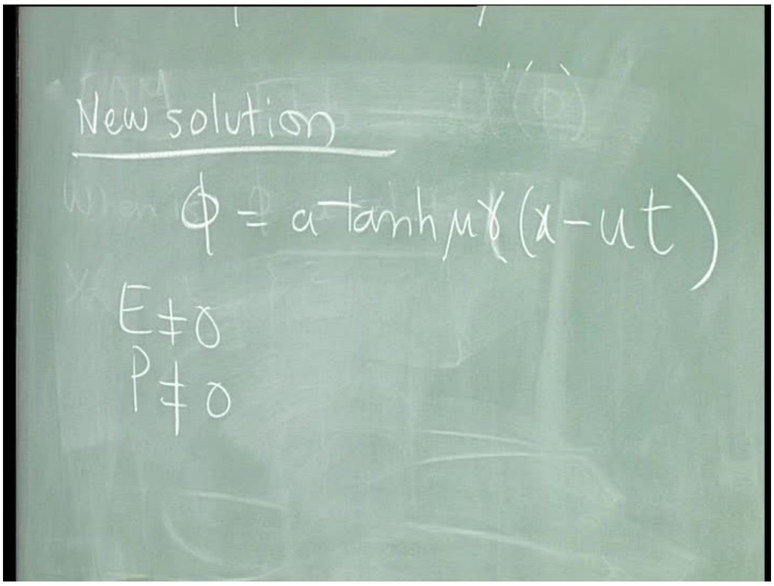

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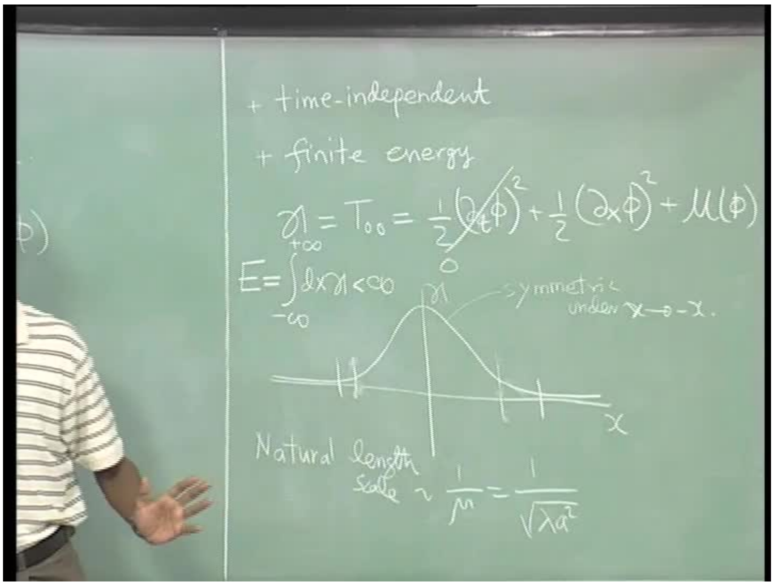

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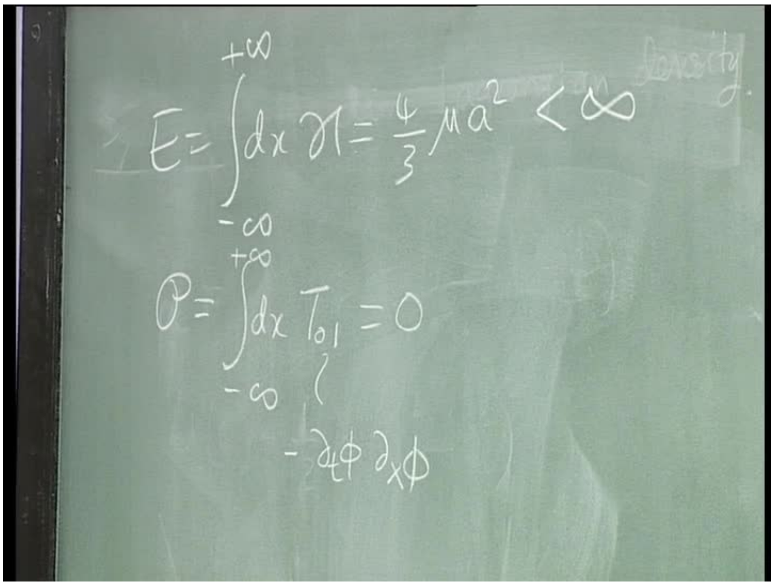

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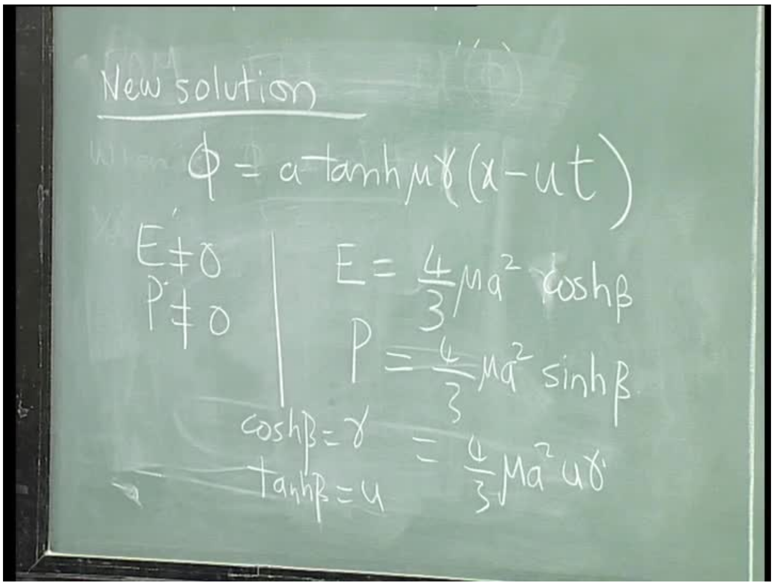

So the new solution under boost, I am dropping the prime, so this is the new solution, and it is fun to actually go back and check, you know you do not have to believe that this is a solution whatever, can go back and check, plug and check, that it does satisfy the equation of motion. But the fun part is to work out what is. Now in this case both e is not zero, p is also not zero, and. So now is where the proof of the pudding or whatever is, to look at the… So I told you we can think of this, as some object which has mass, which is given by this energy. So if you boosted, we know that these things will get changed, and if you use the standard formulae, you would find that. Let us call it e prime, would be 4 by 3 mu a square cos of these things which should be just the gamma factor ,but p would be, what is p; sin hyperbolic of. So wait, let me write this, I always think in term of the rapidity, which is the beta, and this would be sine hyperbolic of beta, where cos beta is just gamma, and tan beta is u. So form this you can work out, what this is. This is what you gamma. Yes, so this is equal to forth third mu a square mu gamma.

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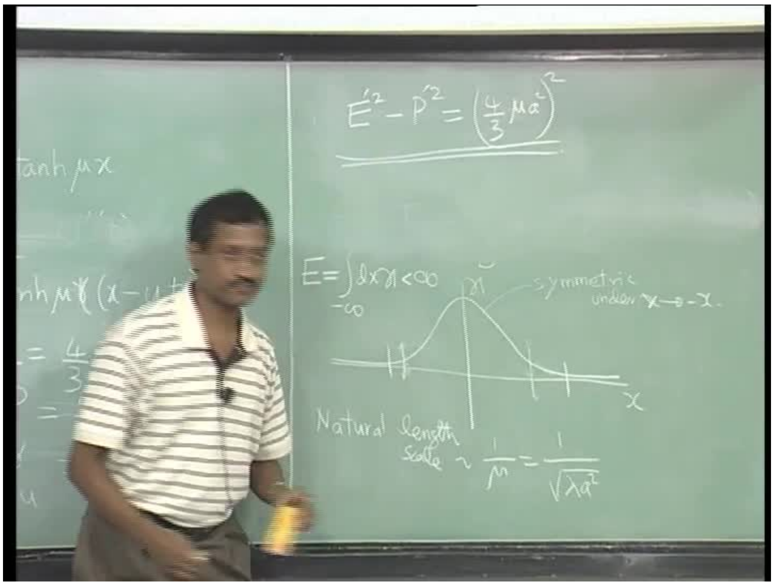

So what this tells you, if you go back and check is, e square minus p square is an invariant, it is equal to the square of this. So what we see, is exactly, the dispersion relation of a particle, which has this property of. So we started out with the solution, which was time independent, but by boosting it we got a solution, and now this object is behaving like, body which is moving with velocity u. So again its short of Lentzthreden to the fact, that we can think of this, as an lump of energy. And so it is in some sense it does behave like a relativistic particle, because dispersion relation is relativistic. So in this course we will see that there are many different kinds of solutions, which we would see, which has similar properties, and there are as I will discuss later, that this particular interpretation may be taken seriously, not for this example, this is very toy example, but we could to do a little bit more complicated solution, but we can no longer write out analytical solutions.

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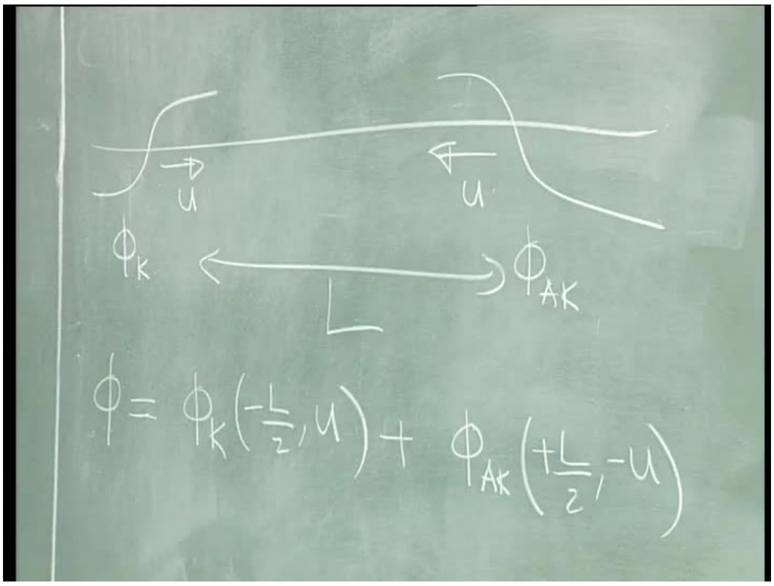

Suppose, so now we know we have boosted solution. Suppose I have a following sort of initial configuration at sometime, so I have. Let me just draw a picture, so that. Yes, this is what I want, yes. So I take a. So in this thing I have a kink, moving in some velocity in this direction, so this is a kink phi kink, moving from left to right, and this would be an anti kink, and let say it is, I could choose a different velocity, just put for simplicity, I could think of this things, and they have separated quite far apart of, this distance of separation between the centers, is very large, to a very good approximation phi of. So let us just write out, phi if I take it to be phi kink, with at location say; minus l by 2 moving with velocity u. So here I am just writing out a kink moving centered at minus l by 2, located at plus, l by 2 plus anti kink located at plus by 2 with velocity minus u. So it is very easy to write out what I meant by these two solutions. So as you can see as l is very large, compare to this length scale, or rather this length scale. There is a very good approximation, it is a good solution.

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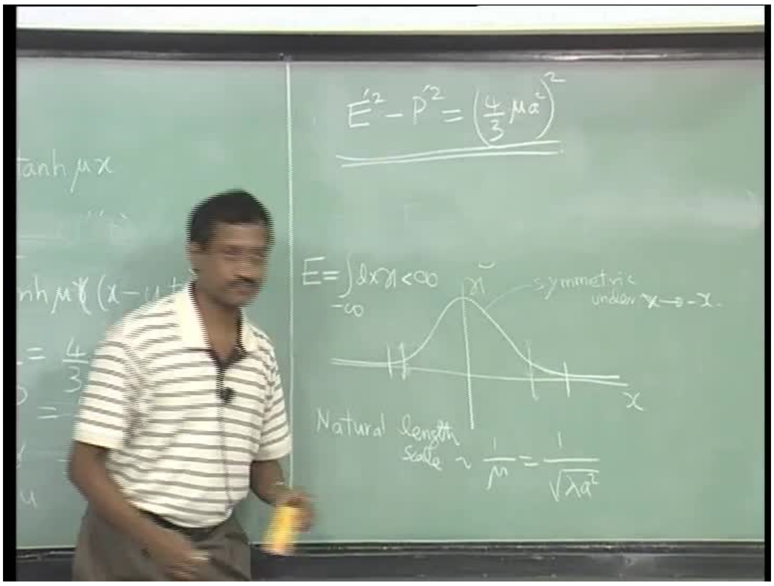

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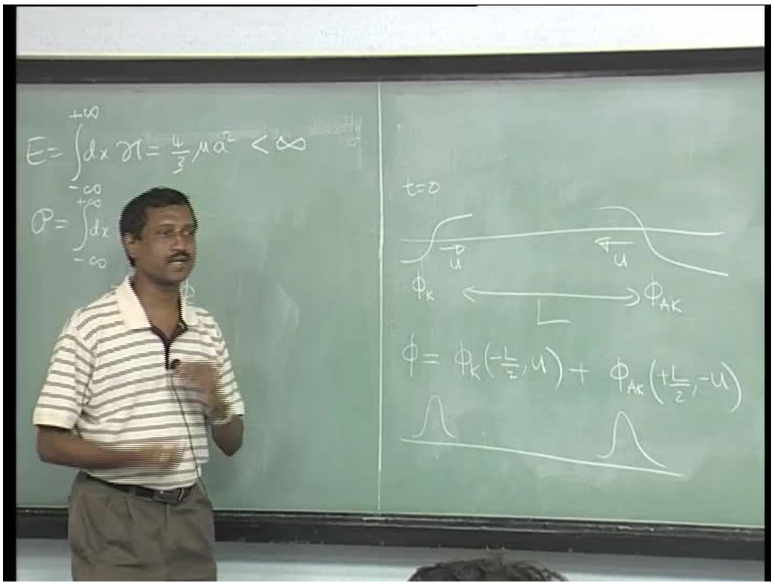

Now, so let say this is your configuration, at some initial time. Now the question is, you let this system evolve, and you can ask what happens to this. So clearly at some point, these two things will collide, and in terms of energy if I draw this profile, I will have something like this. something like this Of course, that no dissipation of energy, in this system energy is conserved. So obviously it cannot go to, it cannot go down to the vacuum. Now the question is, can they go though each other. The answer is no, because they has asymptotic here, are at plus infinity and minus infinity, both it is going to minus a. So that cannot, if they go though each other that would corresponds to ,actually flipping, so that cannot happen. So what can happen, what do you think will happen. They will rebound of each other.

So actually this is wearied, yes that what happens, but it depends on value of u. So here is the fun thing, what you should do, is to actually go ahead, write out a program or whatever, which we do this evolution. What you will find is, there is, this kink has a funny behavior, below some critical values of u, they tend to kind of stick to each other, does not bounce off, but if the u is about the critical values, it actually bounce off. So this is, I have seen a simulation of this, very does these sort of a thing. So but, the, what you have to do is to, convert the problem into the euler lagrange equation into a Hamiltonian kinds of equation, and that is not so hard I mean. So once you do that, you can actually. So here is the case of an approximate solution, but it is pretty good. So you do the evolution, and whatever you get after that, and what you should do, is to basically plot if you wish the energy density, as a function of time. So I will see initially that, I mean these thing come, and they come closer and they collide and go.

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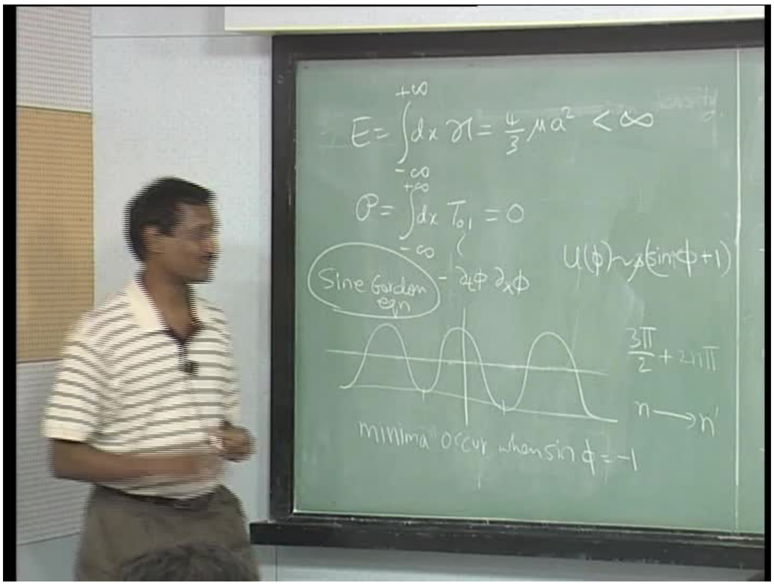

That is another theory, where which is called the sine Gordon theory, where the potential, up to some factor it is sine function. now this is the theory which has infinite number of, because of the potential, I have to shift it obviously, so minus what is it, the lowest values is minus one, so I had one, and there will be some factors etcetera, that is not so relevant, what is important, is to, so this is the kind of potential, you can see that, for every time sin phi hits minus one, you have a out here. So it has infinite number of minima, so unlike the example where we consider, there were only two things, you find that you have so minimum minima occur, when sine phi equal to. So again that might be some dimensional full parameters, but there what I mean, when this sine of this thing takes values, which is equal to minus one, and when dose that happen, when it is equal to 3 pi by 2 plus, so it is like, so there is 3 pi by 2 plus 2 n pi, is it, this is the periodicity of this thing.

So you have a infinity number of vacuum, unlike here where there only two possibilities, now you can see that I can have, I will have a vacuum, I mean I will have solution, which can interpolate between you know, vacuum corresponding to some n going to n prime. So I will have infinite such things, and in this example I can do things like, you know, now you can you can actually have many other possibilities, and this one actually, in some sense here you will find that things behaves elastically in somewhere, things can go though each other, you can have also out of things, because everything. So this th is so this is called the sine Gordon, equation just go ahead and look at Google or whatever, and look up this equation, and even the name of, how it was named sine Gordon modulus; interesting story, I will leave you to find out why it was called that.

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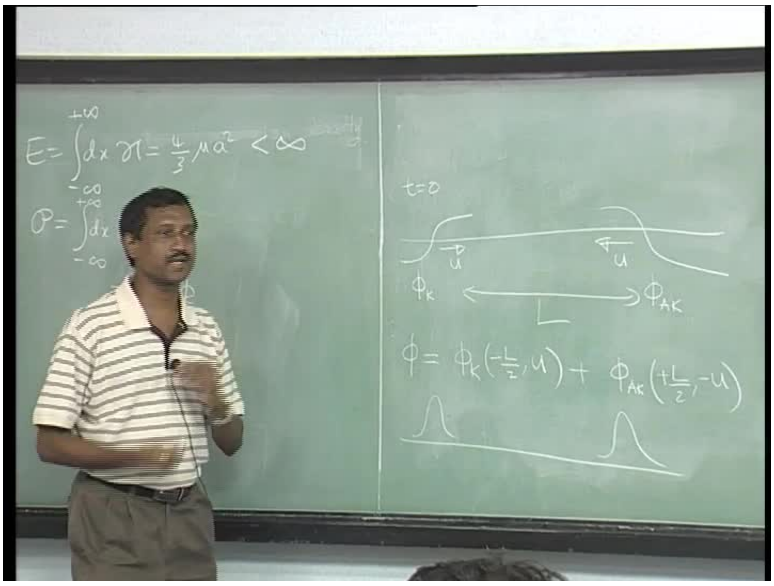

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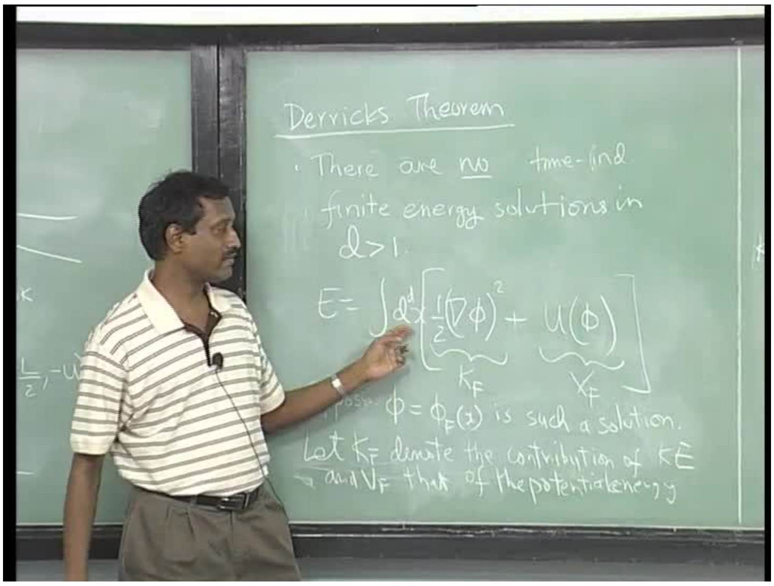

So this is an exercise which I would recommend you to try out. So, this looks very exciting, and it is looks like I can go ahead and, construct all kind of you know, look at time independent solution, in arbitrary dimension, and one made thing that one plus one is, you know we just used it, because it was a toy example etceteras, but there is something which sort of messes up the whole thing, and it says that it goes by the name of derricks theorem. Basically, it says that there is no such solution in high dimension. I will not be, no time independent, finite energy solutions, in space dimension greater than one. The proof is very simple, so I will explain the argument for it, but before I do that I should tell you a little bit about, the fact that the Hamiltonian density, and Lagrangian density are related by a Legendre transform you know that, and the key point to remember, is that Legendre transform map extrema to extrema. So the problem of extremizing the action, or minimizing the energy, actually get become related. So what we will show, the way we will show is to consider the energy, and let say that you give me a solution, if I can show you that there exist, similar solution with lower energy, then obviously that could be closer to the solution, and if I show you that the lowest energy configuration, is where the whole thing goes away, then we are morally, I am telling you that there is no such solutions.

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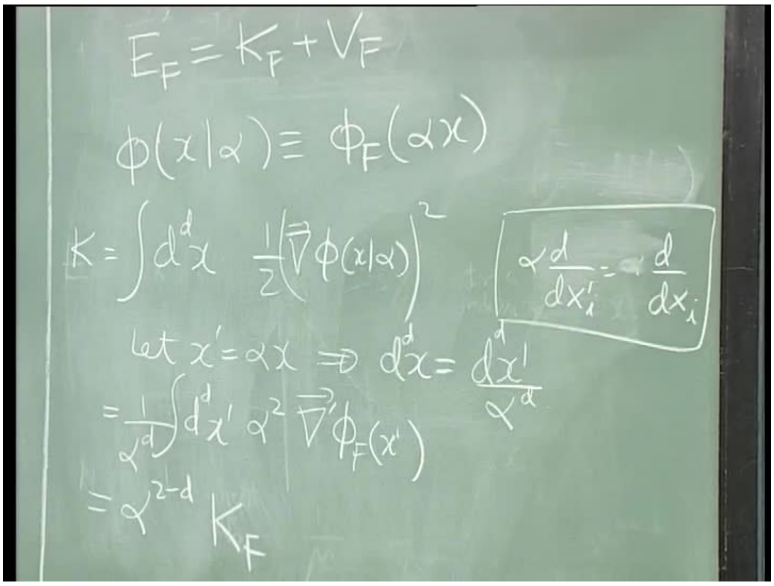

So we could, just what it have to concrete model on mind, take the same Lagrangian density excepts change the dimension to d plus 1. So, let us ask how that goes, so let us look at the energy density, because its time independent as only two components. It has kinetic energy, so not energy; let us look at the energy. So lets us assume, so let just go ahead and call, we will just call this term k f, and we will call this term, what did I call it v f. So suppose phi equal to some phi f of x, is such a solution. And let us, actually k f denote, the contribution of this term; the k e and e f that of the potential energy. So e some finite energy, lets even call it, so let us just write. So what I will do for you, is to give you, consider one parameter family of new solution, and ask what is the lowest energy. So let us just define phi of x, which is a new solution, with some parameters alpha. I just define it to be phi alpha x, alpha is just some parameter. So all I am doing is to rescale the variables. So now we can just go ahead, and ask how this term changes. So let us look at k f, so we want to ask what is the kinetic energy, so we want to do, integral d d x of half gradient of phi of x alpha, gradient, this what we want to do.

But this is just phi f scale, so I can rescale things, and pull things out. So I can redefine, let x prime be equal to alpha x. So this will go to. So, d d implies d d x equal to d x prime d divided by alpha power d, because every special coordinate is scale. So I get 1 by alpha power d, coming from the change of variables, and then there is, also have gradient prime, which should be equal to. Is this correct; x prime is this thing, so no, this is what I get. So what I can do is now, I just want to write everything in term of prime variables.

So what I will get here, is that d by d x. So I will get alpha squares coming from here, times prime variables, but now you can see that this is phi of x, it is like alpha is gone, so that is nothing but, it is phi f of x prime. So if I put these things together, I get alpha power 2 minus d times this is just the original value, which is k f. So by this scaling I have actually got, I can redoing the variables I get this. Similarly, we can check that, what about v f, this is only a scalar function, so all I will get, be a alpha power minus d.

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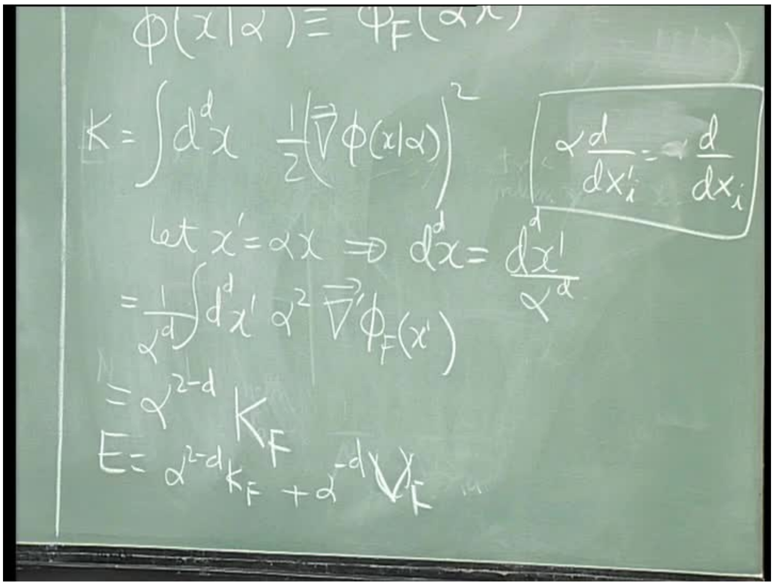

So e of the new solution, is just alpha power 2 minus k d f plus alpha power minus d v f, where k f and v f are just some numbers given.

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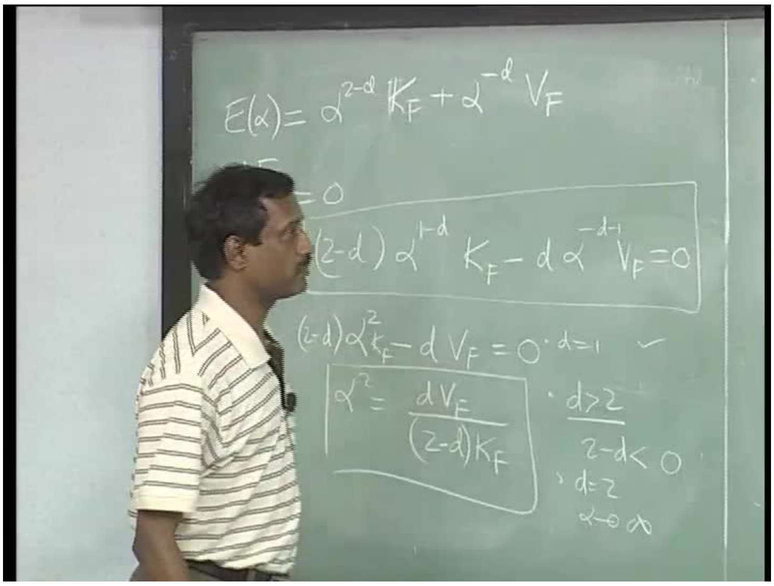

So let me rewrite what I have, and this is the function of alpha. So if you are giving me some time independent finite energy solution. If I want that, I mean I have got a one parameter family, I extremise this with respect to alpha, and if I get the lowest energy configuration, which is lower than that, then I would say that that is, has a greater chance of being, the exact solution, to the equation of motions. So we just go ahead, and do this a one parameter family, I just extremise it. So what do I get here, I get this, so I get 2 minus d. So this is what I get, so I need to solve for alpha, whatever alpha I get here I have to check, if it is, I still need to do one more step, but what are the values. So let us this is correct one minus d, this thing, so. So what do I do next, yes. So I can just pull out these things, alpha power. I multiply everything by alpha power d plus one; is that correct. So actually it is just alpha square. No let me just simple.

So alpha square into 2 minus d minus d, or alpha square equal to d b f by 2 minus d k f. There is, I mean look out here, so first thing is, what happens when b equal to one. So first thing is d greater than 2, this a negative number, for t greater than 2, the 2 minus d is less than zero. These are all posstum numbers, remember that. So this solution would be pure imaginary, so that is not because you have a real kind of field, so that is that has a problem, but for d equal to one, there is indeed a nice solution, and d equal to 2, you can see that the solution corresponds to as for going to infinity.

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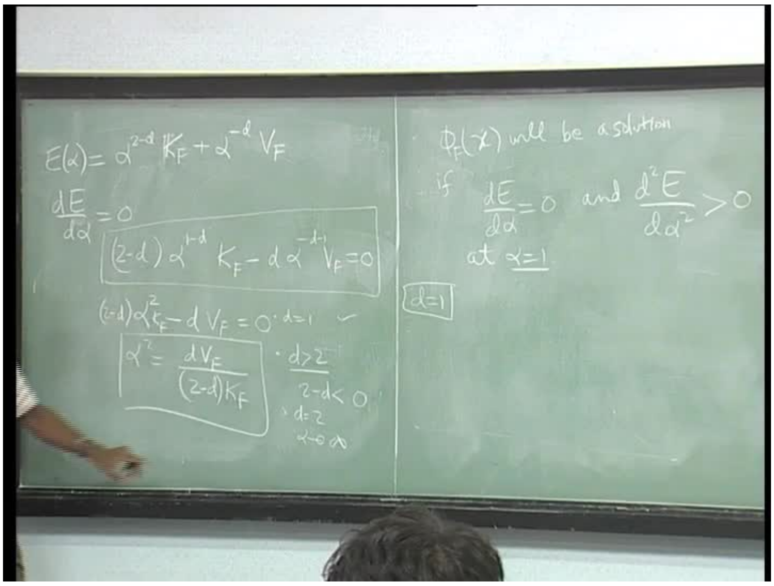

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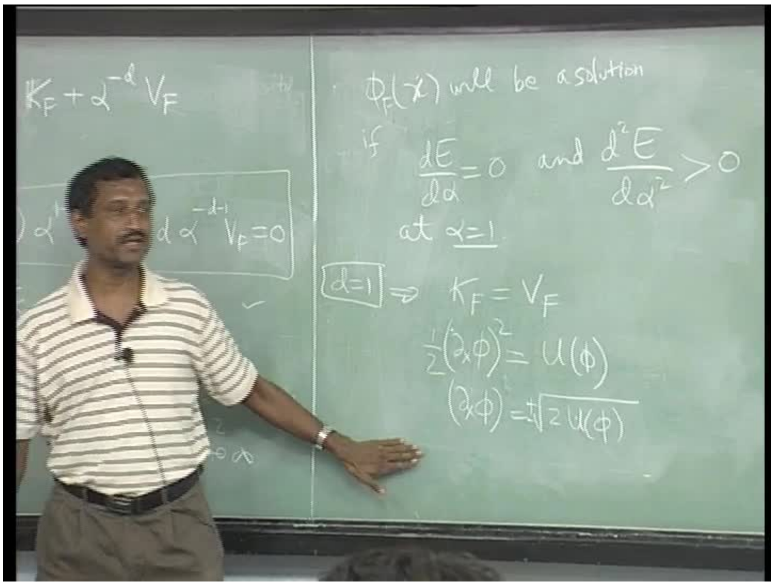

Is this clear? But we can also view it in a slightly different manner, what would happen, let us assume that phi f is indeed a solution, so that would imply, when phi f would be a solution. If f equal to zero, and d square e by d alpha square is greater than zero, these are the two conditions. So, let us add alpha equal to one, because we want, we are claiming somebody is coming up with the solution, relative solution. So question is, can this happen, and so let us do, let us see what happens at d equal to one. So, d equal to one, if you plug this in, so you get one here, and this is also one, and alpha is one. So this will imply, that we need k f equal to b f, and what is that imply, so this. So k f, what was k f, that came from integral of the kinetic energy, and this came from integral of the potential energy, so (( )) no, what I am saying is that, the key point here is that, this can happen if half. (( )) wait, lets we are just doing this, so what we get here, is that grad phi square, or grad phi should be is just one dimension, so I just write it out like this d x of phi should be equal to square root of two plus or minus square root of u of phi.

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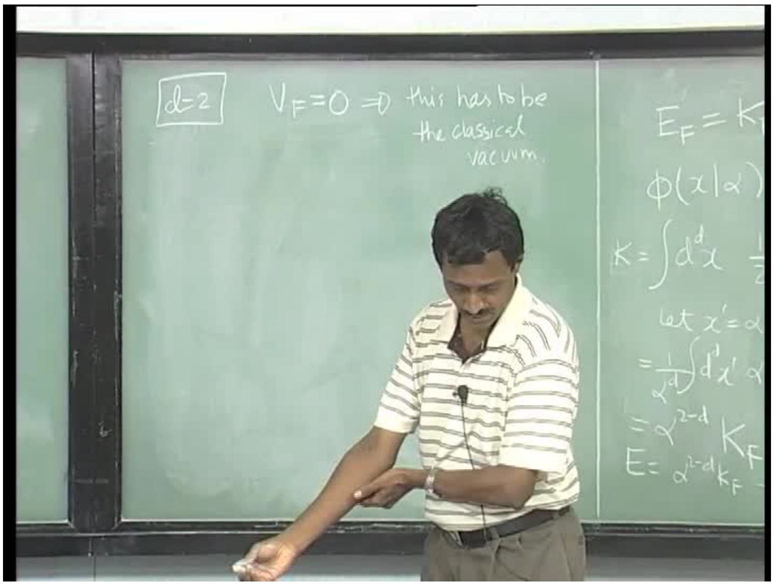

If this is true. Now the need stuff is now, we actually, this is a first order differential equation if. (()) we have a second order equation, but you can go back and actually verify that, the kink solution that we wrote; one of the plus sign is if plus sign is kink, the other is the anti kink. You can check that is actually satisfies, of a first order equation, is so satisfying a second order equation, and you can verify that is indeed the case. But that happens in the other cases, so what happens when d equal to 2. When d equal to 2, and alpha equal to one, you can see that would require v f should be equal to zero, but when is that possible, that implies that it has to be a classical minima everywhere, so there exist, so this implies. If you give me a solution, this implies that, this has to be the classical vacuum. I will come back to my earlier discussion, and what happens for d greater than 2, this becomes sum of two negative terms. This also becomes negative, this also becomes negative. So really the only way it can be zero, is if both are simultaneously zero. So this is the only way this can happen.

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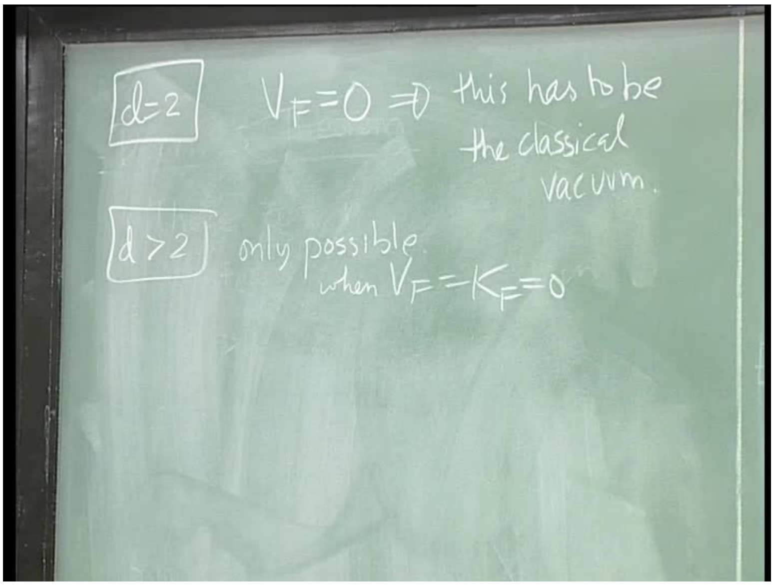

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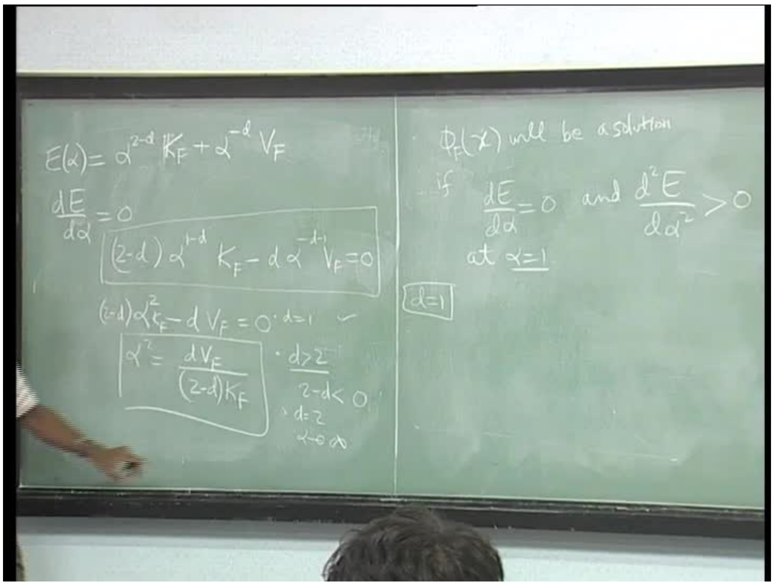

So actually let me come back and discuss what is going on, if we can still go through this process, but what this will show you is that, it its sort of makes. So suppose you have something which has finite energy, and some size, by scaling it you can actually scrunch that thing, and make it smaller and smaller, and then it sort of makes sense for it to completely be a zero size, it will just disappear in some sense, and so that is like saying it become a classical vacuum at the end of the day.

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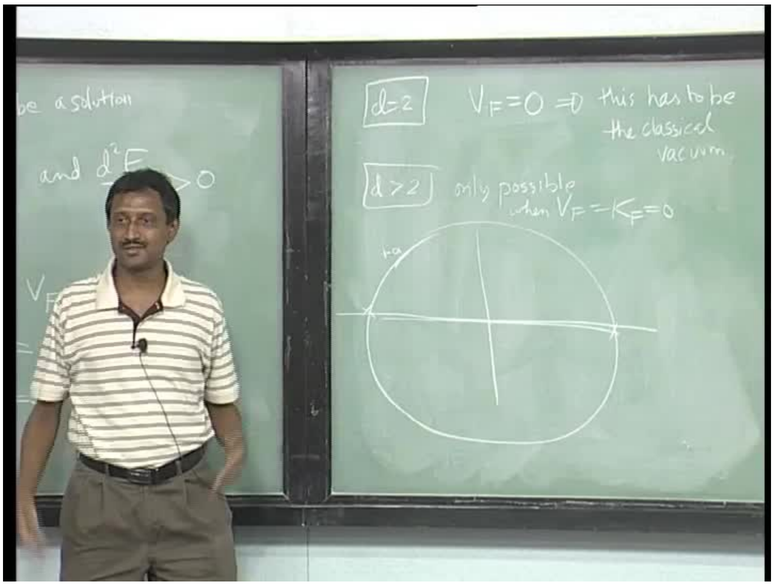

So that cannot be a solution. So this looks like bad news, this tells you that there are no finite energy, time independent solution in high dimension. (( )) there is a scale, there is a scale which one. (( )) why (( )) why, does not it. It does, it has a mass term, and it has, it has, I mean there are things which have dimensions of or length. There are things; that is not the issue. So but, the key point, I mean there is also something which is, if you go to two dimension for instance. I mean you need to specify boundary condition at the circle related infinity. So let say we took the same phi forth potential, which has two vacuum. So when we had one dimension, we could chose plus infinity to be one thing and minus infinity, but here you have to actually set it, you now continuously. So suppose you said that, this region is going to be a plus a, now if you unless you permit some discontinuity, you are forced to choose plus a everywhere. It gets only words in high dimension if you have a sphere, you can uniform. So the point is that, in one dimension there was no way for me to go from this to this plus infinity, but if I go to two dimensions, easily they are smoothly connected by a circular infinity. So, again there is not enough space, and to actually write out solution. So the only solution that you have which are finite energy, are the vacuum solutions, and that is it.

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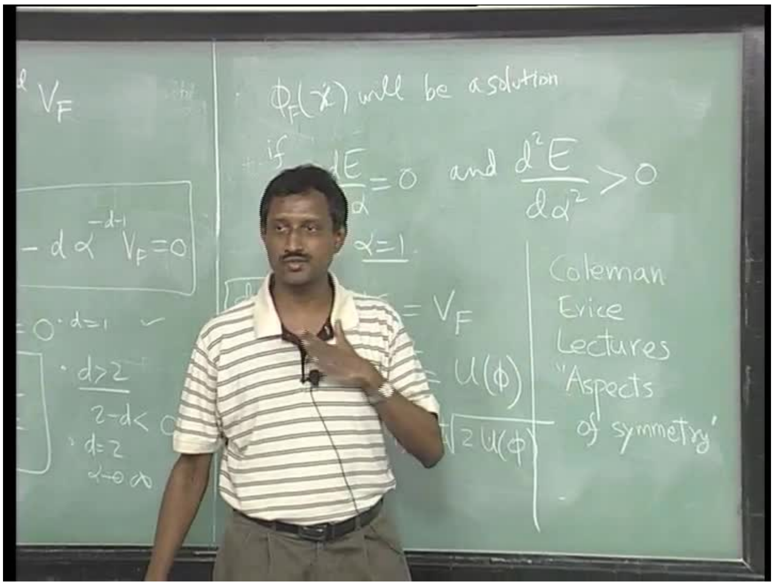

So this is a typical example of what is called a Nogo theorem, but the funniest things, or nicest things about Nogo theorem, is that they can be evaded, and we will see in this course, how this can be evaded, it happens in a very nice manner, but that will you have to wait for may be a few lecture, we will get to that point. So, Derricks theorem is a kind of knells, and it tells you that you do not have solution, which can have finite energy. I mean I am not saying the there are no configuration without, I mean I am sure there are, but they would not be solution your equation of motion. And in fact I would recommend Coleman Erice lecture. So it is a book, titled aspects of symmetry. In fact in that he has a series of lecture on different topics, Erice he used to give a set of lectures over many years, so it was compiled I think four or five years, so it has each lecture is sort of dependent of this thing, but he has something on what he call solitones, or whatever solutions. One of this lectures are is on this topic, and in fact out there he gives a time dependent finite energy solution, just to give you a counter examples. I mean here I clearly said things are time independent, so that is a counter example in some particular model, but just that enough to this thing. So but, what we are talking about is time independent finite energy solution, they are retaining, and that is consequence of derricks theorems.